MODERN PLASMA PHYSICS
Trieste Course 1979
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A BASIC COURSE
GIVEN AT AN AUTUMN COLLEGE
AT THE INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
TRIESTE
16 OCTOBER TO 23 NOVEMBER 1979
AND
SELECTED LECTURES ON
ADVANCED TOPICS
IN FUSION RESEARCH
PRESENTED AT THE COLLEGE

INTERNATIONAL ATOMIC ENERGY AGENCY
VIENNA, 1981
THE INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS (ICTP) in Trieste was established by the International Atomic Energy Agency (IAEA) in 1964 under an agreement with the Italian Government, and with the assistance of the City and University of Trieste.

The IAEA and the United Nations Educational, Scientific and Cultural Organization (UNESCO) subsequently agreed to operate the Centre jointly from 1 January 1970.

Member States of both organizations participate in the work of the Centre, the main purpose of which is to foster, through training and research, the advancement of theoretical physics, with special regard to the needs of developing countries.
FOREWORD

The 1979 Autumn College on Plasma Physics was held at the International Centre for Theoretical Physics, Trieste, from 16 October to 23 November. The Directors were: B.B. Kadomtsev (USSR), B. McNamara (USA), K. Nishikawa (Japan), D. Pfirsch (Federal Republic of Germany), M.N. Rosenbluth (USA) and R.K. Varma (India).

The College catered for a wide variety of needs of participants and lecturers. The first two weeks consisted of a basic course given by leading plasma physicists from developing countries. This proved to be an essential refresher course in preparation for the range of topics to come. The third week offered a change of pace with lectures on several fascinating small experiments, which showed that good and original physics can be done with modest resources. The course on small computers showed that these devices are becoming very cheap and that high-quality computing is possible for a very small capital investment. It will take some time for developing countries to use and appreciate these systems, which must be among the cheapest resources in modern science.

The final three weeks offered lectures on fusion research, computational plasma physics, and space plasmas. The latter were particularly well coordinated, as Professor A. Nishida had arranged with the other lecturers to produce a book reviewing magnetospheric physics.

The present volume comprises the Basic Course and a selection from the advanced lectures.

An International Seminar on Plasma Physics in October 1964 was the first major activity of the International Centre for Theoretical Physics; the Proceedings were published by the International Atomic Energy Agency. The seminar began a tradition of extended contacts between plasma physicists from the United States, West European and Soviet schools. The ICTP College on Theoretical and Computational Plasma Physics, which included the Third International (‘Kiev’) Conference on Plasma Theory, was held in 1977, and the Proceedings, published by the IAEA, included selected lectures from the Kiev Conference.

The 1979 Autumn College began shortly after the announcement that Professor Abdus Salam would share the Nobel Prize for Physics, which enhanced — if that is possible — the normal friendly atmosphere of the Centre. The Directors of the College wish to record their thanks to Professor Salam and his staff for their hospitality and their contribution to international scientific cooperation.
EDITORIAL NOTE

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Part 1

BASIC COURSE
MACROSCOPIC PLASMA PROPERTIES AND STABILITY THEORY

P.H. SAKANAKA
Instituto de Física,
Universidade Estadual de Campinas,
Campinas, São Paulo,
Brazil

Abstract

MACROSCOPIC PLASMA PROPERTIES AND STABILITY THEORY.


1. TWO-FLUID EQUATIONS

1.1. Boltzmann equation

Consider a plasma with charged particles $\alpha$ with mass $m_\alpha$ and charge $e_\alpha$, where $\alpha$ represents the electrons and different ion species. A kinetic representation of this plasma is the one-particle, time-dependent distribution function $f_\alpha(\vec{r}, \vec{v}, t)$, where $f_\alpha$ is defined so that $f_\alpha \, d\vec{r} \, d\vec{v}$ is the probable number of particles in the phase-space volume element $d\vec{r} \, d\vec{v}$ at time $t$. The variation in time of the distribution function is described by the Boltzmann equation [1, 2]:

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \nabla f_\alpha + \frac{e_\alpha}{m_\alpha} (\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}) \cdot \frac{\partial f_\alpha}{\partial \vec{v}} = -f_\alpha$$

(1.1)

where $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$ are the sum of the external electric and magnetic fields and average internal fields originating from the long-range interparticle interactions. The right-hand side of Eq.(1.1) gives the rate of change of the distribution function due to short-range interactions of particles $\alpha$ with particles $\beta$. 
and is called the collision term. The contribution of species $\beta$ on the change of the distribution function of particles $\alpha$ is represented by $C_{\alpha\beta}$, so the total change is given by

$$C_{\alpha} = \sum_{\beta} C_{\alpha\beta}$$

The fields $\vec{E}$ and $\vec{B}$ are described by the Maxwell equations where the current and charge densities, $j(\vec{r}, t)$ and $\rho_q(\vec{r}, t)$, are determined from the velocity moments of the distribution function. The zeroth and first moments give the number density and the average velocity of particles

$$n_{\alpha}(\vec{r}, t) = \int f_{\alpha}(\vec{r}, \vec{v}, t) \, d\vec{v} \quad (1.2)$$

and

$$\vec{u}_{\alpha}(\vec{r}, t) = \frac{1}{n_{\alpha}(\vec{r}, t)} \int f_{\alpha}(\vec{r}, \vec{v}, t) \vec{v} \, d\vec{v} \quad (1.3)$$

The charge and current density are then defined by

$$\rho_q(\vec{r}, t) = \sum_{\alpha} e_{\alpha} n_{\alpha}(\vec{r}, t) \quad (1.4)$$

and

$$j(\vec{r}, t) = \sum_{\alpha} e_{\alpha} n_{\alpha}(\vec{r}, t) \vec{u}_{\alpha}(\vec{r}, t) \quad (1.5)$$

Considering that all particles in the plasma are free, polarization and magnetization are negligible, so one takes $\mu = \epsilon = 1$ in the Maxwell equations. Therefore

$$v \times \vec{E} = - \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (1.6)$$

$$v \times \vec{B} = \frac{4\pi}{c} j + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad (1.7)$$

$$v \cdot \vec{E} = 4\pi \rho_q \quad (1.8)$$

and

$$v \cdot \vec{B} = 0 \quad (1.9)$$
Equations (1.1) and (1.4)—(1.9) form a closed system of equations, except for the collision term which has to be properly modelled. The Boltzmann equation (1.1) is written for each species \( \alpha \).

The collision term may be made more explicit by adopting a collisional model, the choice of which is determined by the need and the features sought for the plasma.

The simplest model is the one which neglects the collision term. This leads to the Vlasov equation:

\[
\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \nabla f_\alpha + \frac{e_\alpha}{m_\alpha} (\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}) \cdot \frac{\partial f_\alpha}{\partial \vec{v}} = 0 \quad (1.10)
\]

The Vlasov equation is applicable for the cases in which the effects of the short-range binary Coulomb collisions are negligible. This situation arises when the plasma parameter \( g \), defined as \( g = 1/nkT \), is small. Here \( \lambda_D \) is the Debye length; \( \lambda_D^2 = kT/4\pi ne^2 \); \( k \) being the Boltzmann constant, \( T \) the temperature, \( n \) the particle density and \( e_\alpha \) the particle charge \([2, 3]\). Therefore \( g \) is the inverse of the number of particles in a Debye sphere.

A simple collision model which describes partially ionized plasmas, where collisions of charged particles with neutrals are important, is the *Krook model*, which is given by

\[
f_\alpha (\vec{r}, \vec{v}, t) = f_{\alpha 0} (\vec{r}, t)
\]

where \( \tau_\alpha \) is the mean free time and \( f_{\alpha 0} \) is the local Maxwell-Boltzmann distribution:

\[
f_{\alpha 0} (\vec{r}, \vec{v}, t) = n_\alpha (\vec{r}, t) \left( \frac{m_\alpha}{2\pi kT_\alpha} \right)^{3/2} e^{-m_\alpha \vec{v}^2/2kT_\alpha} \quad (1.12)
\]

where

\[
n_\alpha (\vec{r}, t) = \int f_\alpha (\vec{r}, \vec{v}, t) d\vec{v} \quad (1.13)
\]

This model conserves the number of particles but does not conserve the momentum or the energy. The distribution function relaxes to a local Maxwellian in the characteristic time \( \tau_\alpha \).

A somewhat more elaborate model is the *BGK model* \([4]\), which can be applied with good approximation to a fully ionized plasma. The BGK model has the form
where $\Pi/\sigma$ is the average collision frequency, $\Pi$ being average density over the configuration space; $f_{a0}$ is the local Maxwell-Boltzmann distribution

$$f_{a0}(x, v, t) = \frac{n_a(x, t)}{2\pi kT_a(x, t)} \left[ \frac{m_a}{2kT_a(x, t)} \right]^{3/2} \exp \left\{ -\frac{m_a}{2kT_a(x, t)} \left[ v^2 - v_a^2(x, t) \right]^2 \right\} \quad (1.15)$$

with

$$v_a(x, t) = \frac{1}{n_a} \int v f_a \, dv \quad (1.16)$$

and

$$kT_a(x, t) = \frac{m_a}{3n_a} \int (v - v_a)^2 f_a \, dv \quad (1.17)$$

This model conserves particles, energy and momentum and may be used to describe collisions in fully ionized plasmas.

A more accurate description of the effect of binary Coulomb collisions in fully ionized plasma is the Fokker-Planck model [2, 3, 5—9]:

$$C_{\alpha\beta} = \Gamma_{\alpha} \left[ -\frac{3}{3v_1} f_{\alpha} \frac{\delta h_{\beta}}{\delta v_1} + \frac{1}{2} \frac{3}{3v_1 3v_j} (f_{\alpha} \frac{\delta^2 g_{\beta}}{\delta v_1 \delta v_j}) \right] \quad (1.18)$$

where $i$ represents the $i$th component of the vector $v$ and the repeated indices mean sum over all $i$'s; $\beta$ represents the colliding particle species,

$$\Gamma_{\alpha} = \frac{4e^2}{m^2} \frac{n \Lambda}{\alpha} \quad \Lambda = 24\pi / g \quad (1.19)$$

$$h_{\beta}(v) = \left( \frac{e_{\beta}}{e_{\alpha}} \right)^2 \frac{m_{\alpha} + m_{\beta}}{m_{\beta}} \left( \frac{m_{\alpha}}{m_{\beta}} \right) \frac{1}{|v - v'|} \int f_{\beta}(v') \, dv' \quad (1.20)$$

and

$$g_{\beta}(v) = \left( \frac{e_{\beta}}{e_{\alpha}} \right)^2 \int f_{\beta}(v') \left| v - v' \right| \, dv' \quad (1.21)$$
In its derivation, the two-body Coulomb scattering is integrated over all particles. The Coulomb divergence for small-angle scattering is eliminated by the Debye screening effect, and the scattering angle is limited to $\pi$, which is a good approximation. The quantity $\Gamma_{\alpha} \partial h_{\alpha}/\partial \tilde{v}$ is the coefficient of dynamical friction, and $\Gamma_{\alpha} \partial^2 g_{\alpha}/\partial \tilde{v} \partial \tilde{v}$ is the dispersion coefficient. The former is the cause of the slowing down of particles streaming through the plasma, which leads to the plasma resistivity, and the latter causes diffusion in velocity space, which leads to the thermalization of a non-Maxwellian plasma.

The expression (1.18) can be put in a more symmetric form:

$$C_{\alpha \beta} = - \left[ \frac{e_B}{e_A} \frac{2}{2} \frac{m_A + m_B}{m_A} \Gamma_{\alpha} \right] \left\{ \int \frac{d \mathbf{k}}{4 \pi} \left( \frac{f_{\alpha} (v)}{m_B^*} - \frac{f_{\beta} (v')}{m_A + m_B} \frac{\partial f_{\alpha} (v')}{\partial v'_{k}} \right) \right\} dv^3$$

(1.22)

where

$$U_{1 k} = \frac{\partial^2 u}{\partial v_{1} \partial v_{k}} = \frac{\delta_{1 k}}{u} - \frac{u_{1} u_{k}}{u^3}$$

(1.23)

and

$$\tilde{u} = \tilde{v} - \tilde{v}'$$

(1.24)

The expression (1.22) is known as the Landau form [9] except for the denominator $m_A + m_B$, which is $m_A$ in this latter form.

All of these models make the system of Boltzmann-Maxwell equations a closed system.

**PROBLEM 1:** Derive (1.22) from (1.18).

**SUGGESTION:** Show first

$$h_{\beta} (v) = \frac{m_A + m_B}{2m_B} \frac{\partial^2 g_{\beta} (v)}{\partial v_{k} \partial v_{k}}$$

1.2. Moments of the Boltzmann equation

In the kinetic formulation, the plasma is represented by the distribution function of particles in the space $(\mathbf{r}, \mathbf{v})$. In the macroscopic formulation, the plasma is represented by macroscopic quantities such as particle density, flux, current density, heat flux, etc., defined in configuration space $\mathbf{r}$, i.e. in a three-dimensional instead of a six-dimensional space. This makes the macroscopic quantities easier to handle, but in compensation there are several quantities to be dealt with instead of only one.
The moments of the Boltzmann equation lead to a system of macroscopic equations. To reproduce all the results of the kinetic equation, one must, in principle, use an infinite set of macroscopic equations derived from all the moments of the Boltzmann equation. This is not at all practical and one has to be content with the limited number of equations with, consequently, a limited set of results. This truncation of the system of equations introduces a problem of closure, as will be shown later.

The moments of the collision term $C_a$ have conservation properties, for the collision of a particle $\alpha$ with a particle $\beta$ does not change their positions but only their velocities. So we have:

*Particle conservation:*

$$C_{\alpha\beta} \, d\vec{v} = 0 \quad (1.25)$$

*Momentum conservation:*

$$\int m_\alpha \, \vec{v} \, C_{\alpha\beta} \, d\vec{v} + \int m_\beta \, \vec{v} \, C_{\beta\alpha} \, d\vec{v} = 0 \quad (1.26)$$

*Energy conservation:*

$$\int \frac{1}{2} m_\alpha v^2 \, C_{\alpha\beta} \, d\vec{v} + \int \frac{1}{2} m_\beta v^2 \, C_{\beta\alpha} \, d\vec{v} = 0 \quad (1.27)$$

When the collision is between particles of the same species, Eqs (1.26) and (1.27) reduce to equations with a single term.

1.2.1. Zeroth moment

The zeroth moment of the Boltzmann equation is obtained by integrating it over velocity space. One has the following terms:

$$\int \frac{\partial f_\alpha}{\partial t} \, d\vec{v} = \frac{2}{\beta t} \int f_\alpha \, d\vec{v} = \frac{\partial}{\partial t} \, n_{\alpha} (\vec{r}, t) \quad (1.28)$$

$$\int \vec{v} \cdot \nabla f_\alpha \, d\vec{v} = \int \vec{v} \cdot \nabla f_\alpha \, d\vec{v} = \vec{v} \cdot n_{\alpha} \vec{u}_{\alpha} (\vec{r}, t) \quad (1.29)$$

$$\int \frac{e_{\alpha}}{m_{\alpha}} \left[ \vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right] \cdot \frac{\partial f_\alpha}{\partial \vec{v}} \, d\vec{v} = 0 \quad (1.30)$$

and

$$\int C_{a} \, d\vec{v} = 0 \quad (1.31)$$
The first term of Eq. (1.30) is zero, if it is assumed that the surface contribution is zero. In general, it is assumed that the surface contribution is always zero, no matter how high the degree of the polynomial which multiplies the function $f_\alpha$, i.e.

$$ P(v_x', v_y', v_z') f_\alpha(\hat{\mathbf{r}}, \hat{\mathbf{v}}, t) \bigg|_{s} = 0 \quad (1.32) $$

where $P$ is a polynomial in $v_x, v_y$, and $v_z$.

The second term of Eq. (1.30) is also zero. This is shown by integrating it by parts and considering that the general term

$$ \int \frac{1}{c} v_i \rho_j \frac{\partial f_\alpha}{\partial v_k} dv^\nu $$

has indices $i, j, k$ in cyclic order. Consequently

$$ \frac{\partial n_\alpha}{\partial t} + v \cdot \nabla n_\alpha = 0 \quad (1.33) $$

which is the *continuity equation* for particles of species $\alpha$.

### 1.2.2. First moment

Multiplying Eq. (1.1) by $\hat{\mathbf{v}}$ and integrating over velocity space results in the following terms:

$$ \int \frac{\partial f_\alpha}{\partial t} \hat{\mathbf{v}} dv^\nu = \frac{\partial}{\partial t} (n_\alpha \hat{\mathbf{v}}_\alpha) \quad (1.34) $$

and

$$ \int \hat{\mathbf{v}} \cdot \nabla f_\alpha \hat{\mathbf{v}} dv^\nu = \nabla \cdot \left( \int \hat{\mathbf{v}} \hat{\mathbf{v}} f_\alpha dv^\nu \right) = \nabla \cdot (n_\alpha \hat{\mathbf{v}}_\alpha) \quad (1.35) $$

where a bar represents the average value over velocity space. For example, the average value of a quantity $g$ is given by

$$ \bar{q} = \frac{1}{n_\alpha} \int q f_\alpha dv^\nu \quad (1.36) $$

The other terms in Eq. (1.1) yield
Multiplying the first moment of Eq. (1.1) by \( m_\alpha \) leads to the equation of momentum conservation:

\[
\frac{3}{3t} \left( n_\alpha m_\alpha \vec{u}_\alpha \right) + \nabla \cdot \left( n_\alpha m_\alpha \vec{v} \vec{v} \right) - e_\alpha n_\alpha \left( \vec{E} + \frac{1}{c} \vec{u}_\alpha \times \vec{B} \right) = \int C_{\alpha \beta} \frac{m_\alpha}{\beta} \vec{v} \vec{v} \ d\vec{v} \quad (1.39)
\]

**1.2.3. Second moment**

Multiplying the Boltzmann equation by \( v^2 \) and integrating over velocity space, we obtain the following terms:

\[
\int \frac{3f_\alpha}{3t} v^2 d\vec{v} = \frac{3}{3t} \left( n_\alpha \overline{v^2} \right) \quad (1.40)
\]

\[
\int \vec{v} \cdot \vec{v} f_\alpha v^2 d\vec{v} = \vec{v} \cdot \left( n_\alpha \overline{v^2} \vec{v} \right) \quad (1.41)
\]

\[
\int \frac{e_\alpha}{m_\alpha} \left( \vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right) \cdot \frac{3f_\alpha}{3\vec{v}} v^2 d\vec{v} = -2 \frac{n_\alpha e_\alpha}{m_\alpha} \vec{E} \cdot \vec{u}_\alpha \quad (1.42)
\]

and

\[
\int \vec{C}_\alpha v^2 d\vec{v} = \int C_{\alpha \beta} \frac{1}{2} m_\alpha v^2 d\vec{v} \quad (\beta \neq \alpha) \quad (1.43)
\]

Multiplying these terms by \( \frac{1}{2} m_\alpha \) results in the equation of energy conservation:

\[
\frac{3}{3t} \left( n_\alpha \frac{1}{2} m_\alpha \overline{v^2} \right) + \nabla \cdot \left( n_\alpha \frac{1}{2} m_\alpha \overline{v^2} \vec{v} \right) - n_\alpha e_\alpha \vec{E} \cdot \vec{u}_\alpha = \int C_{\alpha \beta} \frac{1}{2} m_\alpha v^2 d\vec{v} \quad (1.44)
\]

**1.2.4. Closure problem**

The term \( \vec{v} \cdot \vec{v} f_\alpha \) produces a moment of \( \vec{v} \) one order higher than the moment of the equation being taken and therefore it leads to a system of non-closed equations. To close the system one must either neglect some variables or find relationships between variables.
The average values
\[ n_a m \vec{v} v, \quad \frac{1}{2} n_a m \vec{v}^2 \text{ and } \frac{n_a m}{2} v^2 \vec{v} \]
can be transformed into macroscopically more meaningful terms. To do this one defines the following macroscopic variables:

**Random velocity** (also called peculiar velocity):
\[ \vec{v}_a' \equiv \vec{v} - \vec{u}_a \]

**Stress tensor:**
\[ \vec{\Pi}_a (\vec{r}, t) \equiv n_a m_a \vec{v}_a' \vec{v}_a' = \vec{\Pi}_a + \vec{\pi}_a \]
where
\[ \vec{\Pi}_a (\vec{r}, t) = \frac{1}{3} n_a m_a \vec{v}_a^2 \]
is the **scalar pressure** and \( \vec{\pi}_a \) the traceless part of the stress tensor. \( \vec{\pi}_a \) is symmetric.

**Temperature:**
\[ T_a (\vec{r}, t) = \frac{1}{3 n_a} \int \vec{v}_a'^2 f_a d\vec{v} = \frac{\vec{\Pi}_a}{n_a} \]

**Heat flow:**
\[ \vec{q}_a (\vec{r}, t) = \frac{1}{2} n_a m_a \vec{v}_a' \vec{v}_a' \]
Combining these equations one obtains
\[ n_a m_a \vec{v} v = n_a m_a \vec{u} \vec{u} + \vec{\Pi}_a + \vec{\pi}_a \]
(1.49)
\[ \frac{1}{2} n_a m_a \vec{v}^2 = \frac{n_a m}{2} u^2_a + \frac{3}{2} n_a T_a \]
(1.50)
and
\[ \frac{n_a m}{2} v^2 \vec{v} = \frac{n_a m}{2} u^2_a \vec{u}_a + \frac{3}{2} n_a T_a \vec{u}_a + \vec{u}_a \cdot \vec{\Pi}_a + \vec{q}_a \]
(1.51)
The collision terms may be simplified by separating $\mathbf{v}$ into average and random velocities. The average velocity does not contribute in the first moment. Equation (1.38) may be written as

$$\int C_{\alpha\beta} m_{\alpha} \mathbf{v} \mathbf{d}\mathbf{v} = \int C_{\alpha\beta} m_{\alpha} \mathbf{v}_{a} \mathbf{d}\mathbf{v} = \mathbf{R}_{a}$$

which is the mean momentum transfer from particles of species $\beta$ to particles of species $\alpha$.

The second moment (1.43) may be written:

$$\int C_{\alpha\beta} \frac{1}{2} m_{\alpha} \mathbf{v}^{2} \mathbf{d}\mathbf{v} = \mathbf{u}_{a} \cdot \mathbf{R}_{a} + Q_{a}$$

where

$$Q_{a} = \int C_{\alpha\beta} \frac{1}{2} m_{\alpha} \mathbf{v}_{a}^{2} \mathbf{d}\mathbf{v}$$

is the generated heat in the system of particles $\alpha$ caused by collision with particles $\beta$. The term $\mathbf{u}_{a} \cdot \mathbf{R}_{a}$ is the heat generated among particles $\alpha$ owing to friction with particles $\beta$.

The Lagrangian derivative is defined along the flow line $\mathbf{u}_{a}$ as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_{a} \cdot \nabla$$

The first three moments of the Boltzmann equations (1.33), (1.39) and (1.44) can then be simplified and take the form

$$\frac{dn_{\alpha}}{dt} + n_{\alpha} \mathbf{v} \cdot \mathbf{u}_{a} = 0$$

and

$$\frac{3}{2} n_{\alpha} \frac{d\mathbf{u}_{a}}{dt} + \mathbf{P}_{a} = n_{\alpha} e_{a} \left( \mathbf{E} + \frac{1}{2} \mathbf{u}_{a} \times \mathbf{B} \right) = \mathbf{R}_{a}$$

These are the equations of continuity, motion and heat transfer for particles $\alpha$ and constitute a system of transport equations. It is a system of five equations with nineteen variables $n_{\alpha}, \mathbf{u}_{a}, T_{\alpha}, \mathbf{P}_{a}, \mathbf{E}_{a}, \mathbf{B}_{a}, \mathbf{R}_{a}$ and $Q_{a}$ for each species $\alpha$.

The coupling of equations between species is done through $\mathbf{Q}_{\alpha}, \mathbf{R}_{a}, Q_{a}, \mathbf{E}$ and $\mathbf{B}$. The two latter are solutions of the Maxwell equation in which all the
species are present in the form of \( n_\alpha \) and \( u_\alpha \). The closure of this system of equations is carried out by a simplifying assumption leading to relationships between \( p_\alpha \), \( \pi_\alpha \), \( q_\alpha \), \( R_\alpha \) and \( Q_\alpha \) with the quantities \( n_\beta \), \( u_\beta \) and \( T_\beta \). The terms \( \pi_\alpha \) and \( q_\alpha \) do not originate from the collision term but they depend on collisions.

The system of moment equations may be closed by taking only two moments (1.56) and (1.57) and assuming: (a) \( R_\alpha = 0 \); or (b) \( R_\alpha \propto (u_\alpha - u_\beta) \); (c) \( \pi_\alpha = 0 \); or (d) \( q_\alpha = 0 \) and \( P_\alpha \) given by an equation of state. All four combinations of (a) and (b) with (c) and (d) are possible.

1.3. Two-fluid equations

Consider now a plasma consisting of electrons and only one ion species with charge \( e_i = Ze \). From Eqs (1.26) and (1.27) one can derive the relations:

\[
\begin{align*}
\vec{R}_e + \vec{R}_i &= 0 \quad (1.59) \\
Q_e + \vec{u}_e \cdot \vec{R}_e + Q_i + \vec{u}_i \cdot \vec{R}_i &= 0 \quad (1.60)
\end{align*}
\]

Defining the quantities \( \vec{R} \) and \( \vec{Q} \) as follows:

\[
\vec{R} \equiv \vec{R}_e = -\vec{R}_i \quad (1.61)
\]

and

\[
\vec{Q} \equiv Q_e = -Q_i + (\vec{u}_i - \vec{u}_e) \cdot \vec{R} \quad (1.62)
\]

the two-fluid or transport equations are obtained:

\[
\begin{align*}
\frac{dn_e}{dt} + n_e \vec{v} \cdot \vec{u}_e &= 0 \quad (1.63) \\
\frac{dn_i}{dt} + n_i \vec{v} \cdot \vec{u}_i &= 0 \quad (1.64) \\
\frac{d\vec{u}_e}{dt} &= \nabla p_e + \nabla \cdot \vec{\pi}_e + e_n \left( E + \frac{1}{c} \vec{u}_e \times \vec{B} \right) = \vec{R} \quad (1.65) \\
\frac{d\vec{u}_i}{dt} &= \nabla p_i + \nabla \cdot \vec{\pi}_i - e_n \left( E + \frac{1}{c} \vec{u}_i \times \vec{B} \right) = -\vec{R} \quad (1.66) \\
\frac{3}{2} \ n_e \frac{dT_e}{dt} + p_e \nabla \cdot \vec{u}_e + \vec{\pi}_e : \nabla \vec{u}_e + \nabla \cdot \vec{q}_e &= \vec{Q} \quad (1.67)
\end{align*}
\]
Braginskii [9] calculates the transport coefficients using the method developed by Chapman and Cowling [1] called the successive approximation method. He obtains all the relations necessary to close the above system of equations.

Starting with the collision term of the Boltzmann equation in the Landau form, the transport quantities are calculated assuming that the local distribution function is very close to a Maxwellian. The major features of the transport coefficients are as follows:

(a) The momentum transfer term $\vec{R}$ is made up of two parts: the force of friction $\vec{R}_u$ and a thermal force $\vec{R}_T$, $\vec{R} = \vec{R}_u + \vec{R}_T$ where

$$\vec{R}_u = \frac{en_o}{\sigma_u} \vec{j}_u + \frac{en_o}{\sigma_\perp} \vec{j}_\perp$$

$$\vec{R}_T = -0.71 n_e v_h kT_e - \frac{3}{2} \frac{n_e}{\omega_{ce} \tau_e} (\vec{b} \times \vec{v} kT_e)$$

$$\vec{j} = -en_o (\vec{u}_e - \vec{u}_i)$$

is the current density,

$$\sigma_\perp = \frac{e^2 n_e \tau_e}{m_e} , \quad \sigma_u = 2 \sigma_\perp$$

are the perpendicular and parallel conductivity;

$$\omega_{ce} = \frac{eB}{m_e c}$$

is the Larmor frequency, and

$$\tau_e = \frac{3 \sqrt{m_e} (kT_e)^{3/2}}{4 \sqrt{2\pi} \ln \Lambda \ e \ z^2 n_i}$$

is the electron-ion momentum transfer collision time. $\vec{R}_T$ is usually much smaller than $\vec{R}_u$. This $\sigma_\parallel$ is called the classical or Spitzer conductivity. In many plasma
experiments the measured conductivity is an order or two smaller than this conductivity (called anomalous conductivity), and this is due to plasma fluctuations caused by instabilities. The symbols \( \parallel \) and \( \perp \) are used here to express the components of vectors and coefficients parallel and perpendicular to the magnetic field. The unit vector parallel to the B-field is given by \( \hat{b} = \frac{B}{B} \). The conductivity parallel to the B-field is twice as large as the perpendicular.

(b) The ion heating due to the collisions with electrons is proportional to the difference in the temperatures of the electrons and ions. The electrons are heated by this temperature difference and by the Joule effect:

\[
Q_i = Q_\Delta = \frac{3m_e}{m_i} \frac{n_e}{\tau_e} (kT_e - kT_i)
\]

and

\[
Q_e = \frac{1}{\sigma_n} j_\parallel^2 + \frac{1}{\sigma_L} j_\perp^2 - Q_\Delta
\]

(c) The traceless stress tensor is the viscosity term. In the absence of a B-field it is proportional to the rate-of-strain tensor \( \dot{W} \) defined by

\[
W_{jk} = \frac{\partial u_{aj}}{\partial x_k} + \frac{\partial u_{ak}}{\partial x_j} - \frac{2}{3} \delta_{jk} \nabla \cdot \mathbf{u}
\]

and

\[
\tau_a = - \eta \alpha W_a
\]

when

\[ \mathbf{B} = 0 \]

where

\[ \eta_\alpha = c_\alpha n_\alpha kT_a \tau_a, \quad \alpha = i,e \]

is the viscosity, with \( c_i = 0.96 \) for ions and \( c_e = 0.73 \) for electrons.

The electron-ion momentum transfer time \( \tau_e \) is given by (1.71) and

\[
\tau_i = \frac{3 \sqrt{m_i} (kT_i)^{3/2}}{4 \sqrt{\pi} \Lambda e^0 Z_i n_i}
\]

In the presence of a strong magnetic field there is a tensorial relationship between \( \tau_e \) and \( \dot{W} \), and the viscosity is expressed by a tensor. The important terms in the lowest-order approximations are:
with the z-axis parallel to the B-field. It can be seen that the trace of $\tau_\alpha$ is equal to zero.

(d) The heat flux $\tilde{q}_\alpha$ is a function of the components of $\tilde{u}_\alpha$ and $\nabla T_e$, $\tilde{q}_\alpha = \tilde{q}_\alpha \tilde{u} + \tilde{q}_\alpha \nabla T_e$. Explicitly,

$$\tilde{q}_u = 0.71 n_e T_e \tilde{u} + \frac{3}{2} \frac{n_e T_e}{e_i T_e} (\hat{b} \times \tilde{u})$$

and

$$\tilde{q}_T = \kappa_{a\parallel} \nabla T_a - \kappa_{a\perp} \nabla T_a + \frac{5}{2} \frac{c_n a_{\alpha}}{e_i B} (\hat{b} \times \nabla T_e)$$

where $\kappa_{a\parallel}$ and $\kappa_{a\perp}$ are the thermal conductivities and may be written as

$$\kappa_{a\parallel} = c_{a\parallel} \frac{n_T a_T a}{m_a}$$

and

$$\kappa_{a\perp} = c_{a\perp} \frac{n_T a_T a}{m_a W_a c_a T_a}$$

with $c_{e\|} = 3.16$; $c_{e\perp} = 4.66$; $c_{i\|} = 3.9$; and $c_{i\perp} = 2$. The heat flux $\tilde{q}_{e\parallel}$ is zero.

(e) The term $\tilde{\chi} : \nabla \tilde{u}_\alpha$ expresses the heat generated as a result of viscosity and, in the case of no magnetic field, is expressed as

$$Q_{vis} = - \tilde{\chi} : \nabla \tilde{u}_\alpha = \frac{3}{4} n_o W_{zz}^2$$

(For detailed calculation and results see Ref.[9].)
2. ONE-FLUID EQUATION

2.1. One-fluid variables

There are quantities such as current and charge densities which are more suitable for measurement in plasmas but do not appear naturally in the two-fluid formulation. With linear combinations of Eqs (1.63)—(1.68) they may be transformed into equations for the one-fluid variables, without limiting the range of solutions. However, the best reason for writing the one-fluid equations is that they give a better understanding of the relative values of each term, and thus conclusions may be drawn which lead to a drastic simplification of the equations.

The one-fluid variables are defined as:

\[ \rho(r,t) = n_e m_e + n_i m_i \]  (2.1)

\[ \rho_q(r,t) = e (n_i - n_e), \ Z = 1 \]  (2.2)

\[ \vec{u}(r,t) = \frac{n_e \vec{u}_e + n_i \vec{u}_i}{m_e + m_i} \]  (2.3)

\[ \vec{j}(r,t) = e n_i \vec{u}_i - e n_e \vec{u}_e \]  (2.4)

\[ p(r,t) = p_i + p_e \]  (2.5)

\[ q(r,t) = q_i + q_e \]  (2.6)

\[ \pi(r,t) = \pi_i + \pi_e \]  (2.7)

Inverting the transformation (2.1)—(2.4), one gets the two-fluid variables in terms of the one-fluid variables:

\[ n_e = \frac{\rho - m_i \rho/q/e}{m_i + m_e} \]  (2.8)

\[ n_i = \frac{\rho + m_e \rho/q/e}{m_i + m_e} \]  (2.9)
2.2. One-fluid equations

To obtain the **MHD (magnetohydrodynamic) equations** from two-fluid equations, several approximations have to be imposed, some of them not quite justifiable. The first **two approximations** are:

- **Quasi-neutrality**: \( |n_e - n_i| \ll n_e \)  

- **Small mass ratio**: \( m_e \ll m_i \)

Charge-neutrality is a good approximation for the study of plasma phenomena with a scale length \( L \) much greater than the Debye length \( \lambda_D \), i.e.

\[
L \gg \lambda_D
\]

where \( \lambda_D^2 = \frac{kT_e}{4\pi n e^2} \). For typical thermonuclear fusion parameters, \( n = 10^{14} \) cm\(^{-3}\) and \( T = 10 \) keV, one obtains \( \lambda_D = 0.74 \times 10^{-2} \) cm. Considering that the radius of the plasma is of the order of 50 cm and that the dangerous unstable modes are those which can alter the size of the plasma, this approximation is indeed very good. As a warning, one may say that collisionless ion-acoustic or magnetosonic shock waves have a characteristic length of the order of the Debye length. The small mass ratio is self-evident.

The simplifications which result from these assumptions are

\[
n_e = n_i = n_0 = \frac{\rho}{m_i}
\]

\[
\vec{u}_e = \vec{u} - \frac{1}{n_0 e} \vec{j}
\]

and

\[
\vec{u}_i = \vec{u}
\]

The equation of **continuity of mass** is obtained by multiplying the pair of equations (1.63) and (1.64) by an appropriate mass \( m_\alpha \) and adding

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0
\]
Now, multiplying the same equations by a charge $e_\alpha$ and adding results in the equation of 
continuity of charge,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0 \quad (2.19)$$

Adding the pair of equations (1.65) and (1.66) and using (1.63)–(1.64) and (2.5), (2.7) and (2.15)–(2.18) result in the equation of motion:

$$\frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p + \nabla \cdot \mathbf{\pi} - \frac{1}{c} \mathbf{j} \times \mathbf{B} = 0 \quad (2.20)$$

Now, multiplying the same pair by the charge-to-mass ratio $e_\alpha/m_\alpha$, and adding, one obtains the rate of change of the current density, generally known as the 
generalized Ohm's law:

$$\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (j \mathbf{u} + \mathbf{uj}) - \frac{e}{m_e} \nabla \mathbf{p}_e - \mathbf{v} \cdot \frac{e}{m_e} \mathbf{j} = 0$$

$$+ \frac{e}{m_e c} \mathbf{j} \times \mathbf{B} - \frac{e^2 n_0}{m_e} (E + \frac{1}{c} \mathbf{u} \times \mathbf{B}) = - \frac{e}{m_e} \mathbf{R} \quad (2.21)$$

Finally, adding the heat transfer equations (1.67) and (1.68) one gets the 
heat balance equation:

$$\frac{3}{2} \frac{\partial \mathbf{p}}{\partial t} + \frac{3}{2} \mathbf{u} \cdot \nabla \mathbf{p} + \frac{5}{2} \mathbf{p} \cdot \mathbf{u} - \frac{3}{2} \frac{1}{n_0 e} \mathbf{v} \cdot \mathbf{p}_e \mathbf{j} - \frac{p_e}{n_0 e} \mathbf{v} \cdot \mathbf{j}
- \frac{1}{n_0 e} \mathbf{v}_e \cdot \mathbf{j} + \frac{\mathbf{v}_e \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{q}}{\frac{1}{n_0 e} t_e \mathbf{j} \cdot \mathbf{R} \quad (2.22)$$

Next, one considers three approximations with the objective of simplifying

(2.20)–(2.22).

Low-frequency: $\omega \rightarrow 0 \quad (2.23)$

Long spatial scale: $L$ large but $L \omega \ll c \quad (2.24)$

Intermediate temperature: $T_1 < T < T_2 \quad (2.25)$
TABLE I. ORDER OF MAGNITUDE OF THE TERMS IN Eqs (2.20)–(2.22)

<table>
<thead>
<tr>
<th>Negligible term</th>
<th>Order of magnitude</th>
<th>Reference term</th>
<th>Order of magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ohm's Law:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{\partial}{\partial t} j$</td>
<td>$\omega \tilde{j}$</td>
<td>$\frac{L \tilde{u}}{\delta c}$</td>
<td>$\frac{L \tilde{u}}{\omega \tilde{e} \delta c}$</td>
</tr>
<tr>
<td>$\frac{e}{m_e c} \tilde{j} \times \tilde{B}$</td>
<td>$\left( \frac{L \tilde{u}}{\delta} \right) \frac{j}{n_0 e}$</td>
<td>$\left( \frac{L \tilde{u}}{\delta} \right) \frac{j}{L}$</td>
<td>$\frac{L \tilde{u}}{\omega \tilde{e} \delta c}$</td>
</tr>
<tr>
<td>$\nabla \cdot (\tilde{u} \tilde{j} + \tilde{j} \tilde{u})$</td>
<td>$j \tilde{u} / L$</td>
<td>$\frac{L \tilde{u}^2}{4 \pi e}$</td>
<td>$\frac{L \tilde{u}^2}{4 \pi e}$</td>
</tr>
<tr>
<td>$\frac{e}{m_e} \nabla \tilde{p}_e$</td>
<td>$\frac{\omega \tilde{e} \tilde{u}_e}{4 \pi e} m_e$</td>
<td>$\frac{L \tilde{u}^2}{4 \pi e}$</td>
<td>$\frac{L \tilde{u}^2}{4 \pi e}$</td>
</tr>
<tr>
<td>$\nabla \cdot \frac{e}{m_e} \tilde{p}_e$</td>
<td>$\frac{L \tilde{u}^2 \omega \tilde{e}_e}{4 \pi e}$</td>
<td>$\frac{L \tilde{u}^2 \omega \tilde{e}_e}{4 \pi e}$</td>
<td>$\frac{L \tilde{u}^2 \omega \tilde{e}_e}{4 \pi e}$</td>
</tr>
<tr>
<td>$-\frac{e}{m_e} \tilde{R}_u$</td>
<td>$\frac{j \tilde{u}}{4 \pi \sigma}$</td>
<td>$\omega \tilde{e} \tilde{e}_e \frac{m_e}{4 \pi e}$</td>
<td>$\omega \tilde{e} \tilde{e}_e \frac{m_e}{4 \pi e}$</td>
</tr>
</tbody>
</table>

Equation of motion:

| $\nabla \cdot \tilde{u}$ | $\left( \frac{p}{L} \right) \frac{j \tilde{u}}{L}$ | $\nabla p$ | $\frac{p}{L}$ |

Heat balance equation:

| $\nabla \cdot \tilde{q}$ | $\left( \frac{p}{L} \right) \frac{j}{n_0 e}$ | $\frac{5}{2} p \nabla \cdot \tilde{u}$ | $\frac{p}{L} \tilde{u}$ |
| $-\frac{1}{n_0 e} \tilde{j} : \nabla \tilde{u}$ | $\left( \frac{p}{L} \right) \frac{j \tilde{u}}{n_0 e} L$ | $\frac{5}{2} p \nabla \cdot \tilde{u}$ | $\frac{p}{L} \tilde{u}$ |
| $\tilde{q} \cdot \tilde{u}$ | $\frac{p}{L} \frac{\tilde{u}}{L}$ | $\frac{5}{2} p \nabla \cdot \tilde{u}$ | $\frac{p}{L} \tilde{u}$ |
| $\frac{3}{2} n_0 e \nabla \cdot \tilde{p}_e$ | $\left( \frac{p}{L} \right) \frac{j}{n_0 e}$ | $\frac{5}{2} p \nabla \cdot \tilde{u}$ | $\frac{p}{L} \tilde{u}$ |
(The meaning of these approximations will soon become clear.)

Assumption (2.24) leads to the pre-Maxwell equations; in particular, Ampère’s law is recovered:

\[ \vec{j} = \frac{c}{4\pi} \nabla \times \vec{B} \]  \hspace{1cm} (2.26)

Table I shows the terms of Eqs (2.20)—(2.22) which can be made negligible in relation to the leading term of each equation. The following definitions are used in the table:

- **Collisionless Skin Depth:** \( \delta \equiv \frac{c}{\omega_{pe}} \)
- **Electron Thermal Velocity:** \( v_{\theta e} = \sqrt{\frac{kT_e}{m_e}} \)
- **Electron Mean-Free Path:** \( \lambda_e = v_{\theta e} \tau_e \)
- **Ion Thermal Velocity:** \( v_{\theta i} = \sqrt{\frac{kT_i}{m_i}} \)
- **Ion Mean-Free Path:** \( \lambda_i = v_{\theta i} \tau_i \)
- **Electron Larmor Radius:** \( r_{Le} = \frac{v_{\theta e}}{\omega_{ce}} \)

From (2.26) an order-of-magnitude relationship may be written:

\[ \vec{j} \propto \frac{cB}{4\pi L} \]  \hspace{1cm} (2.27)

which is also used in Table I. It is further assumed that \( \omega_{ce} \tau_e \) and \( \omega_{ci} \tau_i \) are much greater than one.

From Table I, if it is assumed that the terms labelled 'negligible' are much smaller than the reference terms, the following conclusions are made:

(1) From (T-1): \( \omega \ll \omega_{pe} \left( \frac{L_i}{\delta_c} \right) \)  \hspace{1cm} (2.28)

(2) From (T-2): \( \frac{1}{n_0 c} \ll u \)  \hspace{1cm} (2.29)

which is equivalent to

\[ |\vec{u}_i - \vec{u}_e| \ll u \]  \hspace{1cm} (2.30)
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where $R_M$ is called the magnetic Reynolds number

$$R_M = \frac{4\pi \sigma_c \nu}{c^2} \gg 1$$  \hspace{1cm} (2.34)

There is a pattern to all these inequalities. If the temperature of the plasma is lowered, all the inequalities become better except for (2.34). If, instead, the density is increased, the same thing happens. So there is an assumption, namely (2.34), which opposes the rest. This makes the whole set of assumptions valid only for intermediate temperatures and densities.

If the assumption (2.34) is eliminated, this opposition disappears. The resulting set of equations are called the \textit{resistive MHD equations}, and are valid for low temperatures and high densities.

### 2.3. Resistive MHD equations

The set of one-fluid equations (2.18)—(2.22) is simplified by assumptions (2.28)—(2.36), except (2.34). This set, together with the pre-Maxwell equations, forms the closed set of resistive MHD equations for the variables $\rho$, $\vec{u}$, $p$, and $\vec{B}$:

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \rho \vec{u} = 0$$ \hspace{1cm} \text{(continuity)} \hspace{1cm} (2.37)

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} + \nabla p - \frac{1}{c} \vec{j} \times \vec{B} = 0$$ \hspace{1cm} \text{(motion)} \hspace{1cm} (2.38)

$$\frac{3}{2} \frac{\partial \rho}{\partial t} + \frac{3}{2} \vec{u} \cdot \nabla \rho + \frac{5}{2} p \vec{v} \cdot \vec{u} = \eta \vec{j}^2$$ \hspace{1cm} \text{(heat balance)} \hspace{1cm} (2.39)

$$\frac{1}{c} \frac{\partial}{\partial t} \vec{B} + \nabla \times \vec{E} = 0$$ \hspace{1cm} \text{(Faraday)} \hspace{1cm} (2.40)
MACROSCOPIC PLASMA PROPERTIES

\[ \frac{4\pi}{c} j = \nabla \times B \]  
\text{(Ampère)} \hspace{1cm} (2.41)

\[ E = \eta j - \frac{1}{c} \mathbf{u} \times B \]  
\text{(Ohm)} \hspace{1cm} (2.42)

Initially the following condition needs to be satisfied:

\[ \nabla \cdot \mathbf{B} = 0 \]  
\hspace{1cm} (2.43)

For simplicity, and with no suitable justification, the parallel and perpendicular conductivities were considered equal: \( \sigma_p = \sigma_\perp = 1/\eta \), where \( \eta \) is the resistivity.

The equation of continuity of charge is dropped because of charge neutrality.

2.4. Ideal MHD equations

2.4.1. One-adiabatic approximation

Now, including assumption (2.34), which means that the resistivity term is dropped, the troublesome term \( \pi : \nabla \mathbf{u} \) is also dropped, and this results in the ideal MHD equations:

\[ \frac{\partial p}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \]  
\hspace{1cm} (2.44)

\[ \rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \rho \mathbf{u} \mathbf{u} + \nabla p - \frac{1}{c} j \times B = 0 \]  
\hspace{1cm} (2.45)

\[ \frac{3}{2} \frac{\partial p}{\partial t} + \frac{3}{2} \mathbf{u} \cdot \nabla p + \frac{5}{2} \partial \mathbf{u} \cdot \mathbf{u} = 0 \]  
\hspace{1cm} (2.46)

\[ \frac{1}{c} \frac{\partial B}{\partial t} + \nabla \times \mathbf{E} = 0 \]  
\hspace{1cm} (2.47)

\[ \frac{4\pi}{c} j = \nabla \times \mathbf{B} \]  
\hspace{1cm} (2.48)

\[ E = - \frac{1}{c} \mathbf{u} \times \mathbf{B} \]  
\hspace{1cm} (2.49)

where initially \( \nabla \cdot \mathbf{B} = 0 \) must be satisfied.

Equation (2.46) is exactly the adiabatic fluid equation:

\[ \frac{d}{dt} p \rho^{-\gamma} = 0 \]  
\hspace{1cm} (2.50)

with \( \gamma = 5/3 \). This set of equations is therefore called the one-adiabatic equation.

Equation (2.50) may be thought of as an equation of state to close the set
of equations. Instead of the adiabatic equation, another equation of 'state' may be used, such as

\[ \nabla \cdot \mathbf{u} = 0 \], incompressible fluid

and

\[ \frac{d}{dt} \left( \frac{\rho}{\rho} \right) = 0 \], isothermal fluid

2.4.2. Double-adiabatic approximation — CGL

When collisions are infrequent, the inequality (2.35) may not be valid. However, the inequality (2.34) may still hold. Elimination of (2.35) brings in two terms to the set of ideal MHD equations: (a) \( \nabla \cdot \mathbf{u} \) for the equation of motion, and (b) \( \frac{\eta}{\tau} \nabla \mathbf{u} \) for the equation of heat balance. Dropping the equation of heat balance and adding two equations of state, one obtains the double-adiabatic equations:

\[ \frac{3}{\tau} \rho + \nabla \cdot \mathbf{u} = 0 \] (2.53)

\[ \rho \left( \frac{3}{\tau} \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = - \mathbf{v} \cdot \mathbf{p} - \frac{1}{c} \mathbf{j} \times \mathbf{B} \] (2.54)

\[ \frac{1}{c} \frac{3B}{\tau} + \mathbf{v} \times \mathbf{E} = 0 \] (2.55)

\[ \frac{4\pi}{c} \mathbf{j} = \mathbf{v} \times \mathbf{B} \] (2.56)

\[ \mathbf{E} = - \frac{1}{c} \mathbf{j} \times \mathbf{B} \] (2.57)

\[ \frac{d}{dt} \frac{P_\perp^2 P_\parallel}{\rho^5} = 0 \] (2.58)

\[ \frac{d}{dt} \frac{P_\perp}{\rho B} = 0 \] (2.59)

In a coordinate system with the z-axis parallel to \( \mathbf{B} \) at any point, the pressure tensor has the form:

\[ \mathbf{P} = \mathbf{P}_\perp + \pi = \begin{pmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{pmatrix} \] (2.60)

Equations (2.58) and (2.59) are the equations of state added to close the system. Equation (2.58) describes the adiabatic processes in the plane perpendicular
to $\vec{B}$ and along a $\vec{B}$ line. In these two directions there is no coupling and the equation holds. Equation (2.59) expresses the constancy of the magnetic moment $\mu = mv^2/2B$. This is valid for collisionless plasmas, but approximately valid for weakly collisional plasmas.

The stress tensor (2.60) may be written in an arbitrary frame in the form:

$$\vec{P} = P_{\perp} \vec{I} + (P_{\|} - P_{\perp}) \hat{\vec{b}}$$

(2.61)

The divergence of $\vec{P}$ in the parallel and perpendicular directions is written as

$$\nabla \cdot \vec{P} = (\hat{b} \cdot \nabla) P_{\|} + (P_{\|} - P_{\perp}) \nabla \cdot \vec{b}$$

(2.62)

$$\nabla \cdot \vec{P} = \nabla P_{\perp} - (P_{\|} - P_{\perp}) (\hat{b} \cdot \nabla) \hat{b}$$

(2.63)

The set of closed equations (2.53)–(2.59) are called the CGL equations, from early work of Chew, Goldberger and Low [10]. These authors assumed negligible heat flow along the magnetic field and expanded the Vlasov equation in powers of $1/B$ to obtain the above set of macroscopic equations.

3. MHD STABILITY: ENERGY PRINCIPLE

3.1. Linearized ideal MHD equations: force-operator equations

Consider a magnetized plasma in a metal chamber separated from the wall by a magnetic field. It is assumed that this plasma is describable by the ideal MHD equations. The question posed here is: how can one study the stability of such magnetically confined plasmas?

There are three basic methods for determining the stability of a plasma: the intuitive method, the energy principle and the normal mode analysis.

In the intuitive method one considers the forces which arise when a perturbation is made on an equilibrium state. If the forces enhance the perturbation, then the perturbation grows and there is an instability. In the energy principle one calculates the variation in the potential energy of the plasma, $\delta W$, due to the perturbation. If $\delta W$ is negative it means that the plasma can attain a lower energy state, and as a result the perturbation grows. In the normal mode analysis the time dependence of the perturbation is Fourier-analysed in terms of $e^{-i\omega n t}$. The perturbation equation is written as a boundary value problem and one solves for the eigenvalue $\omega_n$. If $\omega_n$ has a negative imaginary part, the system is unstable. (These notes treat only the energy principle [11–14].)
A stability problem is separated into three parts: the equilibrium, the perturbation equation, and the solution of the problem. In any stability analysis the existence of an equilibrium is assumed, the proof of which may in some cases be very difficult.

Consider the ideal MHD equations (2.44)—(2.50), rewritten here:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \tag{3.1}
\]

\[
\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \frac{1}{c} \mathbf{j} \times \mathbf{B} = 0 \tag{3.2}
\]

\[
\frac{d}{dt} \rho \gamma = 0 \tag{3.3}
\]

\[
\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} + \nabla \times \mathbf{E} = 0 \tag{3.4}
\]

\[
\frac{4\pi}{c} \mathbf{j} = \nabla \times \mathbf{B} \tag{3.5}
\]

\[
\mathbf{E} = -\frac{1}{c} \mathbf{u} \times \mathbf{B} \tag{3.6}
\]

and

\[
\nabla \cdot \mathbf{B} = 0 \tag{3.7}
\]

To obtain the equilibrium equations the solutions are assumed to be time-independent, \(\partial / \partial t (\cdot) = 0\) and \(\mathbf{E}_o = \mathbf{u}_o = 0\). Here the subscript zero represents the equilibrium solutions. From the above set of equations the equilibrium equations are obtained:

\[
\frac{1}{c} \mathbf{j}_o \times \mathbf{B}_o = \mathbf{v}_o \tag{3.8}
\]

\[
\frac{4\pi}{c} \mathbf{j}_o = \nabla \times \mathbf{B}_o \tag{3.9}
\]

and

\[
\nabla \cdot \mathbf{B}_o = 0 \tag{3.10}
\]

Combining (3.8) and (3.9) one has the equilibrium equation,

\[
\mathbf{v}_o = \frac{1}{4\pi} (\nabla \times \mathbf{B}_o) \times \mathbf{B}_o \tag{3.11}
\]

where \(\mathbf{B}_o\) is subjected to Eq.(3.10).

Next, one considers the perturbation of the equilibrium quantities:

\[
\rho(\mathbf{r}, t) = \rho_o(\mathbf{r}) + \rho_1(\mathbf{r}, t) \tag{3.12}
\]

\[
\mathbf{u}(\mathbf{r}, t) = \mathbf{u}_o(\mathbf{r}) + \mathbf{u}_1(\mathbf{r}, t), \quad \mathbf{u}_o(\mathbf{r}) = 0 \tag{3.13}
\]
\[ p(r, t) = p_o(r) + p_1(r, t) \]  \( (3.14) \)

\[ \mathbf{B}(r, t) = \mathbf{B}_o(r) + \mathbf{B}_1(r, t) \]  \( (3.15) \)

\[ \mathbf{j}(r, t) = \mathbf{j}_o(r) + \mathbf{j}_1(r, t) \]  \( (3.16) \)

and

\[ \mathbf{E}(r, t) = \mathbf{E}_o(r) + \mathbf{E}_1(r, t) \]  \( (3.17) \)

where the subscript 1 represents the perturbed quantities which are considered much smaller than the corresponding equilibrium quantities. Inserting expressions (3.12)–(3.17) into Eqs (3.1)–(3.7) and subtracting (3.8) from (3.10) result in the perturbation equations:

\[ \rho_1 + \rho_o \mathbf{v} \cdot \mathbf{u}_1 = 0 \]  \( (3.18) \)

\[ \rho_o \frac{\partial}{\partial t} \mathbf{u}_1 = \frac{1}{c} \mathbf{j}_1 \times \mathbf{B}_o + \frac{1}{c} \mathbf{j}_o \times \mathbf{B}_1 - \gamma p_1 \]  \( (3.19) \)

\[ \frac{\partial}{\partial t} \mathbf{P}_1 + \mathbf{u}_1 \cdot \mathbf{V}_0 + \gamma \mathbf{P}_o \mathbf{V} \cdot \mathbf{u}_1 = 0 \]  \( (3.20) \)

\[ \frac{1}{c} \frac{\partial}{\partial t} \mathbf{B}_1 + \mathbf{v} \times \mathbf{E}_1 = 0 \]  \( (3.21) \)

\[ \frac{4\pi}{c} \mathbf{j}_1 = \mathbf{v} \times \mathbf{B}_1 \]  \( (3.22) \)

\[ \mathbf{E}_1 + \frac{1}{c} \mathbf{V}_0 \times \mathbf{B}_o = 0 \]  \( (3.23) \)

and

\[ \mathbf{v} \cdot \mathbf{B}_1 = 0 \]  \( (3.24) \)

This set of fourteen first-order partial differential equations, with variables \( \rho_1, \mathbf{u}_1, \mathbf{P}_1, \mathbf{B}_1, \mathbf{j}_1 \) and \( \mathbf{E}_1 \), can be reduced to one second-order vectorial equation with one variable, \( \mathbf{u}_1 \):

\[ \rho_o \frac{\partial^2}{\partial t^2} \mathbf{u}_1 = \mathbf{V} (\mathbf{u}_1 \cdot \mathbf{V}_0 + \gamma \mathbf{P}_o \mathbf{V} \cdot \mathbf{u}_1) + \frac{1}{4\pi} \left\{ \mathbf{V} \times \left[ \mathbf{V} \times (\mathbf{u}_1 \times \mathbf{B}_o) \right] \right\} \times \mathbf{B}_o \]

\[ + \frac{1}{4\pi} \left( \mathbf{V} \times \mathbf{B}_o \right) \times \left[ \mathbf{V} \times (\mathbf{u}_1 \times \mathbf{B}_o) \right] \]  \( (3.25) \)

The other variables are obtained through time-integration of functions of \( \mathbf{u}_1 \):

\[ \frac{\partial}{\partial t} \mathbf{B}_1 = \mathbf{V} \times (\mathbf{u}_1 \times \mathbf{B}_o) \]  \( (3.26) \)

\[ \frac{\partial}{\partial t} \mathbf{P}_1 = -\mathbf{u}_1 \cdot \mathbf{V}_0 - \gamma \mathbf{P}_o \mathbf{V} \cdot \mathbf{u}_1 \]  \( (3.27) \)

and \( \rho_1, \mathbf{j}_1 \) and \( \mathbf{E}_1 \) are solved from (3.18), (3.22) and (3.23). With a given set of
initial conditions \( \bar{u}_1(\vec{r}, 0) \) and \( \bar{u}_1^\prime(\vec{r}, 0) \), and appropriate boundary conditions, Eq. (3.25) can be solved.

The above equations are written in Eulerian formalism. The variable \( \bar{u}_1(\vec{r}, t) \) and the equilibrium quantities, such as \( P_0(\vec{r}) \), are expressed in terms of the Eulerian position \( \vec{r} \), i.e., for a fixed position \( \vec{r} \) the fluid is passing by with a velocity \( \bar{u}_1(\vec{r}, t) \), pressure \( p(\vec{r}, t) \), etc.

It is more convenient, however, to write Eq. (3.25) in terms of the Lagrangian displacement \( \vec{\xi} \) as in Fig. 1. This is the displacement of any element from its equilibrium position and is expressed in terms of the equilibrium position \( \vec{r}_0 \) and time \( t \) as \( \vec{\xi}(\vec{r}_0, t) \) [15, 16]. The time derivative of \( \vec{\xi}(\vec{r}_0, t) \) at fixed \( \vec{r}_0 \) describes the velocity of the same fluid element as it varies in time:

\[
\frac{d}{dt} \vec{r}(t) = \frac{\partial}{\partial t} \vec{\xi}(\vec{r}_0, t) = \vec{\dot{\xi}}(\vec{r}_0, t)
\]  

(3.28)

The Eulerian velocity at \( \vec{r} \), \( \bar{u}(\vec{r}, t) \) coincides with \( \vec{\dot{\xi}}(\vec{r}_0, t) \) within the same order of magnitude for

\[
\bar{u}(\vec{r}, t) = \bar{u}(\vec{r}_0, t) + (\vec{\xi}, \vec{v}) \bar{u}(\vec{r}_0, t) + \ldots
\]  

(3.29)

by expanding it for small \( \vec{\xi} \) around \( \vec{r}_0 \). The second term on the right-hand side is a second-order term, for \( \vec{\xi} \) and \( \vec{u} \) are both first order; therefore

\[
\bar{u}(\vec{r}_0, t) = \frac{\partial}{\partial t} \vec{r}_0 + \vec{\xi}(\vec{r}_0, t)
\]  

(3.30)

(For a complete description of this subject see Ref. [16].)
Equations (3.25)–(3.27) may be written in terms of the Lagrangian displacement $\xi$. Integration of Eq.(3.26) results in

$$\vec{B}_1(\vec{r}_o, t) = \nabla \times (\vec{\xi}(\vec{r}_o, t) \times \vec{B}_o(\vec{r}_o))$$  \hspace{1cm} (3.31)

For convenience later, the quantity $\vec{B}_1$ is renamed $\vec{Q}$:

$$\vec{Q} = \nabla \times (\vec{\xi} \times \vec{B}_o)$$  \hspace{1cm} (3.32)

From (3.27) we have

$$p_1 = -\vec{\xi} \cdot \nabla p_o - \gamma p_o \nabla \cdot \vec{\xi}$$  \hspace{1cm} (3.33)

and from (3.25) we obtain the equation of motion

$$\rho_o \frac{\partial^2}{\partial t^2} \vec{\xi} = \vec{F}(\vec{\xi})$$  \hspace{1cm} (3.34)

where the force-operator is defined:

$$\vec{F}(\vec{\xi}) = \nabla \times \nabla p_o (\vec{\xi} \cdot \vec{B}_o) + \frac{1}{4\pi} \left[ (\nabla \times \vec{Q}) \times \vec{B}_o + (\vec{Q} \times \vec{B}_o) \times \vec{Q} \right]$$  \hspace{1cm} (3.35)

The equation of motion (3.35) may be solved when initial conditions $\vec{\xi}(\vec{r}_0,0)$ and $\vec{\xi}(\vec{r}_0,0)$ and boundary conditions are given.

3.2. Boundary conditions

Assuming that the plasma considered here is confined by the magnetic field, there could be a vacuum region between the plasma and the wall. It is therefore necessary to set up plasma/vacuum interface boundary conditions and vacuum/conducting wall boundary conditions[16]:

**Plasma/vacuum interface**

$$\left[ p + \frac{B^2}{8\pi} \right] = 0$$  \hspace{1cm} (3.36)

$$n \times [E] = \frac{1}{c} (n \cdot u) [B]$$  \hspace{1cm} (3.37)

$$n \cdot [B] = 0$$  \hspace{1cm} (3.38)

**Vacuum/perfectly conducting wall interface**

$$n \times \vec{E} = 0$$  \hspace{1cm} (3.39)
\[ \vec{n} \cdot \frac{\partial}{\partial t} B = 0 \quad (3.40) \]

The symbol \([q]\) means the jump of the value of \(q\) across the interface, i.e. \([q] = q - q'\) where \(q\) is the value just inside the plasma and \(q'\) the value just outside. The vector \(\vec{n}\) is a unit vector normal to the interface pointing into the vacuum. Its counterpart is \(\vec{n}'\) which points into the plasma.

Equation (3.35) is written in terms of the equilibrium position \(\vec{r}_0\). When there is a displacement \(\xi\), the interface changes position. This has to be taken into account in these boundary conditions, but the quantities have still to be written in terms of \(\vec{r}_0\). This complicates the equations a great deal.

Equation (3.36) may be written explicitly:

\[
\left[ P_0(\vec{r}) + P_1(\vec{r}, t) + \frac{1}{8\pi} \left[ \vec{B}_0^0(\vec{r}) + \vec{Q}(\vec{r}, t) \right] \right]^2 = \frac{1}{8\pi} \left[ \vec{B}_0^0(\vec{r}) + \vec{B}_1^1(\vec{r}, t) \right]^2
\]

(3.41)

Since \(\vec{r} = \vec{r}_0 + \vec{\xi}\), each equilibrium term may be expanded around \(\vec{r}_0\) with displacement \(\vec{\xi}\):

\[
P_0(\vec{r}) = P_0(\vec{r}_0) + \vec{\xi} \cdot \nabla P_0(\vec{r}_0) + \ldots \quad (3.42)
\]

and

\[
\vec{B}_0^0(\vec{r}) + 2 \vec{B}_0^0(\vec{r}) \cdot \vec{Q}(\vec{r})
\]

\[
= \vec{B}_0^0(\vec{r}_0) + \vec{\xi} \cdot \nabla \vec{B}_0^0(\vec{r}_0) + 2 \vec{B}_0^0(\vec{r}_0) \cdot \vec{Q}(\vec{r}_0) \quad (3.43)
\]

Collecting terms and using (3.33), one obtains the first-order boundary conditions:

\[
- \gamma P_0 \nabla \cdot \vec{\xi} + \frac{1}{4\pi} \vec{B}_0 \cdot \vec{\xi} + \frac{1}{8\pi} \vec{\xi} \cdot \nabla \vec{B}_0^2 = \frac{1}{4\pi} \vec{B}_0^0 \cdot \vec{B}_1^1 + \frac{1}{8\pi} \vec{\xi} \cdot \nabla \vec{B}_0^1 \quad (3.44)
\]

Now, from (3.37) and Ohm's law (3.6) the following relation is obtained:

\[
\vec{n} \times \vec{E}' = \frac{1}{c} (\vec{n} \cdot \vec{u}) \vec{B}' \quad (3.45)
\]

Since \(\vec{E}\) and \(\vec{u}\) are first-order quantities, and considering the vector potential \(\vec{A}\) with Coulomb gauge \((\nabla \cdot \vec{A} = 0)\), defined as

\[
\vec{E} = - \frac{1}{c} \frac{\partial}{\partial t} \vec{A}, \quad \vec{B}'_1 = \nabla \times \vec{A}
\]

(3.46)

one has, after integrating in time,

\[
- \vec{n}_0 \times \vec{A} = (\vec{n}_0 \cdot \vec{E}) \vec{B}_0'
\]

(3.47)
Condition (3.39) at the interface vacuum/wall becomes
\[ \hat{n}_o \times \hat{A} = 0 \]  

(3.48)

Finally, collecting the equations of motion and the boundary conditions, and dropping the subscript zero, since there is no more need for it, we have
\[ \rho \frac{\partial^2 \xi}{\partial t^2} = \hat{F}(\xi) \]  

(3.49)

where
\[ \hat{F}(\xi) = \nabla (\hat{\xi} \cdot \hat{\nu} \rho + \gamma_p \hat{\nu} \rho \cdot \hat{\nu} \xi) + \frac{1}{4\pi} \left[ (\nu \times \hat{Q}) \times \hat{B} + (\nu \times \hat{B}) \times \hat{Q} \right] \]  

(3.50)

with the boundary conditions
\[ -\gamma_p \nabla \cdot \hat{\xi} + \frac{1}{4\pi} \hat{B} \cdot \hat{Q} + \frac{1}{8\pi} \nabla \cdot \hat{B}^2 = \frac{1}{4\pi} \hat{B}^t \cdot \nu \times \hat{A} + \frac{1}{8\pi} \xi \cdot \nu \hat{B}^t \]  

(3.51)

\[ -\hat{n} \times \hat{A} = (\hat{n} \cdot \hat{\xi}) \hat{B}^t \text{ and } \hat{n}_o \times \hat{A} = 0 \]  

(3.52)

This completes the specifications and the problem is now well defined.

3.3. Self-adjointness of the force operator

It may be proved that the force operator is a self-adjoint operator. To prove it one must show first that an energy integral exists and then proceed to demonstrate the self-adjointness. The system of equations (3.1)—(3.7) may be written in the conservation forms. In particular, the equation of state may be transformed into the energy conservation equation (see e.g. Ref. [16]):
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho \hat{u}^2 + \frac{\rho}{\gamma - 1} + \frac{\hat{B}^2}{8\pi} \right) + \nabla \cdot \left[ \left( \frac{1}{2} \rho \hat{u}^2 + \frac{\rho}{\gamma - 1} + p + \frac{\hat{B}^2}{4\pi} \right) \hat{u} - \frac{1}{4\pi} \hat{u} \cdot \hat{BB} \right] = 0 \]  

(3.53)

**PROBLEM:** Obtain (3.53)

Integrating (3.53) over all space, and assuming that the outgoing energy flux is zero at the wall, one obtains the energy integral:
\[ U = \int_p \left( \frac{1}{2} \rho \hat{u}^2 + \frac{\rho}{\gamma - 1} + \frac{\hat{B}^2}{8\pi} \right) \, d\tau = \text{const} \]  

(3.54)
where $d\tau$ is the volume element, $P$ represents the total plasma space (including
the vacuum region), and $U$ is the total energy. $U$ may be separated into kinetic $K$
and potential $W$ energy:

$$U = K + W$$ (3.55)

with

$$K = \int \frac{1}{2} \rho u^2 d\tau$$ (3.56)

and

$$W = \int \left( -\frac{P}{\gamma - 1} + \frac{B^2}{8\pi} \right) d\tau$$ (3.57)

Since $\dot{u}_0 = 0$ the variation of the kinetic energy $\delta K$ due to the perturbation
is exactly

$$K = \int_P \frac{1}{2} \rho u^2 d\tau = \int_P \frac{1}{2} \rho \dot{\xi} \cdot \dot{\xi} d\tau$$ (3.58)

The variation of the potential energy due to the perturbation, represented by
$\delta W$, depends on $P$ and $B$, and since these variables and their perturbations do not
depend on $\dot{\xi}$ but only on $\ddot{\xi}$, it can be assumed that $\delta W$ is a functional of $\ddot{\xi}$ as
$\delta W[\ddot{\xi}, \dot{\xi}]$. From the conservation of energy

$$\dot{K} = -\delta W$$ (3.59)

and from (3.58) with (3.49)

$$\dot{K} = \int_P \rho \ddot{\xi} \cdot \ddot{\xi} d\tau = \int_P \ddot{\xi} \cdot F(\ddot{\xi}) d\tau$$ (3.60)

But

$$-\delta \dot{W} = -\delta W[\ddot{\xi}, \dot{\xi}] - \delta W[\ddot{\xi}, \dot{\xi}]$$ (3.61)

and, inserting (3.60) and (3.61) into (3.59), one has

$$\int_P \ddot{\xi} \cdot F(\ddot{\xi}) d\tau = -\delta W[\ddot{\xi}, \dot{\xi}] - \delta W[\ddot{\xi}, \dot{\xi}]$$ (3.62)

Since the boundary conditions (3.51) and (3.52) are satisfied for $\ddot{\xi}$ as well as
$\ddot{\xi}$, $\ddot{\xi}$ may be considered as another perturbation, $\ddot{\eta} \equiv \ddot{\xi}$, and it satisfies the
equation. Therefore one may construct
Since the right-hand side of Eqs (3.63) and (3.64) are equal, one obtains the proof of the self-adjointness of the force operator, i.e.,

\[ \int \vec{n} \cdot \vec{F}(\xi) \, d\tau = \int \vec{\xi} \cdot \vec{F}(\vec{n}) \, d\tau \tag{3.65} \]

Furthermore

\[ \delta W[\vec{\xi}, \vec{\xi}] = -\frac{1}{2} \int \vec{\xi} \cdot \vec{F}(\vec{\xi}) \, d\tau \tag{3.66} \]

The self-adjointness of the force operator may be proved directly from the definition of \( \vec{F} \) (see Goedbloed [16]), but this is very long and tedious work.

Now, substituting (3.49) into (3.66), the variation of the potential is obtained:

\[
\begin{align*}
\delta W[\vec{\xi}, \vec{\xi}] &= -\frac{1}{2} \int \left\{ \vec{\xi} \cdot \frac{1}{4\pi} \left[ (\nabla \times \vec{Q}) \times \vec{B} + (\nabla \times \vec{B}) \times \vec{Q} \right] 
+ \vec{\xi} \cdot \nabla \left[ \vec{\xi} \cdot \vec{V} + \gamma p \vec{V} \right] \right\} \, d\tau 
\end{align*}
\tag{3.67}
\]

The integration is performed over the space occupied by the plasma only, excluding the vacuum regions.

Using the following vector identities:

\[
\begin{align*}
\nabla \cdot \left[ \vec{\xi} \left( \vec{\xi} \cdot \vec{V} + \gamma p \vec{V} \right) \right] &= \vec{\xi} \cdot \nabla \left( \vec{\xi} \cdot \vec{V} + \gamma p \vec{V} \right) + (\nabla \cdot \vec{V}) \left[ \vec{\xi} \cdot \vec{V} + \gamma p \vec{V} \right] 
\tag{3.68}
\end{align*}
\]

\[
\begin{align*}
\nabla \cdot \left[ (\vec{\xi} \times \vec{B}) \times \vec{Q} \right] &= \vec{Q} \cdot \nabla \times (\vec{\xi} \times \vec{B}) - (\vec{\xi} \times \vec{B}) \cdot (\nabla \times \vec{Q}) 
+ \vec{Q}^2 
+ \vec{\xi} \cdot \left[ (\nabla \times \vec{Q}) \times \vec{B} \right] 
\tag{3.69}
\end{align*}
\]

and

\[
\begin{align*}
\vec{\xi} \cdot \left[ (\nabla \times \vec{B}) \times \vec{Q} \right] &= - (\nabla \times \vec{B}) \cdot (\vec{\xi} \times \vec{Q}) 
\tag{3.70}
\end{align*}
\]

in (3.66) and applying the divergence theorem, result in two surface terms.
Using (3.51) and (3.52) and rearranging the terms, one obtains a more interesting form of the variation of the potential energy:

$$\delta W(\xi) = \delta W_P(\xi) + \delta W_s(\xi) + \delta W_v(\xi)$$ (3.71)

where

$$\delta W_P(\xi) = \frac{1}{2} \int_P \left[ \gamma P (v \cdot \xi)^2 + (v \cdot \xi)(\xi \cdot v_P) + \frac{\alpha^2}{4\pi} \right] \, d\tau$$

$$+ \frac{1}{4\pi} (\xi \times \partial) \cdot (\nu \times B) \right] \, d\tau$$ (3.72)

$$\delta W_s(\xi) = \frac{1}{2} \int_S (\xi \cdot n)^2 \, n \cdot \left[ v_P + \frac{1}{8\pi} B^2 \right] \, d\tau$$ (3.73)

and

$$\delta W_v(\xi) = \int_V \frac{1}{8\pi} (v \times A)^2 \, d\tau$$ (3.74)

The subscript $P$ represents the plasma volume, $V$ the vacuum volume and $s$ the plasma/vacuum interface. The boundary conditions

$$\vec{n} \times \vec{A} = -(\vec{n} \cdot \xi) \vec{B}$$ (3.75)

for the plasma/vacuum interface and

$$\vec{n} \times \vec{A} = 0$$ (3.76)

for the vacuum/perfectly conducting wall still have to be satisfied.

Equation (3.66) means that the variation of the potential energy is the work done against the force $\vec{F}(\xi)$. This work leads to an increased potential energy of the plasma $W_P$, of the surface $W_s$ and of the vacuum $W_v$.

It is interesting to note that from (3.67) it seems that the variation in the potential energy does not depend on the vacuum region. This is only apparent because its dependence is hidden under the form of boundary conditions (3.51) and (3.52), since (3.71) shows that $\delta W$ has contributions from the plasma, the interface and the vacuum.

Consider now a normal-mode solution with an exponential time-dependence $\exp(-i\omega t)$. The time derivative of the equation of motion (3.34) may be eliminated so that

$$\rho^{-1} \ddot{A}(\xi) = -\omega^2 \xi$$ (3.77)

where $\xi = \xi(\tau)$ is a function of $\tau$ only, and with the boundary conditions it becomes an eigenvalue problem. The natural definition for the inner or scalar product of the two vector fields $\vec{A}$ and $\vec{n}$ for the study of this problem is
One may then talk about the spectrum of the operator $\rho^{-1} \mathbf{F}$. Its spectrum consists of the collection of eigenvalues $\{-\omega_n^2\}$, corresponding to the eigenfunctions $\xi_n^\prime (\mathbf{r})$.

Since the force operator is self-adjoint (or Hermitian) its eigenvalue should necessarily be real, positive or negative, and therefore $\omega_n$ is real or purely imaginary. The eigenvalue $\omega_n$ being real means that the mode is a \textit{stable oscillation}, and $\omega_n$ imaginary means an exponentially growing mode, i.e. an \textit{unstable mode}.

The Rayleigh-Ritz variational principle may be applied to this eigenvalue problem. The eigenfunctions of the operator $\rho^{-1} \mathbf{F}$ are obtained for functions $\xi$ for which the functional

$$\delta W(\xi) = \frac{\langle \xi, \rho^{-1} \mathbf{F}(\xi) \rangle}{\langle \xi, \xi \rangle} = \frac{\delta W(\xi)}{I(\xi)}$$

becomes stationary [16], where the virial

$$I(\xi) = \langle \xi, \xi \rangle = I(t)$$

and $\delta W(\xi)$ is given in (3.66). The virial is written $I(t)$ when it is necessary that the parameter $t$ be explicit.

If it is assumed that the eigenfunctions of the operator $\rho^{-1} \mathbf{F}$ are a subset of the class of square-integrable functions, one may say that the eigenfunction $\xi$ is a linear combination of a base $\{n_1, n_2, \ldots\}$ of the square-integrable functions.

Starting with a trial function

$$\xi = \sum_{n=1}^{\infty} a_n \mathbf{n}_n$$

using the variational method, a set of coefficients $\{a_n\}$ may be found which extremizes the functional $\Omega^2(\xi)$. The eigenfunction is then given by

$$\xi_1 = \sum_{n=1}^{\infty} a_n \mathbf{n}_n$$

and its corresponding eigenvalue $\omega_1^2 = \Omega^2(\xi_1)$. In particular, the ground state is that which makes $\Omega^2(\xi)$ minimum. Thus a set of eigenvalues $\{\omega_n\}$ and their corresponding eigenfunctions can be found.

However, the representative spectrum of a typical MHD plasma contains both discrete and continuous regions, e.g. in the models studied by Goedbloed [16–18]. Grad [19] and Kruskal and Oberman [20] also showed earlier that the operator $\rho^{-1} \mathbf{F}$ does not have a complete set of eigenfunctions in the space of square-integrable functions. This introduces complications for the analysis of our next subject: the energy principle.
3.4. The energy principle

The energy principle states that an equilibrium is stable if and only if \( \delta W(\xi) > 0 \) for all displacements \( \xi(\tau) \) that are bound in norm and satisfy the boundary conditions. This is the Rayleigh-Ritz variational principle applied to marginal stability analysis, i.e. \( \omega^2 = 0 \). To apply this principle one uses a finite class of trial functions covering a subspace of the Hilbert space of the system to calculate \( \delta W \) given in (3.71). If any one of \( \delta W \) is negative, that is enough to say that the equilibrium considered is unstable. If one has a good physical intuition and constructs a trial function which renders \( \delta W < 0 \), the problem of instability is solved.

If the modes are all normal modes and have exponential dependence in time \( \exp(-\omega_n t) \), demonstration of the principle is straightforward. In this case one considers a complete basis \( \{ \xi_n(\tau) \} \):

\[
\rho^{-1} F(\xi_n) = -\omega_n^2 \xi_n
\]

and the orthonormality

\[
(\xi_m, \xi_n) = \delta_{nm}
\]

An arbitrary trial function is expanded in the eigenfunctions:

\[
\tilde{\xi} = \sum_{n=1}^{\infty} a_n \xi_n
\]

which, therefore, applied to \( \delta W \), result in:

\[
\delta W = - (\tilde{\xi}, \rho^{-1} F(\tilde{\xi})) = \sum_{n=1}^{\infty} a_n^2 \omega_n^2
\]

Thus \( \delta W \) can be made negative if and only if there exists at least one \( \xi_n(\tau) \) with eigenvalue \( \omega_n^2 < 0 \). This completes the proof.

A correct proof of necessity and sufficiency is given by Laval, Mercier and Pellat [12], in which no assumption of a complete basis of discrete eigenfunctions is made. Besides the proof, they provide the lower and upper bounds for the growth rate of the unstable perturbation.

They proceed to use the virial and the energy-conservation equations to demonstrate that the kinetic energy of the possible perturbation remains bound for all values of time if and only if \( \delta W > 0 \) for all \( \xi \). They use the concepts of unstable equilibrium according to Lyapunoff's definition.

**Virial equation:**

\[
\frac{\delta^2}{\delta t^2} I(\xi) = \frac{\delta^2}{\delta t^2} (\xi, \xi) = 2 (\dot{\xi}, \dot{\xi}) + 2 (\ddot{\xi}, \xi) = 2\kappa - 2\varepsilon W
\]
Energy-conservation equation:

\[ H = K + \delta W, \quad \dot{H} = 0 \tag{3.87} \]

**PROOF:**

**Sufficiency:** If \( \delta W > 0 \) for all \( \vec{\xi} \), then from (3.87) one concludes that \( K < H \), therefore bound for all \( t \).

**Necessity:** If for a displacement \( \vec{\eta} \), \( \delta W(\vec{\eta}) < 0 \), the system will present an unbounded kinetic energy.

Assume

\[ \delta W(\vec{\eta}) = -\gamma^2 I(\vec{\eta}), \gamma \text{ real} \tag{3.88} \]

One chooses

\[ \vec{\xi}(\vec{r},0) = \vec{\eta}(\vec{r}) \text{ and } \vec{\xi}(\vec{r},0) = \gamma \vec{\eta}(\vec{r}) \tag{3.89} \]

Inserting (3.89) in (3.87), one obtains

\[ K(\vec{\xi}(\vec{r},t)) + \delta W(\vec{\xi}(\vec{r},t)) = H(\vec{\xi}(\vec{r},0)) \]

\[ = K(\vec{\xi}(\vec{r},0)) + \delta W(\vec{\xi}(\vec{r},0)) = 0 \tag{3.90} \]

and from (3.86) and (3.87)

\[ I(t) = 2K - 2\delta W = 4K - 2H = 4K > 0 \tag{3.91} \]

The Schwartz inequality gives

\[ \dot{\mathbf{i}}(t) = 4(\vec{\xi},\vec{\xi})^2 < 4(\vec{\xi},\vec{\xi}) (\vec{\xi},\vec{\xi}) = 4I(t)K = \mathbf{i}(t)I(t) \tag{3.92} \]

Then

\[ \frac{\dot{\mathbf{i}}(t)}{\mathbf{i}(t)} \leq \frac{\dot{\mathbf{i}}(0)}{\mathbf{i}(0)} \tag{3.93} \]

\[ \ln \frac{\mathbf{i}(t)}{\mathbf{i}(0)} \leq \ln \frac{\mathbf{i}(0)}{\mathbf{i}(0)} \tag{3.94} \]

\[ \frac{\dot{\mathbf{i}}(t)}{\mathbf{i}(t)} \geq \frac{\dot{\mathbf{i}}(0)}{\mathbf{i}(0)} \tag{3.95} \]
From (3.87)
\[ \frac{I(t)}{I(0)} = 2\gamma \] (3.96)

From (3.95) and (3.96)
\[ I(t) \geq I(0) \exp(2\gamma t) \] (3.97)

consequently \( \xi \) grows at least as fast as \( \exp(\gamma t) \). q.e.d.

The upper bound for the growth rate is demonstrated by:

THEOREM: If a positive number \( \Gamma \) can be found such that for all functions \( \xi \)
\[ -r^2 \leq -\gamma^2 \xi = \delta W(\xi)/I(\xi) \] (3.98)
then \( \xi(t) \) cannot grow faster than \( \exp(\Gamma t) \).

PROOF:
\[ I(t) = 2k(t) - 2\delta W(t) = 2H(t) - 4\delta W(t) \leq 2H(t) + 4r^2 I(t). \] (3.99)

Hence,
\[ \ddot{I}(t) - 4r^2 I(t) \leq 2H(t) = 2H(0) \] (3.100)

Therefore \( I(t) \) cannot grow faster than \( \exp(2\Gamma t) \) and \( \xi(t) \) cannot grow faster than \( \exp(\Gamma t) \). q.e.d.

The energy principle is a very powerful tool to study the stability of MHD plasmas. The extension of this principle to the double-adiabatic equations is made by Bernstein et al. [11].

4. STABILITY PROBLEMS: APPLICATION OF THE ENERGY PRINCIPLE —
STABILITY OF SHARP-BOUNDARY PLASMAS

Consider a non-magnetized plasma with a constant pressure, confined by a vacuum magnetic field. Assume that the plasma is representable by the ideal MHD equations and that there is an equilibrium.

What is the stability of this equilibrium?

One may apply the energy principle to answer this question. With no magnetic field, \( \vec{B} = 0 \), and constant pressure, \( \nabla p = 0 \), the equation for \( \delta W \) (3.71) becomes
MACROSCOPIC PLASMA PROPERTIES

\[ \delta W = \frac{1}{2} \int_{s} \gamma p (v \cdot \xi) ^{2} d\tau \]

\[ + \frac{1}{2} \int_{s} (\nabla \cdot \bar{n}) ^{2} \bar{n} \cdot \frac{\nabla B'}{8\pi} dS + \int_{V} \frac{1}{8\pi} (v \times \bar{A}) ^{2} d\tau \quad (4.1) \]

Observing that \( \nabla \cdot \xi \) contributes with a positive definite term in \( \delta W \), the most unstable cases should be those with \( \xi \) which are divergence-free, i.e. incompressible perturbations of the fluid,

\[ \nabla \cdot \xi = 0 \quad (4.2) \]

Then \( \delta W \) becomes

\[ \delta W = \frac{1}{16\pi} \int_{s} (\nabla \cdot \bar{n}) ^{2} \bar{n} \cdot \nabla B'^{2} dS + \frac{1}{8\pi} \int_{V} (v \times \bar{A}) ^{2} d\tau \quad (4.3) \]

The vacuum term is a stabilizing contribution since it is also a positive definite term. One cannot set \( \nabla \times \bar{A} = 0 \) arbitrarily here because \( \bar{A} \) must satisfy the boundary conditions with (3.75) and (3.76) and the Maxwell equations \( \nabla \times \nabla \times \bar{A} = 0 \) with \( \nabla \cdot \bar{A} = 0 \). However, one can minimize \( \delta W_{v} \) by choosing the trial functions \( \xi \) and \( \bar{A} \) such that \( \delta W_{v} \) is negligible compared to \( \delta W_{s} \).

The surface term is a destabilizing term, since \( \bar{n} \cdot \nabla B'^{2} \) is negative when the field lines are curved out from the plasma, and so its magnitude decreases away from plasma. If \( \xi \) and \( \bar{A} \) may be chosen such that \( \delta W_{v} \) is negligible compared to \( \delta W_{s} \) and \( \bar{n} \cdot \nabla B'^{2} < 0 \) on some surface region, then there is a \( \xi \) which makes \( \delta W < 0 \), i.e. the equilibrium is unstable.

4.1. Stable curvature

If \( n \cdot \nabla B'^{2} > 0 \) everywhere on interface, the equilibrium is stable for \( \delta W > 0 \). If the confined plasma in Fig.2(a) has an axial symmetry, such as the topology obtained by rotation round the \( xx \)-axis, or a toroidal symmetry, such as the figure obtained by rotation round the \( zz \)-axis, the plasma is surrounded everywhere by a stable magnetic configuration. It is presumed that the equilibrium is stable. However, for plasma confinement purposes this is not enough because of the loss of particles through the magnetic point or line cusps. These cusps have windows the size of the ion Larmor radius.

In Fig.2(b) the radius of curvature \( \bar{R} \) of the vacuum magnetic field lines has to be defined. In vacuum, \( \nabla \times \bar{B}' = 0 \) for \( \bar{J}' = 0 \). Therefore, from Stokes' theorem, the integral round the closed curve \( ABCD \) gives \( S_{1}B'_{1} = S_{0}B'_{0} \). The arc
S$_i$ is proportional to $R_i$, so

$$B_1 R_1^2 = B_0 R_0^2$$  \hspace{1cm} (4.4)

Therefore

$$\frac{1}{2} \vec{n} \cdot \nabla B^2 = \vec{n} \cdot \frac{\vec{R} B^2}{R^2}$$  \hspace{1cm} (4.5)

The stability of the confining field is then related to the sign of $\vec{n} \cdot \vec{R}$.

4.2. Unstable curvature

If $\vec{n} \cdot \nabla B^2 < 0$ for some region of the interface, the equilibrium is unstable since $\delta W < 0$ for some $\vec{\xi}$.

This statement has to be proved.

Consider a co-ordinate system $(x,y,z)$ such that the plasma/vacuum interface is tangent to the plane $x=0$ touching at the point $\vec{r}_0 = (0,0,0)$. Let $\vec{B}'(\vec{r}_0) = B(0,0)\vec{e}_z$, and $\vec{n}(\vec{r}_0) = \vec{e}_x$ be the normal-to-the-equilibrium interface. Choose the trial displacement $\vec{\xi}$ and the vector potential $\vec{A}$ so that

$$\xi_x(0,y,z) = \xi_0 f(y,z) \sin ky$$  \hspace{1cm} (4.6)

and

$$\vec{A}(x,y,z) = f(y,z) \nabla \left( \frac{\xi_0 f(x,y)}{k} \cos ky e^{-kx} \right)$$  \hspace{1cm} (4.7)
where $f$ is a function of order unity which falls to zero in a characteristic distance $a$ such that

$$ka \gg \frac{R}{a} \gg 1 \quad (4.8)$$

The vector potential $A$ satisfies the boundary condition (3.75), and satisfies $\nabla \times \nabla \times \vec{A} = 0$ with $\nabla \cdot \vec{A} = 0$, up to the order of $O(1/ka)$. The function $f(y,z)$ is determined from an equation of the order of $O(1/ka)$. The other components of $\vec{\xi}$ may be chosen to satisfy $\nabla \cdot \vec{\xi} = 0$. The vacuum contribution is written

$$\delta W_v = \int \frac{1}{8\pi} (\nabla \times \vec{A})^2 d\tau$$

$$= \int \frac{1}{8\pi} \left[ \nabla \times \nabla \times \left( \frac{\xi_0}{k} B'(x,y) \cos ky e^{-kx} \right) \right]^2 d\tau$$

$$= \int \frac{1}{8\pi} \frac{\xi_0^2 B'^2}{a^2} e^{-2kx} d\tau = \frac{\xi_0^2 B'^2}{8\pi a^2} \frac{\Delta S}{2k} \quad (4.9)$$

while the surface contribution is

$$\delta W_s = \frac{1}{16\pi} \int (\vec{\xi},\vec{\eta})^2 \vec{n} \cdot B'^2 dS = \frac{\xi_0^2 B'^2}{16\pi R} \Delta S \quad (4.10)$$

Therefore

$$\frac{\delta W_v}{\delta W_s} = \frac{R/a}{ka} \ll 1 \quad (4.11)$$

Then, if $n \cdot \nabla B'^2 < 0$, a trial function $\vec{\xi}$ may be chosen so that $\delta W < 0$. Therefore the equilibrium is unstable.

The lower bound value of the growth rate can then be calculated from (3.97). Expression (4.6) is the value of $\xi_x$ for $x = 0$. An $x$-dependence of $\xi_x$ may be chosen within the validity of the above analysis, so that one defines

$$\xi_z = 0$$

$$\xi_y = \xi_0 \xi e^{kx} \cos ky$$

$$\xi_x = \xi_0 \xi e^{kx} \sin ky \quad (4.12)$$

The virial results in

$$I(\vec{\xi}) = \frac{1}{2} \int \rho \vec{\xi}^2 dz = \frac{\rho \xi_0^2}{2k} \Delta S \quad (4.13)$$
Therefore

\[ n(\xi) = - \frac{\delta W(\xi)}{I(\xi)} = - \frac{B_1^2}{\delta \pi \rho} \frac{k}{R} \]  

(4.14)

In conclusion, the growth rate is at last \( \Omega \), that is to say:

\[ \gamma \approx \Omega = V_A \frac{1}{4\pi R} \]  

(4.15)

where \( V_A^2 = \frac{B'^2}{4\pi \rho} \) is the Alfvén velocity in a plasma of density \( \rho \) in the magnetic field \( B' \).

5. THERMODYNAMIC APPROACH FOR STABILITY OF PLASMAS: NEWCOMB AND ROSENBLUTH'S STABILITY CRITERIA

The MHD theory of stability is limited to modes with very long characteristic times and lengths in relatively low-temperature plasmas. The fluid theory may lift some of the limitations but it still deals with macroscopic modes. For higher-temperature plasmas, kinetic theory is required to study shorter time- and length-scale macroscopic modes.

In general, any mode of oscillation is driven unstable when free energy is available in the plasma, i.e. there is some extra energy above the ‘ground-state’ energy of the plasma. The free energy may be converted into energy of the electromagnetic fields or another distribution of energy among the particles of plasma. In principle, all bound plasmas have free energy which may drive some of the macroscopic modes unstable. An unbound and uniform plasma usually has the least free energy, except when the particle energy distribution is very peculiar, such as the case of two humps, and then it may drive microscopic modes unstable.

One asks: is there any stable distribution of energy in plasmas? The answer is yes. Newcomb [21] has shown that a Maxwellian velocity distribution is stable against small electromagnetic field perturbations, using the properties of entropy. Rosenbluth [22] has extended the definition of entropy and proved that a distribution monotonically decreasing with energy is stable. Others ([23–26] and p.447 of Ref.[3]) have presented the same criterion in similar or different forms.

Rosenbluth’s criterion for the stability of plasmas is as follows: a uniform and unbounded ideal plasma, with or without static uniform magnetic field, and in which the distribution function is monotonically decreasing with particle energy, is stable.
**Proof:**

This may be proved, first, by showing that the state of minimum kinetic energy of a system subject to some constraints is the one with a distribution function which is monotonically decreasing with particle energy, and, second, by pointing out that any departure from the minimum state increases the kinetic energy of the system and that any electric and magnetic fields which may develop must also increase in energy. Since all three forms of energy are initially at this minimum value, there cannot be any growth of electric or magnetic fields because no source of energy is available.

The constraints are that the total number of particles

\[ N = \int f \, d\mathbf{r}d\mathbf{v} \quad (5.1) \]

and the generalized entropy

\[ S = \int G(f) \, d\mathbf{r}d\mathbf{v} \quad (5.2) \]

be constant.

The generalized entropy has the form given in (5.2) where \( G = G(f) \) is a function to be determined. \( S \) is constant, because \( f \) is constant by virtue of the Vlasov equation, \( df/dt = 0 \), and the phase-space volume element is also constant, from Liouville's theorem.

**Problem:** Using the Vlasov equation, prove that the entropy given as (5.2) is constant, where \( G(f) = f \ln f \).

The total kinetic energy with \( E = mv^2/2 \) is given by

\[ K = \int Ef \, d\mathbf{r}d\mathbf{v} \quad (5.3) \]

The minimization is carried out by employing Lagrange's undetermined multipliers and defining the functional

\[ I = K + \alpha N + \beta S = \int \left[ Ef + \alpha f + \beta G(f) \right] d\mathbf{r}d\mathbf{v} \quad (5.4) \]

By taking the first variation, \( f = f_0 + \delta f \), one has

\[ \delta I = I - I_0 = \int \left[ E + \alpha + \beta \frac{dG}{df} \right] df \, d\mathbf{r}d\mathbf{v} \quad (5.5) \]

from which one obtains

\[ E + \alpha + \beta \frac{dG}{df} = 0 \quad (5.6) \]

in order to extremize the kinetic energy.
To prove that this extremum is a minimum requires a tedious calculation but it can be done. Solving for $G$, one obtains

$$dG = - \left( \frac{E + a}{\beta} \right) df = - \left( \frac{E(f) + a}{\beta} \right) df$$

(5.7)

which may be integrated unambiguously only if the inverse function $E(f_0)$ is single-valued, i.e. $E(f_0)$ is a monotonic function of $f_0$. Since $E$ as energy has an upper bound, $f_0$ should be a monotonically decreasing function.

Thus it is proved that any distribution which is a monotonically decreasing function of energy is in the lowest energy state.

There are three important conclusions:

(a) A definition is obtained of the generalized entropy $G(f)$. In particular, if $f_0$ is Maxwellian, then $G(f) = f\beta nf$.

(b) The collective modes by themselves do not produce the Maxwell-Boltzmann distribution.

(c) In a uniform and isotropic plasma there are no instabilities which transfer energy between ions and electrons and no energy is available to drive the instability.

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PLASMA KINETIC EQUATIONS

M.H.A. HASSAN
School of Mathematical Sciences,
University of Khartoum,
Khartoum,
Sudan

Abstract

PLASMA KINETIC EQUATIONS.
1. The Liouville equation and the BBGKY hierarchy of equations.
2. The Vlasov equation.
3. Kinetic equation for a uniform magnetized plasma.
4. Unmagnetized kinetic equations: the Balescu-Lenard equation; the Landau equation; Fokker-Planck form of Balescu-Lenard equation (the RMJ equation).
5. Derivation of a simple usable magnetized kinetic equation (generalized magnetic Rosenbluth potentials).

INTRODUCTION

In general, two formulations have been followed in the derivation of kinetic equations for magnetized plasmas. The more straightforward of these is based directly upon BBGKY theory and uses one of a variety of procedures to truncate the hierarchy. Rostoker and Rosenbluth [1], Rostoker [2], Silin [3] and Schram[4] based their theories on the Mayer cluster expansion together with some assumptions on the magnitudes of the correlation functions. Rostoker and Rosenbluth [1] and Rostoker [2] assumed that the two-particle correlation function was of first order in the plasma parameter $1/A_0$ (the inverse of the number of particles in a Debye sphere) and that other correlations were of higher than first order in $1/A_0$. Retaining those terms that are of first order in $1/A_0$, Rostoker and Rosenbluth [1] obtained a magnetized kinetic equation applicable to a test particle in a Maxwellian plasma, while Rostoker [2] generalized the result to arbitrary distribution functions that are independent of azimuthal angles. Silin [3] obtained a Landau-type equation by ignoring the three- and higher-particle correlation functions and terms that describe collective effects. A Landau-type equation for a single-component plasma was also obtained by Schram [4], using the multi-time-scale theory developed by Frieman and Sandri. Sundaresan and Wu [5] based their theory on Bogolyubov's method in which Bogolyubov's functional ansatz is used instead of the cluster expansion. They obtained a magnetized kinetic equation similar to that of Rostoker [2]. The other approach was initiated by Thompson and Hubbard [6] and made more rigorous by Rostoker [7] in his 'dressed test particle' theory. In this method one calculates the autocorrelation function in terms of fluctuating electric fields and obtains expressions for the coefficients of friction and diffusion. Prominent among those
who adopt this procedure are Sitenko and Yu [8], Klimontovich [9] and recently Ichimaru and Rosenbluth [10]. A more general treatment of this kind, incorporating quantum effects, has been given by Eleonskij et al. [11].

All these results are closely related. Apart from notation, the kinetic equations independently derived by Rostoker [2] and Sundarasan and Wu [5] are identical. Schram’s equation, restricted to isotropic distribution functions, can be derived from Rostoker’s equation by setting $\epsilon(k, \omega)$ ($\epsilon$ being the plasma dielectric function), which appears in Rostoker’s equation, equal to unity. Comparison with Silin’s equation is only possible if we take moments. Silin’s expression for the temperature relaxation time can be derived from Rostoker’s equation by setting $\epsilon(k, \omega)$ equal to unity and taking the appropriate moment. The magnetized friction and diffusion coefficients derived independently by Sitenko and Yu [8], Klimontovich [9] and Ichimaru and Rosenbluth [10] are identical to each other. Furthermore, if we substitute these coefficients into the Fokker-Planck equation we obtain the same magnetized collision integral as that derived by Rostoker [2] and by Sundaresan and Wu [5]. The quantum collision integral derived by Eleonskij et al. [11] is shown to possess the Rostoker form in the classical limit.

Here we shall adopt the BBGKY theory approach and derive a magnetized kinetic equation for a stable uniform non-relativistic plasma using an operator method originally developed by Dupree [12] for the unmagnetized plasma. The magnetized kinetic equation obtained generalizes that derived by Rostoker [2], who made the assumption that the one-particle distribution function was independent of the gyration angle, and all previous results can be derived from it. An additional advantage of this approach is that the collision integral is obtained in a form which is free from infinite sums of Bessel functions, and this greatly facilitates calculations based on it.

When the limit of zero magnetic field is taken, the equation reduces to the familiar Balescu-Lenard equation. For a two-component ion-electron plasma in which the ions are unmagnetized, the collision integral of the equation can be written in terms of two ‘generalized magnetic Rosenbluth potentials’. These potentials simplify substantially if wave effects are neglected (i.e. if we set $|\epsilon(k, \omega)|^2 = 1$ in these potentials) and reduce to the standard Rosenbluth potentials in the limit of vanishing magnetic field.

1. THE LIOUVILLE EQUATION AND THE BBGKY HIERARCHY OF EQUATIONS

We consider a multi-species plasma consisting of $N$ charged particles, confined in a volume $V$ and subjected to an external uniform magnetic field $B$. Let $N_\sigma$ be
the number of particles of mass $m_\sigma$ and charge $e_\sigma$; with $\sigma = 1, \ldots, \mu$, and

$$\sum_{\sigma=1}^{\mu} N_\sigma = N$$

The $k$th particle ($k \in \{1, \ldots, N\}$) has position $x_k$, velocity $v_k$ and type $\sigma_k$ ($\sigma_k \in \{1, \ldots, \mu\}$). Let $X_k$ designate the six-dimensional particle co-ordinates ($x_k, v_k$).

We now define a function $D_N(X_1, \ldots, X_N, t)$, known as the Liouville density function, with the following properties:

(i) $D_N(X_1, \ldots, X_N, t)dX_1 \ldots dX_N$ is the probability of finding the system in the state $(X_1, X_1 + dX_1, \ldots, X_N, X_N + dX_N)$.

(ii) $D_N$ is normalized to one; $\int D_N dX_1 \ldots dX_N = 1$.

(iii) $D_N = 0$ at $x_k = \pm \infty$; no particles at infinite distances away.

(iv) $D_N = 0$ at $v_k = \pm \infty$; no particles have infinite velocities.

(v) $D_N(X_1, \ldots, X_N, t = 0)$ is given.

It is well known in statistical mechanics that $D_N$ develops in time according to Liouville's theorem:

$$\frac{\partial D_N}{\partial t} + \sum_{k=1}^{N} v_k \cdot \frac{\partial D_N}{\partial x_k} + \sum_{k=1}^{N} \frac{e_{\sigma_k}}{m_{\sigma_k}} v_k \times B \cdot \frac{\partial D_N}{\partial v_k} - \sum_{k \neq \ell} \frac{1}{m_k} \frac{\partial \phi_{k\ell}}{\partial x_k} \frac{\partial D_N}{\partial v_\ell} = 0$$

(1.1)

$\phi_{k\ell}$ is the potential energy of interaction between particle $k$ and particle $\ell$.

Neglecting the contribution of transverse fields to the interaction, $\phi_{k\ell}$ takes the Coulombic form:

$$\phi_{k\ell} = \frac{1}{4\pi \epsilon_0} \frac{e_{\sigma_k} e_{\sigma_\ell}}{|x_k - x_\ell|}$$

Let us now define the reduced $s$-particle distribution function $f_s$ by

$$f_s(X_1, \ldots, X_s, t) = V^s \int \ldots \int D_N d^3 X_{s+1} \ldots d^3 X_N$$

A differential equation governing the time development of $f_s$ can be generated by integrating the Liouville equation (1.1) over the sub-space $X_{s+1} \ldots X_N$. Using the definition of $f_s$ and properties (iii) and (iv), we arrive at the well known
BBGKY hierarchy of equations (BBGKY stands for Bogolyubov, Born, Green, Kirkwood and Yvon, who independently recognised and developed the equations between 1939 and 1946):

\[
\frac{\partial f_s}{\partial t} + \sum_{k=1}^{s} v_k \cdot \frac{\partial f_s}{\partial x_k} + \sum_{k=1}^{s} \frac{e_k}{m_k} v_k \times B \cdot \frac{\partial f_s}{\partial v_k} - \sum_{k=1}^{s} \sum_{l=1}^{s} \frac{1}{m_k m_l} \frac{\partial \varphi_{kl}}{\partial x_k} \cdot \frac{\partial f_s}{\partial v_l} = \sum_{k=1}^{s} \sum_{\sigma=1}^{k} \frac{n_\sigma}{m_k} \frac{\partial \varphi_{k\sigma}}{\partial x_k} \cdot \frac{\partial f_{s+1}}{\partial v_k} d^3 X_\sigma
\]

where

\[
n_\sigma = \lim_{N_\sigma \to \infty} \frac{(N_\sigma - s)/V}{N \to \infty} \quad (s \ll N)
\]

This limit allows us to consider an infinite system and remove surface effects. Equation (1.2) without the right-hand side is the Liouville equation for s-particles. The right-hand side represents the effect of the other N-s particles and hence embodies collective effects.

The first member of the hierarchy (s = 1) bears a marked resemblance to Boltzmann's equation, except that the collision term in Boltzmann's equation (obtained by making numerous assumptions) has been replaced now by the integral on the right-hand side. However, it differs in that the right-hand side is no longer a functional of \( f_1 \) alone, but involves \( f_2 \): the penalty we pay for the rigour of the BBGKY approach. In like manner, each of the higher-order equations relates one of the distribution functions to the next higher one. Thus the entire set of equations must be solved in order to determine \( f_1 \), which is equivalent to solving the Liouville equation.

The usefulness of the hierarchy, however, stems from the fact that it allows for a systematic expansion procedure in terms of a small parameter (inverse number of particles in the Debye sphere), which results in breaking the chain of equations. To find the appropriate small parameter that governs the decoupling process, it is useful to express the BBGKY chain in dimensionless form. To this end we take the reciprocal plasma frequency \( \omega_p^{-1} = (e^2 n_0/\epsilon_0 m_0)^{-1/2} \) and the Debye length \( \lambda_D = (v_0/2 \omega_p) \) (associated with an arbitrarily chosen species) as typical units of time and distance. We then introduce the non-dimensional variables, denoted by tildes:

\[
v = \lambda_D \omega_p \tilde{v}, \quad f_s = (\lambda_D \omega_p)^{3s} \tilde{f}_s, \quad dX = d\tilde{X} = \lambda_D \omega_p^3 \tilde{d}X
\]

\[
n_\sigma = n_0 \tilde{n}_\sigma, \quad \phi_{ij} = \frac{\omega_p^2 m_0}{n_0 \lambda_D} \tilde{\phi}_{ij}, \quad p = m v = m_0 \lambda_D \omega_p \tilde{P}
\]
Inserting these new variables into the hierarchy equations (1.2) gives (dropping the tilde notation)

\[ \frac{df_{s}}{dt} + \sum_{k=1}^{i} v_{k} \cdot \frac{df_{s}}{dx_{k}} + \sum_{k=1}^{i} \eta_{k} v_{k} \times \hat{B} \cdot \frac{df_{s}}{dv_{k}} - \frac{1}{\Lambda_{0} k} \sum_{k=1}^{i} \sum_{l=1}^{s} \frac{\partial \phi_{kl}}{\partial x_{k}} \cdot \frac{df_{s}}{dp_{k}} \]

\[ = \sum_{k=1}^{i} \sum_{\sigma=1}^{k} \int n_{\sigma} \frac{\partial \phi_{k\sigma}}{\partial x_{k}} \cdot \frac{df_{s+1}}{dp_{k}} d^{3}x_{\sigma} \]  

(1.3)

where

\[ \eta_{k} = \frac{e_{k}B}{m_{k}\omega_{p}} = \Omega_{k}/\omega_{p} \]

is the ratio of the cyclotron frequency, \( \Omega_{k} \), to the plasma frequency \( \omega_{p} \); \( \hat{B} \) is a unit vector in the direction \( B \) and \( \Lambda_{0} \) is the number of particles in a Debye sphere, \( \Lambda_{0} = n\lambda_{D}^{3} \).

2. THE VLASOV EQUATION

For most plasmas of interest, \( \Lambda_{0} \) is a large quantity (of order \( 10^{6} \) for thermonuclear plasmas). In the limit \( \Lambda_{0} \to \infty \) we can ignore the interaction term. The resultant hierarchy will then have the solution

\[ f_{s} = \prod_{i=1}^{s} f_{1}(X_{i}, t) \]

provided that \( f_{s} \) satisfies the Vlasov equation:

\[ \frac{\partial f_{1}(X_{1}, t)}{\partial t} + v_{1} \cdot \frac{\partial f_{1}}{\partial x_{1}} + \eta_{1} v_{1} \times \hat{B} \cdot \frac{\partial f_{1}}{\partial v_{1}} \]

\[ - \sum_{\sigma=1}^{\mu} \int \frac{n_{\sigma}}{m_{1}} \frac{\partial \phi_{1\sigma}}{\partial x_{1}} \cdot \frac{\partial f_{1}}{\partial v_{1}} (X_{1}, t) f_{1}(X_{\sigma}, t) dX_{\sigma} = 0 \]
Restoring the dimensional form of this equation and introducing the notation

\[ f_\alpha(X_\alpha, t) = f_1(X_1, t), \quad f_\beta(X_\beta, t) = f_1(X_\alpha, t) \]

we can write the above equation in the form:

\[
\frac{\partial f_\alpha}{\partial t} + v_\alpha \cdot \frac{\partial f_\alpha}{\partial x_\alpha} + \frac{e_\alpha}{m_\alpha} (E + v_\alpha \times B) \cdot \frac{\partial f_\alpha}{\partial v_\alpha} = 0 \tag{2.1}
\]

where we have defined the electric field:

\[
E = \sum_{\beta} \frac{e_\beta}{4\pi \varepsilon_0} \int \frac{x_\alpha - x_\beta}{|x_\alpha - x_\beta|^3} f_\beta(X_\beta, t) d^3X_\beta \tag{2.2}
\]

2.1. The linearized Vlasov equation

Equations (2.1) and (2.2) are linearized in the usual way by assuming that

\[ f_\alpha = f_\alpha^0 + f_\alpha' \, , \quad E = E_1 \, , \quad B = B_0 + B_1 \]

and treating \( f_\alpha' \), \( E_1 \) and \( B_1 \) as small compared to their equilibrium values. Furthermore, we assume that the equilibrium distribution function \( f_\alpha^0 \) is isotropic.

Substituting those perturbations into Eq.(2.1) and neglecting terms of higher order than the first in perturbations, we obtain the linearized Vlasov equation (writing \( f_\alpha' = f_\alpha \), \( E_1 = E \), \( B_0 = B \)):

\[
\frac{\partial f_\alpha}{\partial t} + v_\alpha \cdot \frac{\partial f_\alpha}{\partial x_\alpha} + \frac{e_\alpha}{m_\alpha} E \cdot \frac{\partial f_\alpha^0}{\partial v_\alpha} + \frac{e_\alpha}{m_\alpha} (v_\alpha \times B) \cdot \frac{\partial f_\alpha}{\partial v_\alpha} = 0 \tag{2.3}
\]

The linear equation (2.3) can be solved using Fourier-Laplace transforms.

We first introduce the Fourier transform in space:

\[
\tilde{f}_\alpha(k, v_\alpha, t) = \int f_\alpha(x_\alpha, v_\alpha, t) \exp(-i k \cdot x_\alpha) d^3x_\alpha
\]

Taking the Fourier transform of (2.3), we obtain:

\[
\frac{\partial \tilde{f}_\alpha}{\partial t} + i(k \cdot v_\alpha) \tilde{f}_\alpha + \frac{e_\alpha}{m_\alpha} E \cdot \frac{\partial \tilde{f}_\alpha^0}{\partial v_\alpha} + \frac{e_\alpha}{m_\alpha} (v_\alpha \times B) \cdot \frac{\partial \tilde{f}_\alpha}{\partial v_\alpha} = 0 \tag{2.4}
\]
where $\mathbf{E}$ is given by the Fourier transform of Eq. (2.2):

$$
\mathbf{E} = -\frac{ik}{\varepsilon_0 k^2} \sum_\gamma e_\gamma n_\gamma \int \mathbf{f}_\gamma(k, v_\alpha, t) \, d^3v_\gamma
$$

(2.5)

Equation (2.4) can be written in the more compact form:

$$
\frac{\partial \mathbf{f}_\alpha}{\partial t} + \mathbf{H}_\alpha \mathbf{f}_\alpha = 0
$$

(2.6)

where the operator $\mathbf{H}_\alpha$ is given by

$$
\mathbf{H}_\alpha = i(k \cdot v_\alpha) \mathbf{f}_\alpha - \frac{i}{m_\alpha} \frac{\partial f_\alpha}{\partial v_\alpha} \cdot k \sum_\gamma n_\gamma \Phi_{\alpha \gamma} \int \mathbf{f}_\gamma \, d^3v_\gamma + \frac{e_\alpha}{m_\alpha} (v_\alpha \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial v_\alpha}
$$

(2.7)

We now introduce the modified Laplace transforms in time:

$$
\hat{f}_\alpha = \int_0^\infty \mathbf{f}_\alpha(k, v_\alpha, t) \exp(i\omega t) \, dt
$$

with the inverse:

$$
\mathbf{f}_\alpha = \frac{1}{2\pi} \int_C \exp(-i\omega t) \hat{f}_\alpha(k, v_\alpha, \omega) \, d\omega
$$

where the contour $C$ is parallel to the real axis and above all singularities of $\hat{f}_\alpha$.

Taking the Laplace transform of (2.4), (2.5) and (2.6), we find:

$$
i(k \cdot v_\alpha - \omega) \hat{f}_\alpha + \frac{e_\alpha}{m_\alpha} \mathbf{E} \cdot \frac{\partial f_\alpha}{\partial v_\alpha} + \frac{e_\alpha}{m_\alpha} (v_\alpha \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial v_\alpha} = \hat{f}_\alpha(k, v_\alpha, 0)
$$

(2.8)

$$
\mathbf{E} = -\frac{ik}{\varepsilon_0 k^2} \sum_\gamma e_\gamma n_\gamma \int \mathbf{f}_\gamma \, d^3v_\gamma
$$

(2.9)
and
\[
\hat{f}_\alpha = \frac{1}{H_\alpha - i\omega} \hat{f}_\alpha(k, v_\alpha, 0) \tag{2.10}
\]

It is convenient to choose a cylindrical coordinate system with the z-axis along the magnetic field. We thus write

\[ B = (0, 0, B) \]

\[ v_\alpha = (v_{\alpha\perp}\cos\phi, v_{\alpha\perp}\sin\phi, v_{\alpha\parallel}) \tag{2.11} \]

\[ v_\gamma = (v_{\gamma\perp}\cos\xi, v_{\gamma\perp}\sin\xi, v_{\gamma\parallel}) \]

and

\[ k = (k_{\perp}\cos\chi, k_{\perp}\sin\chi, k_{\parallel}) \]

Noting that
\[
e_{\alpha} (v_{\alpha}\times B) \cdot \frac{\partial \hat{f}_\alpha}{\partial v_{\alpha}} = -\Omega_\alpha \frac{\partial \hat{f}_\alpha}{\partial \phi}
\]

Eq.(2.8) can be written in the form:

\[
\frac{\partial \hat{f}_\alpha}{\partial \phi} - \frac{i}{\Omega_\alpha} (k \cdot v_\alpha - \omega) \hat{f}_\alpha = \frac{1}{\Omega_\alpha} \left[ \frac{e_\alpha}{m_\alpha} \hat{E} \cdot \frac{\partial e_\alpha^0}{\partial v_\alpha} - \hat{f}_\alpha(k, v_\alpha, 0) \right] \tag{2.12}
\]

Equation (2.12) is a first-order linear differential equation. Following Bernstein, we introduce the primed notation:

\[ v'_\alpha = (v_{\alpha\perp}\cos\phi', v_{\alpha\perp}\sin\phi', v_{\alpha\parallel}\sin\phi', v_{\alpha\parallel}) \]

and note that

\[
\exp \left[ \frac{i}{\Omega_\alpha} \int_{\phi'}^{\phi} (k \cdot v''_{\alpha} - \omega) \right] = \exp \left[ \frac{i}{\Omega_\alpha} \right] \left( \Omega_\alpha v_{\alpha\parallel} - \omega \right) (\phi - \phi') - i k_{\perp} v_{\alpha\perp} \]

is an integrating factor for (2.12). If we now make use of the fact that on physical grounds \( f_\alpha \) must be bounded and periodic in \( \phi \) with a period \( 2\pi \), and that \( \omega \) is defined with a positive imaginary part, the solution of (2.12) can readily be written as:
From Eqs (2.9) and (2.13) we now have

\[ \mathbf{k} \cdot \mathbf{E} = -\frac{i}{\epsilon_0} \sum_{\gamma} e_{\gamma} n_{\gamma} \int_f \mathbf{r} \, d^3 \mathbf{v}_{\gamma} \]

\[ = \frac{i}{\epsilon_0} \sum_{\gamma} \frac{e_{\gamma} n_{\gamma}}{\Omega_{\gamma}} \int d^3 \mathbf{v}_{\gamma} \int_0^\infty d\eta'' e^{X_\gamma} \left[ \frac{e_{\gamma}}{m_{\gamma}} \mathbf{E} \cdot \frac{\partial f^0_{\gamma}}{\partial v_{\gamma}'} - \tilde{f}_{\gamma}(k, v_{\gamma}, 0) \right] \quad (2.14) \]

where

\[ X_\gamma(\eta'', \omega, k) = -\frac{i}{\Omega_{\gamma}} \left\{ (k_{||} v_{\gamma||} - \omega) \eta'' + k_{\perp} v_{\gamma\perp} \{ \sin(\nu'' + \eta'') - \sin \nu'' \} \right\} \quad (2.15) \]

\[ v^0_{\gamma} = (v_{\gamma\perp} \cos(\xi'' + \eta''), v_{\gamma\perp} \sin(\xi'' + \eta''), v_{\gamma||}) \]

and we have made the transformations:

\[ \eta'' = (\phi' - \xi), \quad \nu'' = (\xi - \chi) \]

Restricting ourselves to longitudinal waves only, we have

\[ \tilde{E} = \frac{k}{k^2} (k \cdot \tilde{E}) \]

in view of which we deduce from Eq.(2.14)

\[ \mathbf{k} \cdot \tilde{E} = -\frac{i}{\epsilon_0} \frac{1}{e(k, \omega)} \sum_{\gamma} \frac{e_{\gamma} n_{\gamma}}{\Omega_{\gamma}} \int d^3 \mathbf{v}_{\gamma} \int_0^\infty d\eta'' e^{X_\gamma} f_{\gamma}(k, v^0_{\gamma}, 0) \quad (2.16) \]
where \( \varepsilon(k, \omega) \) is the plasma dielectric function:

\[
\varepsilon(k, \omega) = 1 - \frac{i}{\varepsilon_0 k^2} \sum_{\gamma} \frac{e_{\gamma}^2 n_\gamma}{\Omega_\gamma m_\gamma} \int d^3 v_\gamma \int_0^\infty d\eta'' e^{X_\gamma k \cdot \frac{\partial f_\gamma^0}{\partial \nu_\gamma}} (2.17)
\]

Substitution of (2.16) into (2.13) yields, after introducing the variable of integration \( \eta = \phi' - \phi \):

\[
\hat{f}_\alpha = \frac{1}{\Omega_\alpha} \int_0^\infty d\eta e^{X_\alpha} \left[ f_\alpha(k, \nu_\alpha^0, 0) + \frac{A_\alpha}{\varepsilon(k, \omega)} \sum_{\gamma} \frac{e_\gamma n_\gamma}{\Omega_\gamma} \int_0^\infty d\eta'' e^{X_\gamma f_\gamma(k, \nu_\gamma^0, 0)} \right]
\]

where

\[
A_\alpha = \frac{ie_\alpha}{\varepsilon_0 m_\alpha k^2} \cdot \frac{\partial f_\alpha^0}{\partial \nu_\alpha^0} (2.19)
\]

\[
X_\alpha(\eta, \omega, k) = -\frac{i}{\Omega_\alpha} [(k_\parallel \nu_\alpha \parallel - \omega)\eta + k_\perp \nu_\alpha \perp \{\sin(\nu + \eta) - \sin\nu\}] (2.20)
\]

and \( \nu = \phi - x \).

From Eqs (2.10) and (2.18) we now have the following expression for the operator \( 1/(H_\alpha - i\omega_1) \) (writing \( \omega = \omega_1 \)):

\[
\frac{1}{H_1} = \frac{1}{\Omega_1} \int_0^\infty d\eta e^{X_1} \left[ 1 + \frac{A_1}{\varepsilon(k, \omega_1)} \sum_{\gamma} \frac{e_\gamma n_\gamma}{\Omega_\gamma} \int d^3 v_\gamma \int_0^\infty d\eta'' e^{X_\gamma} \right]
\]

(2.21)

Replacing \( \alpha, \omega_1, k, \eta \) and \( \nu \) by \( \beta, \omega_2, -k, \eta' \) and \( \nu' \), respectively, we obtain from (2.21)

\[
\frac{1}{H_\beta} = \frac{1}{\Omega_\beta} \int_0^\infty d\eta' e^{X_\beta} \left[ 1 + \frac{A_\beta}{\varepsilon(-k, \omega_2)} \sum_{\gamma} \frac{e_\gamma n_\gamma}{\Omega_\gamma} \int d^3 v_\gamma \int_0^\infty e^{X_\gamma d\eta''} \right]
\]

(2.22)
In the case when the distribution function $f^{0}_\alpha$ is independent of the gyration angle $\phi$, the plasma dielectric function (2.17) takes the form:

$$
\varepsilon(k, \omega) = 1 - \frac{i}{k^2} \sum_{\gamma} \omega^{2}_\gamma \int d^3v_{\gamma} \int_0^\infty dt \\
\times \exp[iY_{\gamma}(t, \omega, k)] \left[ k_{||} \frac{\partial f^{0}_\gamma}{\partial v_{0||}} + k_{\perp} \cos(\nu'' - \Omega_{\gamma} t) \frac{\partial f^{0}_\gamma}{\partial v_{\gamma\perp}} \right] 
$$

(2.23)

where we have made the change of variable $\eta'' = \Omega_{\gamma} t$. $\omega_{\gamma}$ is the plasma frequency; $\omega^{2}_\gamma = e^2 n_{\gamma}/(\epsilon_0 m_{\gamma})$ and

$$
Y_{\gamma}(t, \omega, k) = a_{\gamma}[\sin\nu'' - \sin(\nu'' + \Omega_{\gamma} t) + t(\omega - k_{||} v_{\gamma||})]
$$

Furthermore, the relations

$$
\exp(ia \sin \theta) = \sum_{n = -\infty}^{\infty} J_n(a) \exp in \theta
$$

$$
\cos \theta \exp(ia \sin \theta) = \sum_{n = -\infty}^{\infty} \frac{n}{a} J_n(a) \exp in \theta
$$

enable us to perform the $\xi$ and $t$ integrations in Eq.(2.23). The result is

$$
\varepsilon(k, \omega) = 1 + \sum_{\gamma} \frac{\omega^{2}_\gamma}{k^2} \int d^3v_{\gamma} \frac{\text{Im}(a_{\gamma}) O^m_{\gamma} f^{0}_\gamma}{(\omega - k_{||} v_{\gamma||} - m\Omega_{\gamma})}
$$

(2.24)

where

$$
O^m_{\gamma} = \left( k_{||} \frac{\partial}{\partial v_{\gamma||}} + \frac{m\Omega_{\gamma}}{v_{\gamma\perp}} \frac{\partial}{\partial v_{\gamma\perp}} \right)
$$

and

$$
a_{\gamma} = k_{\perp} v_{\gamma\perp}/\Omega_{\gamma}
$$

(2.25)
Bear in mind that \( \omega \) is originally defined with positive imaginary part. The imaginary part of (2.24) can readily be found upon employing the Plemelj formula:

\[
\delta m \varepsilon(k, \omega) = -\pi \sum_{m} \frac{\omega_{m}^{2}}{k^{2}} d^{3}v_{\gamma} \sum_{m} J_{m}(a_{\gamma}) \delta(\omega - k_{\parallel} v_{\gamma\parallel} - m\Omega_{\gamma}) O_{m}^{\gamma} f_{y}^{0}
\]

(2.26)

where \( \omega \) is now real. Carrying out the \( m \) summation, we find

\[
\delta m \varepsilon(k, \omega) = -\sum_{\gamma} \frac{\omega_{\gamma}^{2}}{2k^{2}} \int d^{3}v_{\gamma} \int_{-\infty}^{\infty} dt \, e^{iY_{\gamma}} \left[ k_{\parallel} \frac{\partial f_{0}}{\partial v_{\gamma\parallel}} + k_{\perp} \cos(\nu'' + \Omega_{\gamma} t) \frac{\partial f_{0}}{\partial v_{\gamma\perp}} \right]
\]

(2.27)

Another form of \( \varepsilon(k, \omega) \) that contains the zero-order Bessel function is obtained from (2.23) by performing the integral over the angle \( \xi \). The result is

\[
\varepsilon(k, \omega) = 1 - \frac{i}{k^{2}} \sum_{\gamma} 2\pi \omega_{\gamma}^{2} \int_{0}^{\infty} v_{\gamma\perp} dv_{\gamma\perp} \int_{-\infty}^{\infty} dv_{\gamma\parallel}
\]

\[
\times \int_{0}^{\infty} dt \left[ k_{\parallel} \frac{\partial f_{0}}{\partial v_{\gamma\parallel}} J_{0} \left( 2a_{\gamma} \sin \left( \frac{\Omega_{\gamma} t}{2} \right) \right) \right]
\]

\[
+ \frac{i}{v_{\gamma\perp}} \frac{\partial f_{0}}{\partial v_{\gamma\perp}} J_{0} \left( 2a_{\gamma} \sin \left( \frac{\Omega_{\gamma} t}{2} \right) \right) \exp i(\omega - k_{\parallel} v_{\gamma\parallel}) t
\]

(2.28)

When the distribution function \( f_{\gamma} \) is Maxwellian, integration of (2.24) over \( v_{\gamma\perp} \) can be accomplished with the help of the relation

\[
\int_{0}^{\infty} e^{-a^{2}x^{2}} J_{m}^{2}(bx) x \, dx = \frac{1}{2a^{2}} \exp(-b^{2}/2a^{2}) \Im^{2}(b^{2}/2a^{2})
\]

(2.29)

where \( \Im^{2} \) is the modified Bessel function of the first kind of imaginary argument. Integration over \( v_{\gamma\parallel} \) leads to the well known plasma dispersion function. After some manipulations we obtain
PLASMA KINETIC EQUATIONS

\[ e(k, \omega) = 1 + \sum_{\gamma} k_{\gamma}^2 \left[ 1 - \sum_{m} \frac{\omega}{\omega - m\Omega_{\gamma}} \Lambda_{m}(b_\gamma) \{Z(z_m^\gamma - i\sqrt{\pi}z_m^\gamma \exp[(z_m^\gamma)^2])\} \right] \]

(2.30)

where

\[ k_{\gamma}^2 = \frac{2\omega_{\gamma}^2}{v_{\theta\gamma}}, \quad \nu_{\theta\gamma} = \frac{2\kappa T_{\gamma}}{m_{\gamma}}, \quad b_{\gamma} = \frac{k_{\perp}^2 \nu_{\theta\gamma}}{2\Omega_{\gamma}^2}, \quad z_m^\gamma = \frac{\omega - m\Omega_{\gamma}}{k_{\perp} \nu_{\theta\gamma}} \]

(2.31)

and

\[ Z(x) = 2x e^{-x^2} \int_0^x e^{t^2} \, dt \]

(2.32)

3. KINETIC EQUATION FOR A UNIFORM MAGNETIZED PLASMA

According to the theory of probability the solution

\[ f_s = \prod_{i=1}^{n} f_1(X_i, t) \]

(3.1)

introduced in the previous section means that the s-particles are independent or uncorrelated. The next step is to introduce some correlation by considering solutions that deviate from the factored form (3.1). Our procedure is to use the Mayer-Cluster expansion in which we assume that the hierarchy distribution functions \( f_s \) can be written as

\[ f_1 = f_\alpha(X_\alpha, t) \]

\[ f_2 = f_\alpha(X_\alpha, t)f_\beta(X_\beta, t) + g^{(2)}_{\alpha\beta}(X_\alpha, X_\beta, t) \]

(3.2)

\[ f_3 = f_\alpha f_\beta f_\gamma + f_\alpha g^{(2)}_{\alpha\beta}(X_\beta, X_\gamma, t) + f_\beta g^{(2)}_{\alpha\beta}(X_\alpha, X_\gamma, t) + f_\gamma g^{(2)}_{\alpha\beta}(X_\alpha, X_\beta, t) + g^{(3)}_{\alpha\beta\gamma}(X_\alpha, X_\beta, X_\gamma, t) \]
and so on. Such an expansion can always, of course, be formally used, but it is only useful if the correlation functions $g^{(i)}$ are small compared to the factorable parts of the hierarchy distribution functions.

We are mainly concerned here with the Balescu-Lenard theory, which is based on the following assumptions:

(a) The weak coupling approximation, in which the two-particle correlation function $g^{(2)}$ is of lowest order in $1/\Lambda_0$ and $g^{(i)}$, $i > 2$, are of higher order in $1/\Lambda_0$. Furthermore, we retain only those terms that are of lowest order in $1/\Lambda_0$. (The phraseology should not be taken to mean that $g^{(2)}_{\alpha\beta}$ is expandable in $1/\Lambda_0$: the 'lowest-order' dependence is in fact $1/\Lambda_0$ — the point is discussed by DeWitt in Ref. [13].)

(b) The Bogolyubov adiabatic approximation, in which the one-particle distribution function is assumed stationary during a time required for $g_{\alpha\beta}$ to approach its asymptotic value.

(c) The plasma is stable and spatially homogeneous.

(d) The particles are initially uncorrelated ($g_{\alpha\beta}(t = 0) = 0$).

On substituting the expansion (3.2) for the distribution functions into the hierarchy equations (3.1) and using assumption (a), we find that all the hierarchy equations are satisfied except the first two. Restoring the dimensional form of these equations and introducing spatial homogeneity in the functions $f$ and $g$, the equations simplify to:

$$
\frac{\partial f_\alpha}{\partial t} (v_\alpha, t) + \frac{e_\alpha}{m_\alpha} v_\alpha \times B \cdot \frac{\partial f_\alpha}{\partial v_\alpha} (v_\alpha, t)
$$

$$
= \frac{1}{m_\alpha} \sum_\beta n_\beta \int \frac{\partial \varphi_{\alpha\beta}}{\partial x_\alpha} \cdot \frac{\partial g_{\alpha\beta}}{\partial v_\alpha} (x_\alpha - x_\beta, v_\alpha, v_\beta, t) \, d^3 x_\beta = \left( \frac{\partial f_\alpha}{\partial t} \right)_c
$$

(3.3)

$$
\left[ \frac{\partial}{\partial t} + v_\alpha \cdot \frac{\partial}{\partial x_\alpha} + v_\beta \cdot \frac{\partial}{\partial x_\beta} + \frac{e_\alpha}{m_\alpha} v_\alpha \times B \cdot \frac{\partial}{\partial v_\alpha} + \frac{e_\beta}{m_\beta} v_\beta \times B \cdot \frac{\partial}{\partial v_\beta} \right] g_{\alpha\beta}
$$

$$
- \frac{1}{m_\alpha} \frac{\partial f_\alpha}{\partial v_\alpha} \cdot \sum_\gamma n_\gamma \int \frac{\partial \varphi_{\alpha\gamma}}{\partial x_\alpha} g_{\alpha\gamma} \, d^3 x_\gamma - \frac{1}{m_\beta} \frac{\partial f_\beta}{\partial v_\beta} \cdot \sum_\gamma n_\gamma \int \frac{\partial \varphi_{\beta\gamma}}{\partial x_\beta} g_{\beta\gamma} \, d^3 x_\gamma
$$

$$
= \frac{\partial \varphi_{\alpha\beta}}{\partial x_\alpha} \cdot \left( \frac{1}{m_\alpha} \frac{\partial}{\partial v_\alpha} - \frac{1}{m_\beta} \frac{\partial}{\partial v_\beta} \right) f_\alpha f_\beta
$$

(3.4)
To obtain a kinetic equation we should solve (3.4) for $g_{\alpha\beta}$ in terms of $f_\alpha$ and substitute the result into the right-hand side of (3.3). To this end we first introduce a Fourier transform with respect to the variable $x = x_\alpha - x_\beta$, defined by

$$g_{\alpha\beta} = \int g_{\alpha\beta} e^{-ik \cdot x} \, d^3x$$

and

$$\tilde{g}_{\alpha\beta} = \int \tilde{g}_{\alpha\beta} e^{-ik \cdot x} \, d^3x = \frac{e_\alpha e_\beta}{\epsilon_0 k^2}$$

(3.5)

In terms of the transformed quantities, (3.3) and (3.4) become

$$\frac{\partial f_\alpha}{\partial t} = - \frac{\partial}{\partial v_\alpha} \cdot J$$

(3.6)

$$\frac{\partial \tilde{g}_{\alpha\beta}}{\partial t} + (\tilde{H}_\alpha + \tilde{H}_\beta) \tilde{g}_{\alpha\beta} = \tilde{S}_{\alpha\beta}$$

(3.7)

where

$$J = \frac{i}{(2\pi)^3} m_\alpha \sum_\beta n_\beta \int k \varphi_{\alpha\beta} \tilde{g}_{\alpha\beta}(k, v_\alpha, v_\beta, t) \, dk \, dv_\beta$$

(3.8)

and we have defined the operators

$$\tilde{H}_\alpha \tilde{g}_{\alpha\beta} = i (k \cdot v_\alpha) \tilde{g}_{\alpha\beta} - \frac{i}{m_\alpha} k \cdot \frac{\partial f_\alpha}{\partial v_\alpha} \sum_\gamma \varphi_{\alpha\gamma} \int \tilde{g}_{\alpha\gamma} \, dv_\gamma$$

$$+ \frac{e_\alpha}{m_\alpha} v_\alpha \times B \cdot \frac{\partial \tilde{g}_{\alpha\beta}}{\partial v_\alpha}$$

(3.9)

$$\tilde{H}_\beta \tilde{g}_{\alpha\beta} = -i (k \cdot v_\beta) \tilde{g}_{\alpha\beta} + \frac{i}{m_\beta} k \cdot \frac{\partial f_\beta}{\partial v_\beta} \sum_\gamma \varphi_{\beta\gamma} \int \tilde{g}_{\beta\gamma} \, dv_\gamma$$

$$+ \frac{e_\beta}{m_\beta} v_\beta \times B \cdot \frac{\partial \tilde{g}_{\alpha\beta}}{\partial v_\beta}$$

(3.10)

The operators $\tilde{H}_\alpha$ and $\tilde{H}_\beta$ are the same as those appearing in the previous section (see Eq.(2.7)) and since they operate on different orthogonal coordinates, they commute. The method we employ to solve (3.7) is originally due to Dupree [12]
and is well described by Montgomery and Tidman [14] for the case of no external magnetic field.

Equation (3.7) is a first-order inhomogeneous differential equation in \( t \). Its formal solution can therefore be written down and we obtain, using assumptions (b) and (d) and the fact that \( H_\alpha \) and \( H_\beta \) commute,

\[
g_{\alpha\beta}(k, v_\alpha, v_\beta, t) = \int_0^t R_\alpha(\tau) R_\beta(\tau) S_{\alpha\beta} \, d\tau
\]  

(3.11)

where

\[
R_\alpha(\tau) = e^{-H_\alpha \tau}, \quad R_\beta(\tau) = e^{-H_\beta \tau}
\]

(3.12)

and we have dropped the bar notation.

We now introduce the modified Laplace transform in time:

\[
\hat{R}_\alpha(\omega_1) = \int_0^\infty R_\alpha(\tau)e^{i\omega_1 \tau} \, d\tau, \quad \hat{R}_\beta(\omega_2) = \int_0^\infty R_\beta(\tau)e^{i\omega_2 \tau} \, d\tau
\]

(3.13)

with the inverse,

\[
R_\alpha(\tau) = \frac{1}{2\pi} \int_{c_1} e^{-i\omega_1 \tau} \hat{R}_\alpha(\omega_1) \, d\omega_1, \quad R_\beta(\tau) = \frac{1}{2\pi} \int_{c_2} e^{-i\omega_2 \tau} \hat{R}_\beta(\omega_2) \, d\omega_2
\]

(3.14)

The contours \( c_1 \) and \( c_2 \) are parallel to the real axis and above all singularities of \( \hat{R}_\alpha(\omega_1) \) and \( \hat{R}_\beta(\omega_2) \) respectively.

From (3.11)—(3.14) we can easily deduce

\[
g_{\alpha\beta}(k, v_\alpha, v_\beta, t) = \frac{1}{(2\pi)^2} \int_{c_1} \int_{c_2} \frac{\exp[-i(\omega_1 + \omega_2)\tau]}{(H_\alpha - i\omega_1)(H_\beta - i\omega_2)} S_{\alpha\beta} \, d\omega_2 \, d\omega_1 \, d\tau
\]

(3.15)

The \( \tau \) integration can easily be performed to give

\[
g_{\alpha\beta}(t) = \frac{1}{(2\pi)^2} \int_{c_1} \int_{c_2} \{\exp[-i(\omega_1 + \omega_2)\tau] - 1\} \frac{d\omega_1}{(H_\alpha - i\omega_1)} \frac{d\omega_2}{(H_\beta - i\omega_2)} \frac{S_{\alpha\beta}}{(\omega_1 + \omega_2)}
\]

(3.16)
The operators \( \frac{1}{H_\alpha - i\omega_1} \) and \( \frac{1}{H_\beta - i\omega_2} \) are given by Eq. (3.13):

\[
\frac{1}{H_\alpha - i\omega_1} = \int_0^\infty ds \exp [iY_\alpha(s, \omega_1, k, \nu)] \\
\times \left[ 1 + \frac{A_\alpha^0}{\epsilon(k, \omega_1)} \sum_\gamma e_\gamma n_\gamma \int d^3v_\gamma \int_0^\infty ds'' \exp [iY_\gamma(s'', \omega_1, k, \nu'')] \right]
\]

\[
\frac{1}{H_\beta - i\omega_2} = \int_0^\infty ds' \exp [iY_\beta(s', \omega_2, -k, \nu')]
\times \left[ 1 + \frac{A_\beta^0}{\epsilon(-k, \omega_2)} \sum_\gamma e_\gamma n_\gamma \int \exp [iY_\gamma(s'', \omega_2, -k, \nu'')] ds'' d^3v_\gamma \right]
\]

where

\[
Y_\alpha(s, \omega, k, \nu) = \left[ (k_\parallel v_\alpha - \omega) s + \frac{k_\perp v_\alpha}{\Omega_\alpha} (\sin(\nu + \Omega_\alpha s) - \sin \nu) \right]
\]

with similar expressions for \( Y_\beta \) and \( Y_\gamma \), and where

\[
A_\alpha^0 = \frac{ie_\alpha}{\epsilon_0 m_\alpha k^2} \frac{\partial f_\alpha}{\partial v_\alpha^0}, \quad A_\beta^0 = -\frac{ie_\beta}{\epsilon_0 m_\beta k^2} \frac{\partial f_\beta}{\partial v_\beta^0}
\]

is the longitudinal plasma dielectric function and is given by

\[
\epsilon(k, \omega) = 1 - \frac{i}{\epsilon_0 k^2} \sum_\gamma \frac{e_\gamma^2 n_\gamma}{m_\gamma} \int d^3v_\gamma \int_0^\infty ds'' \exp [iY(s'', \omega, k, \nu'')] [k \cdot \frac{\partial f_\gamma}{\partial v_\gamma^0}]
\]

(3.20)

The vectors \( v_\alpha, v_\beta, v_\gamma, k, v_\alpha^0, v_\beta^0, v_\gamma^0 \) are expressed in cylindrical polar coordinates with polar angles \( \phi, \psi, \xi, \chi, (\phi + \Omega_\alpha s'), (\psi + \Omega_\beta s') \) and \( (\xi + \Omega_\gamma s'') \) respectively, so that, for example, \( v_\alpha^0 = (v_{\alpha\parallel} \cos(\phi + \Omega_\alpha s'), v_{\alpha\perp} \sin(\phi + \Omega_\alpha s'), v_{\alpha\parallel}) \). Finally

\[
\nu = \phi - \chi, \quad \nu' = \psi - \chi, \quad \nu'' = \xi - \chi
\]

(3.21)

On substituting (3.17) into (3.16) we find, after some manipulations,
\[
\sum_{\beta} n_\beta e_\beta \int g_{\alpha}(t) \, d^3 v_\beta = \frac{i}{(2\pi)^2} \int_{c_1} d\omega_1 \int_{c_2} d\omega_2 \int_{0}^{\infty} ds \frac{\exp[-i(\omega_1 + \omega_2)s] - 1}{(\omega_1 + \omega_2)} 
\times \exp[iY_\alpha(s, \omega_1, k, \nu)] 
\times \left[ \left(1 - \frac{1}{\epsilon(-k, \omega_2)} \right) e_{\alpha f_{\alpha}} - A_{\alpha}^0 \frac{U(k, \omega_1) + U(-k, \omega_2)}{\epsilon(k, \omega_1)\epsilon(-k, \omega_2)} \right. 
+ \left. A_{\alpha}^0 \frac{U(k, \omega_1)}{\epsilon(k, \omega_1)} \right] d^3 v_\beta
\] (3.22)

where we have defined the function

\[
U = \sum_{\beta} n_\beta e_\beta^2 \int d^3 v_\beta \int_{0}^{\infty} ds' \exp[iY_\beta(s', \omega, k, \nu)] f_\beta
\] (3.23)

The analytic properties of the functions \( U \) and \( \epsilon \) have been extensively investigated by many authors in connection with the theory of plasma oscillations (e.g. Bernstein [15]). It is known that for distribution functions that are Hölder-continuous and vanish sufficiently fast when velocity tends to infinity, \( U \) and \( \epsilon \) will define functions that are analytic in the upper half-plane. Furthermore, if we assume with Landau that the distribution function is the restriction to the real axis of a function which is analytic in the complex velocity plane, we can analytically continue \( U \) and \( \epsilon \) into the lower half-plane. It follows that the only singularities of the integrand of Eq.(3.22) are at the zeros of the plasma dielectric function \( \epsilon(k, \omega) \). Moreover, for a stable plasma, there are no zeros of \( \epsilon \) in the upper half-plane.

Let us first consider the \( \omega_2 \) integration. The contour \( c_2 \) is defined in the upper half-plane. In the expression \( \left[ \exp[-i(\omega_1 + \omega_2)] - 1 \right] \) the contour for the \(-1\) term is closed by a large semicircle in the upper half-plane. No singularities are enclosed and the result is zero. For the exponential term we move the contour into the lower half-plane, using the analytic continuations of the functions \( U \) and \( \epsilon \). The contour is closed by a large semicircle enclosing all zeros of \( \epsilon(-k, \omega_2) \) and the pole \( \omega_2 = -\omega_1 \). In the asymptotic limit \( t \to \infty \) (assumption (b)), the contributions from the zeros of \( \epsilon \) are all damped by the exponential. The only undamped contribution comes from the pole \( \omega_2 = \omega_1 \). Thus, upon applying Cauchy’s theory, (3.23) reduces to
\sum_{\beta} n_\beta e_\beta \int g_{\alpha\beta}(k, v_\alpha, v_\beta, \infty) d^3 v_\beta = \frac{1}{2 \pi} \int_{\epsilon_1} d \omega_1 \int_{0}^{\infty} ds \exp \left[ i Y_{\beta}(s, \omega_1, k, \nu) \right]

\left( J_1 \right) \quad e_\alpha f_\alpha \left[ 1 - \frac{1}{\epsilon(-k, -\omega_1)} \right] + \frac{1}{\epsilon(k, \omega_1)} \left[ -2 A_{\alpha_0} \Re U(k, \omega_1) + A_{\alpha_0} U(k, \omega_1) \right]

(3.24)

We now turn to the \( \omega_1 \) integration. The last term (labelled \( J_3 \)) gives no contribution upon closing the contour by a large semicircle in the upper half-plane. The two terms \( J_1 \) add to a function which is integrable by completing the contour by a large semicircle in the lower half-plane. The term \( J_2 \) has poles everywhere in the complex plane, due to the presence of \( |\epsilon(k, \omega)|^2 \) in the denominator. We may, however, deform the contour by pulling it down towards the real axis and may employ the Plemelj formula to write the integral in terms of its principal value and its residues. After a lengthy but straightforward analysis, we arrive at

\[
\left( \frac{\partial f_\alpha}{\partial t} \right) = \frac{1}{e_0(2 \pi)^3 m_\alpha} \frac{\partial}{\partial v_\alpha} \cdot \int \frac{k}{k^2} d^3 k \int_{-\infty}^{\infty} d \omega \int_{-\infty}^{\infty} ds \frac{\exp \left[ i Y_\alpha(s, \omega, k, \nu) \right]}{|\epsilon(k, \omega)|^2}

\times \left[ \Theta(s) A_{\alpha_0} \Re U(k, \omega) + e_\alpha f_\alpha \epsilon(k, \omega) \right]

(3.25)

where

\[
\Theta(s) = [1 + \text{sgn}(s)]
\]

and the sign function is defined by

\[
\text{sgn}(s) = \begin{cases} 
1, & \text{when } s > 0 \\
0, & \text{when } s = 0 \\
-1, & \text{when } s < 0
\end{cases}
\]

(3.26)

The collision integral (3.25) inserted into (3.3) yields our basic magnetized kinetic equation. It generalizes the equation derived by Rostoker, who made the assumption that the one-particle distribution function was independent of the gyration phase \( \phi \). When this assumption \( f_\alpha = f_\alpha(v_{\alpha||}, v_{\alpha\perp}) \) is made, the integral over \( s \) in (3.25) can be easily evaluated upon using the expansions
\[ e^{i\alpha \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(a) e^{i\alpha} \quad , \quad \cos \theta e^{i\alpha \sin \theta} = \sum_{n=-\infty}^{\infty} \frac{n}{a} e^{i\alpha} J_n(a) \]  

(3.27)

and the value of the integral

\[ \int_{-\infty}^{\infty} \exp[i(\omega - k \parallel v_{\parallel} - n\Omega)] \Theta(s) \, ds = 4\pi \delta_+(\omega - k \parallel v_{\parallel} - n\Omega) \]  

(3.28)

where \( \delta_+ \) is the Heisenberg delta function

\[ \delta_+(x) = \frac{1}{2}\delta(x) - \frac{1}{2\pi i} \]

(3.25) becomes

\[ \left( \frac{\partial f_\alpha}{\partial t} \right)_k = -\frac{i}{\varepsilon_0(2\pi)^3 n_\alpha} \frac{\partial}{\partial n} \left\{ \frac{k}{k^2} \int_{-\infty}^{\infty} \frac{d\omega}{|\varepsilon(k, \omega)|^2} \sum_{n=-\infty}^{\infty} \exp\left[i\alpha \sin \nu - i\nu\right] J_n(a_\alpha) \right\} \times \left[ \frac{2ie_\alpha}{\varepsilon_0 n_\alpha k^2} O_\alpha^n f_\alpha \delta_+(\omega - d_\alpha^n) \Re U(k, \omega) + e_\alpha f_\alpha \varepsilon(k, \omega) \delta(\omega - d_\alpha^n) \right] \]  

(3.29)

where

\[ O_\alpha^n = \left( k_\parallel \frac{\partial}{\partial v_{\parallel}} + \frac{n k_\perp}{a_\alpha} \frac{\partial}{\partial v_{\perp}} \right), \quad d_\alpha^n = k_\parallel v_{\parallel} + n\Omega \]

and

\[ a_\alpha = \frac{k_\perp v_{\perp}}{\Omega} \]

We observe at this point that since the left-hand side of the kinetic equation is real, only the real part of the right-hand side of (3.29) contributes to the kinetic equation. The exponential term in (3.29) gives real quantities (Bessel functions) if the \( \chi \) integration is performed. It follows therefore, by virtue of (3.27), that only the real part of the Heisenberg delta function contributes and only the imaginary part of \( \varepsilon(k, \omega) \) is needed. The imaginary part of \( \varepsilon(k, \omega) \) and the real part of \( U \) can easily be calculated from (3.20) and (3.23). Hence (3.29) reduces, after carrying out the sum over \( n \), to:
\[
\left(\frac{\partial f_\alpha}{\partial t}\right)_c = -\frac{1}{2\epsilon_0^2(2\pi)^2} \sum_\beta e_\beta^2 n_\beta \frac{\partial}{\partial v_\alpha} \int d^3v_\beta \int_\infty^\infty \frac{k}{k^4} d^3k \int^\infty_{-\infty} d\omega \int^\infty_{-\infty} ds \int^\infty_{-\infty} dt
\]

\[
\times \exp \left[ iZ_\alpha(s, \omega, k, \nu) + iY_\beta(t, \omega, k, \nu') \right] \left( \frac{O_\beta}{m_\beta} - \frac{O_\alpha}{m_\alpha} \right) \frac{f_\alpha f_\beta}{|\epsilon(k, \omega)|^2} \tag{3.30}
\]

where

\[
O_\alpha = \left( k_\parallel \frac{\partial}{\partial v_\alpha} + k_\perp \cos (\nu + \Omega_\alpha t) \frac{\partial}{\partial v_\parallel} \right)
\tag{3.31}
\]

\[
O_\beta = \left( k_\parallel \frac{\partial}{\partial v_\beta} + k_\perp \cos (\nu' + \Omega_\beta t) \frac{\partial}{\partial v_\parallel} \right)
\tag{3.32}
\]

The collision integral (3.30) reduces to Rostoker's collision integral if we use the expansions (3.27), and carry out the \(\psi, \chi, t, s\) and \(\omega\) integrals. The upshot is

\[
\left(\frac{\partial f_\alpha}{\partial t}\right)_c = -\frac{1}{2\epsilon_0^2(2\pi)^2} \sum_\beta e_\beta^2 n_\beta \int d^3v_\beta \int_\infty^\infty \frac{d^3k}{k^4} \sum_{n,m=-\infty}^{\infty} \frac{O_\alpha^* J_n^2(a_\alpha) J_m^2(a_\beta)}{|\epsilon(k, \omega)|^2} \delta(d_\alpha^n - d_\beta^m)
\tag{3.33}
\]

where \(O_\alpha^n, d_\alpha^n, a_\alpha\) are given by Eq. (3.29) and

\[
O_\beta^m = \left( k_\parallel \frac{\partial}{\partial v_\beta} + m k_\perp \frac{\partial}{\partial v_\parallel} \right), \quad d_\beta^m = k_\parallel v_\beta + m \Omega_\beta
\tag{3.34}
\]

\[
a_\beta = \frac{k_\perp v_\beta}{\Omega_\beta}
\]

The results of Schram [4] and Silin [3] can be derived from the collision integrals (3.25) and (3.30) by setting \(|\epsilon(k, \omega)|^2 = 1\) and performing the \(\omega\) and \(s\) integrations.

We thus obtain:

\[
\left(\frac{\partial f_\alpha}{\partial t}\right) = -\frac{1}{\epsilon_0^2(2\pi)^2} \sum_\beta e_\beta^2 \frac{\partial}{\partial v_\alpha} \int d^3v_\beta \int_0^\infty \frac{k k}{k^4} d^3k \int^\infty_0 dt
\]

\[
\times \exp \left[ iZ_{\alpha\beta} + ik_\parallel \varphi(v_\alpha - v_\beta) \right] \left[ \frac{f_\alpha}{m_\alpha} \frac{\partial f_\beta}{\partial v_\parallel} - \frac{f_\beta}{m_\beta} \frac{\partial f_\alpha}{\partial v_\parallel} \right]
\tag{3.35}
\]
and
\[
\left( \frac{\partial f_\alpha}{\partial t} \right)_c = -\frac{e_\alpha^2}{2e_0^2(2\pi)^3 m_\alpha} \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \sum_\beta e_\beta^2 n_\beta \int \frac{d^3 \mathbf{v}_\beta}{k^4} d^3 k \int_{-\infty}^{\infty} \times \exp \left[ iZ_{\alpha\beta} + ik_\parallel t(v_{\alpha\parallel} - v_{\beta\parallel}) \right] \left( \frac{Q_\beta}{m_\beta} - \frac{Q_\alpha}{m_\alpha} \right) f_\alpha f_\beta \, dt
\]
\[\text{(3.36)}\]

where
\[
Z_{\alpha\beta} = a_\alpha [\sin \nu - \sin (\nu - \Omega_\alpha t)] + a_\beta [\sin \nu' - \sin (\nu' + \Omega_\beta t)]
\]

and
\[
\dot{\alpha} = \left( k_\parallel \frac{\partial}{\partial v_{\alpha\parallel}} + k_\perp \cos (\nu - \Omega_\alpha t) \frac{\partial}{\partial v_{\alpha\perp}} \right)
\]
\[\text{(3.37)}\]

Equation (3.35), restricted to a one-component plasma, is identical to Schram's (Eqs (4.3)–(4.5)) provided that the asymptotic limit \( t \to \infty \) is taken in his equation. Silin's expression for the relaxation time can be obtained by multiplying (3.36) by \( \frac{1}{2} m_v^2 \) and integrating it with respect to \( v_\alpha \), taking \( f_\alpha \) and \( f_\beta \) to be Maxwellian.

4. UNMAGNETIZED KINETIC EQUATIONS

4.1. The Balescu-Lenard equation

In the limit of zero magnetic field, \( \Omega \to 0 \), the collision integral (3.25) reduces to
\[
\left( \frac{\partial f_\alpha}{\partial t} \right)_c = -\sum_\beta R_{\alpha\beta} \frac{\partial}{\partial v_i} \int Q_{ij} \left\{ \frac{1}{m_\alpha} \frac{\partial f_\beta}{\partial v_i} f_\alpha - \frac{1}{m_\alpha} \frac{\partial f_\alpha}{\partial v_i} f_\beta \right\} d^3 v'
\]
\[\text{(4.1)}\]

where
\[
Q_{ij} = \int \frac{k_\mu k_\nu \delta(k \cdot v - k' \cdot v')}{{k^4}^2 \epsilon(k, k \cdot v')} \, d^3 k
\]
\[\text{(4.2)}\]

\[
\epsilon(k, \omega) = 1 + \sum_\beta \frac{\omega_\beta^2}{k^2} \int \frac{k \cdot (\partial f_\beta / \partial v')}{(\omega - k \cdot v')} \, d^3 v'
\]
\[\text{(4.3)}\]
The expression (4.1) is the well-known Balescu-Lenard collision integral (see for example, Ref.[14]). The plasma dielectric function appearing in (4.3) plays a fundamental role in the Balescu-Lenard collision integral. It properly includes collective wave effects in the theory and, in particular, it removes the divergence at small values of \( k \) that occurs in Landau's equation (see Section 4.2). At large values of \( k \), however, the dielectric function approaches unity and the integral over \( k \) in (4.3) diverges logarithmically. The source of this divergence is that in deriving the kinetic equation in the previous section the interaction term was regarded as a small perturbation (of order \( 1/A_0 \)). While this assumption may be valid over most of the phase space, it does not hold in certain regions where interparticle separations are small. This small region in phase space corresponds to large values of \( k \) in the Fourier transform. A cut-off at the inverse distance of closest approach is usually invoked to render the integral convergent.

4.2. The Landau equation

If we approximate \( \varepsilon(k, \omega) \) in (4.2) by its static value \( \varepsilon(k, 0) = (1 + k^2/k_D^2) \), the integrals over \( d^3k \) can be evaluated to give

\[
Q_{ij} = \pi \log A_0 \left[ \frac{w^2 \delta_{ij} - w_i w_j}{w^3} \right]
\]

where

\[
w = |v - v'|
\]

The tensor (4.5) when substituted into (4.1) yields the famous Landau equation with two cut-offs \( k_0 \) and \( k_D \).

4.3. Fokker-Planck form of Balescu-Lenard equation (the RMJ equation)

If we cast the tensor \( Q_{ij} \) in the form:

\[
Q_{ij} = \frac{1}{4\pi^3} \int \int \int \int \frac{k_i k_j}{k^4 \varepsilon(k, \omega)^2} \exp\{is(\omega - k \cdot v) + it(\omega - k \cdot v')\} ds dt d\omega d^3k
\]
(which reduces to (4.2) when the trivial integrations over s, t and \( \omega \) are carried out), it is then readily observed that Eq.(4.1) can be written in the standard Fokker-Planck form:

\[
\left( \frac{\partial f_{v}}{\partial t} \right) = \frac{\partial}{\partial v_{i}} \left( f_{v} \frac{\partial h}{\partial v_{i}} \right) + \frac{1}{2} \frac{\partial}{\partial v_{i}} \left( \frac{\partial f_{v}}{\partial v_{j}} \frac{\partial^{2} g}{\partial v_{j} \partial v_{i}} \right)
\]

(4.6)

where the coefficients of friction and diffusion are written in terms of two scalar potentials:

\[
A_{i} = \frac{\partial h}{\partial v_{i}} = \frac{1}{4 \pi^{2}} \sum_{B} \frac{R_{\alpha B}}{m_{\beta}} \int \int \int \int \int \int \exp[i(\omega - k \cdot v) + it(\omega - k \cdot v')] \frac{1}{|\epsilon(k, \omega)|^{2}}
\]

\[
\times \frac{\partial f_{B}}{\partial v_{i}} ds \, dt \, d\omega \, d^{3}k \, d^{3}v'
\]

(4.7)

\[
D_{ij} = \frac{\partial^{2} g}{\partial v_{i} \partial v_{j}} = \frac{1}{2 \pi^{2}} \sum_{B} \frac{R_{\alpha B}}{m_{\alpha}} \int \int \int \int \int \int \int \int \int \int \exp[i(\omega - k \cdot v) + it(\omega - k \cdot v')] \frac{1}{|\epsilon(k, \omega)|^{2}}
\]

\[
\times f_{B} ds \, dt \, d\omega \, d^{3}k \, d^{3}v'
\]

(4.8)

The 'generalized Rosenbluth potentials' \( h \) and \( g \) can easily be derived from (4.7) and (4.8) upon integrating over the velocity \( v \).

\[
h = \frac{i}{4 \pi^{2}} \sum_{B} \frac{R_{\alpha B}}{m_{\beta}} \int \int \int \int \int \int \int \int \int \int \exp[i(\omega - k \cdot v) + it(\omega - k \cdot v')] \frac{1}{s}
\]

\[
\times \frac{k \cdot (\partial f_{B}/\partial v')}{{k'}^{2}|\epsilon(k, \omega)|^{2}} \, ds \, dt \, d\omega \, d^{3}k \, d^{3}v'
\]

(4.9)

\[
g = \frac{1}{2 \pi^{2}} \sum_{B} \frac{R_{\alpha B}}{m_{\alpha}} \int \int \int \int \int \int \int \int \int \int \left\{ \frac{1 - \exp[i(\omega - k \cdot v) + it(\omega - k \cdot v')]}}{s^{2}} \right\}
\]

\[
\times \frac{\exp[it(\omega - k \cdot v')]f_{B}}{k^{4}|\epsilon(k, \omega)|^{2}} ds \, dt \, d\omega \, d^{3}k \, d^{3}v'
\]

(4.10)

Here, the constant of integration in the expression for \( g \) has been chosen such as to render the \( s \)-integral convergent for small values of \( s \).
We may easily satisfy ourselves that if we neglect 'wave effects', i.e. if we replace $e(k, \omega)$ by its static value $e(k, 0)$, the potentials $g$ and $h$ reduce to the familiar Rosenbluth potentials:

$$h_c = 2\pi \ln \Lambda \sum_p \frac{R_{p\beta}}{m_p} \int \frac{f_p(v')}{|v - v'|} d^3v' = \sum_p h_{c\beta}^p$$  \hspace{1cm} (4.11)

$$g_c = 2\pi \ln \Lambda \sum_p \frac{R_{p\beta}}{m_p} \int |v - v'|f_p(v') d^3v' = \sum_p g_{c\beta}^p$$  \hspace{1cm} (4.12)

If we use the result:

$$\left(\frac{1}{e(k, \omega)^2} - \frac{1}{e(k, 0)^2}\right) = \frac{|e(k, 0)|^2 - |e(k, \omega)|^2}{|e(k, 0)|^2 |e(k, \omega)|^2}$$  \hspace{1cm} (4.13)

We can write the potentials $h$ and $g$ in the form:

$$h = h_c + h_w = h_c(1 + H)$$  \hspace{1cm} (4.14)

$$g = g_c + g_w = g_c(1 + G)$$  \hspace{1cm} (4.15)

where

$$H = \frac{h_w}{h_c}, \quad G = \frac{g_w}{g_c}$$  \hspace{1cm} (4.16)

The parts $h_c$ and $g_c$ (given by Eqs (4.11) and (4.12)) are the contributions to the potentials due to near Coulomb collisions. The parts $h_w$ and $g_w$ describe the interaction of particles with plasma waves. They are given by

$$h_w = \frac{i}{4\pi^2} \sum_p \frac{R_{p\beta}}{m_p} \int \int \int \int \int \exp\left(\text{i}s(\omega - k \cdot v) + \text{i}t(\omega - k \cdot v')\right)$$

$$\times \frac{W(k, \omega)}{k^4} \frac{k \cdot \frac{\partial f_p}{\partial v'}}{d^3v'} ds \, dt \, d\omega \, d^3k \, d^3v' = \sum_p h_{w\beta}^p$$  \hspace{1cm} (4.17)

$$g_w = \frac{1}{2\pi^2} \sum_p \frac{R_{p\beta}}{m_p} \int \int \int \int \int \left\{\frac{1 - \exp\left[\text{i}s(\omega - k \cdot v)\right]}{s^2} \right\}$$

$$\times \exp\left(\frac{W(k, \omega)}{k^4}\right) f_p ds \, dt \, d\omega \, d^2k \, d^3v' = \sum_p g_{w\beta}^p$$  \hspace{1cm} (4.18)
where

$$W(k, \omega) = \frac{\varepsilon_0^2 - |\varepsilon(k, \omega)|^2}{\varepsilon_0^2 |\varepsilon(k, \omega)|^2}, \quad \varepsilon_0^2 = \left( 1 + \frac{k_B^2}{k^2} \right)$$

(4.19)

4.4. The case of Maxwellian field particles

This section is devoted to the calculations of the correction terms $H$ and $G$ assuming the field particles distribution function $f_\beta$ to be Maxwellian, i.e.

$$f_\beta = \frac{1}{\pi^{3/2}} \frac{1}{v_{\beta}^3} e^{-v^2/v_{\beta}^2}$$

(4.20)

and restricting attention to a two-component plasma. We confine ourselves in particular to the cases for which our calculations give $H$ and $G$ greater than or equal to unity. The substitution of (4.20) into (4.11) and (4.12) leads to the results:

$$h_\beta^B = 2\pi \frac{R_{\alpha\beta} \ln \Lambda}{m_\beta} \frac{\Phi \left( \frac{v}{v_{\beta}} \right)}{v}$$

(4.21)

$$g_\beta^B = 2\pi \frac{R_{\alpha\beta} \ln \Lambda}{m_\alpha} \left\{ \frac{v_{\beta}^2}{2v} \Phi \left( \frac{v}{v_{\beta}} \right) + \frac{v_{\beta}^2}{\sqrt{\pi}} e^{-v^2/v_{\beta}^2} \right\}$$

(4.22)

where $\Phi$ is the error function. Noting that

$$\int f_\beta e^{-ik \cdot v} v^3 dv' = e^{-i/4k^2v^2_{\beta}}$$

and performing the $t$ integration and the two angle integrations in $\int d^3k$, (4.17) and (4.18) reduce to

$$h_\omega^B = -\frac{2i}{\sqrt{\pi}} \frac{R_{\alpha\beta}}{m_\beta v_{\beta}^3} \frac{1}{v} \int \frac{dk}{k^2} \int \frac{ds}{s^2} \int \frac{d\omega}{\omega} W(k, \omega) e^{i\omega - \omega^2/2k^2} \right] d\omega$$

(4.23)
where, for a two-component ion-electron plasma, the dielectric function $\epsilon(k, \omega)$ takes the form:

$$\epsilon(k, \omega) = \epsilon_r + i \epsilon_i$$  \hspace{1cm} (4.25)

$$\epsilon_r = 1 + \frac{k_p^2}{k^2} \{1 - Z(x)\} + \frac{k_p^2}{k^2} t\{1 - Z(\mu x)\}$$  \hspace{1cm} (4.26)

$$\epsilon_i = \sqrt{\frac{\pi}{2}} \frac{k_p^2}{k} x \{e^{-x^2} + \mu t e^{-\mu^2 x^2}\}$$  \hspace{1cm} (4.27)

$Z(x)$ is the plasma dispersion function

$$Z(x) = 2 x e^{-x^2} \int_0^x e^{t^2} \, dt$$  \hspace{1cm} (4.28)

which has the asymptotic values

$$Z(x) \approx 2 x^2 \hspace{1cm} x \ll 1$$  \hspace{1cm} (4.29)

$$Z(x) \approx 1 + \frac{1}{2 x^2} \hspace{1cm} x \gg 1$$  \hspace{1cm} (4.30)

$$x = \frac{\omega}{k v_{te}}, \hspace{0.5cm} t = \frac{T_e}{T_i}, \hspace{0.5cm} k_e = \frac{\omega_e}{v_{te}} \sqrt{2} \hspace{1cm} \text{and} \hspace{1cm} \mu^2 = \frac{T_e}{T_i} \frac{m_i}{m_e}$$  \hspace{1cm} (4.30)

Turning now to the $\omega$ integral in (4.23) and (4.24), we first observe that the integrand becomes exponentially small if $\omega/k v_{te} \beta \ll 1$. Thus for the ion case ($\beta = i$), most of the contribution to the integral comes from $0 < x < 1/\mu$. In this region, however, noting that for most plasmas of interest $\mu \gg 1$, and using (4.29), we deduce, to first order of approximation, that $\epsilon(k, \omega) \approx \epsilon_0$. As a result $W(k, x) \approx 0$ and our calculations will give both $H$ and $G$ less than unity. On the other hand, if $\beta = e$, the dominant contribution to the integral comes from $0 < x < 1$. In this region the interval $0 < x < 1/\mu$, as in the ion case, gives a small contribution, whereas for $1/\mu < x < 1$ the asymptotic values (4.29) give

$$\epsilon_r = 1 + \frac{k_e^2}{k^2} - \frac{k_e^2}{k^2} \frac{t}{2 x^2}$$  \hspace{1cm} (4.31)
and

\[ \epsilon_i \ll \epsilon_r \]  

(4.32)

The condition (4.32) enables us to use the approximate result

\[ \frac{1}{|\epsilon|^3} \approx \pi \frac{\delta(\epsilon_r)}{|\epsilon|} \]  

(4.33)

in view of which

\[ W \approx \pi \left\{ \frac{\epsilon_i^2 - |\epsilon|^2}{\epsilon_0^2} \right\} \frac{\delta(\epsilon_r)}{|\epsilon|} = \pi \frac{\delta(\epsilon_r)}{|\epsilon|} \]  

(4.34)

By virtue of (4.31) and (4.34) the \( \omega \) integral in (4.23) and (4.24) can now easily be performed to obtain

\[ h^e = \frac{2R_{ac}}{m_0v_0} \int_0^{k_0} \frac{k^2 dk}{(k^2 + k_0^2)[1 + \mu t e^{-\mu t^{2i} + 1}]} \int_0^\infty \frac{\sin(skuv) \sin(skvb)}{s^2} ds \]  

(4.35)

\[ g^e = \frac{8}{\sqrt{2}} \frac{R_{ac}}{m_0k_e} \int_0^{k_0} \frac{k dk}{(k^2 + k_e^2)[1 + \mu t e^{-\mu t^{2i} + 1}]} \times \int_0^\infty \frac{skv - \sin(skv) \cos(skvb)}{s^2} ds \]  

(4.36)

where

\[ b = k_e t^{1/2} / 2 \sqrt{(k^2 + k_e^2)^{1/2}} \]

and, as usual, the upper limit in the \( k \) integral has been replaced by \( k_0 \), the inverse distance of closest approach, to avoid the divergence for large values of \( k \). Making the change of variables

\[ y = skv, \quad x = \frac{k_e}{(k^2 + k_e^2)^{1/2}} \]  

(4.37)
and performing the s integrals, (4.35) and (4.36) reduce to

\[ h_x^s = \pi \frac{R_{as}}{m_e v_k} \left[ t \alpha \int_{1/\Lambda_0}^{1/\alpha} \frac{(1 - x^2) \, dx}{\{1 + \mu t \, e^{-\beta x^2}\}} \right] \text{ if } \alpha \leq 1 \]

\[ h_x^s = \pi \frac{R_{as}}{m_e v_k} \left[ t \alpha \int_{1/\Lambda_0}^{1/\alpha} \frac{(1 - x^2) \, dx}{\{1 + \mu t \, e^{-\beta x^2}\}} + \int_{1/\alpha}^{1} \frac{(1 - x^2) \, dx}{x\{1 + \mu t \, e^{-\beta x^2}\}} \right] \]

\[ \text{if } \Lambda_0 > \alpha > 1 \]

\[ g_x^s = -\frac{4\pi}{\sqrt{2}} \frac{R_{as}}{m_a} t^{1/2} \alpha \int_{1/\Lambda_0}^{1/\alpha} \frac{(1 - x^2) \, dx}{x\{1 + \mu t \, e^{-\beta x^2}\}} \text{ if } \alpha \leq 1 \]

\[ g_x^s = -\frac{4\pi}{\sqrt{2}} \frac{R_{as}}{m_a} t^{1/2} \left[ \alpha \int_{1/\Lambda_0}^{1/\alpha} \frac{(1 - x^2) \, dx}{x\{1 + \mu t \, e^{-\beta x^2}\}} + \frac{\alpha^2}{2} \int_{1/\alpha}^{1} \frac{(1 - x^2) \, dx}{\{1 + \mu t \, e^{-\beta x^2}\}} \right] \]

\[ \text{if } \Lambda_0 > \alpha > 1 \]

where

\[ \alpha = \frac{v_k}{v} \left( \frac{t}{2} \right)^{1/2}, \quad \beta = \frac{1}{2} \left( \mu^2 + 1 \right) = \frac{1}{2} \mu_2 \]

Approximate values of the above integrals can readily be derived if we observe that the dominant contribution to the integrals comes from \( x \leq \lfloor (\ln \mu t/\beta) \rfloor^{1/2} \). Substituting these values into (4.16) and making use of (4.21) and (4.22), we finally obtain, after some algebra, the asymptotic results shown in Table I.

**TABLE I. ASYMPTOTIC RESULTS**

<table>
<thead>
<tr>
<th>( v_0 )</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( \Lambda_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_0 \gg 1 )</td>
<td>( x_0 &lt; v_0 &lt; 1 )</td>
<td>( x_0 &lt; y_0 &lt; x_0 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>( t \ln x_0 )</th>
<th>( t x_0 )</th>
<th>( t \ln (x_0/y_0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H )</td>
<td>( \frac{t x_0}{4 \ln \Lambda} )</td>
<td>( \frac{\sqrt{\pi}}{8} \frac{t x_0}{v_0 \ln \Lambda} )</td>
<td>( \frac{\sqrt{\pi}}{8} \frac{t \ln (x_0/y_0)}{v_0 \ln \Lambda} )</td>
</tr>
<tr>
<td>( G )</td>
<td>( \frac{t \ln (y_0 \Lambda_0)}{v_0 \ln \Lambda} )</td>
<td>( \frac{\sqrt{\pi}}{8} \frac{t \ln (y_0 \Lambda_0)}{v_0 \ln \Lambda} )</td>
<td>( \frac{\sqrt{\pi}}{8} \frac{t x_0}{v_0 \ln \Lambda} )</td>
</tr>
</tbody>
</table>
Thus the correction terms are small for ion field particles. In the case of electron field particles, the correction terms are significant if the plasma is strongly non-isothermal, such that the condition $m_i/m_e > T_e/T_i \gg 1$ is satisfied — the dominant correction is then given by the terms for which $v_0 < x_0$ (see Table I).

It should be noted that such a non-isothermal plasma can support weakly damped oscillations (ion sound waves) with phase velocities much less than the electron thermal velocity but much greater than the ion thermal velocity, i.e.

$$v_0 \ll \omega/k \ll v_e, \quad \text{i.e.} \; 1/\mu < x < 1 \quad (x = \omega/kv_e)$$

(4.41)

Such waves can easily interact with the electrons, whereas the interaction of these waves with ions is relatively weak. The condition (4.41) is exactly the frequency domain that gives the dominant contribution to the integrals (4.23) and (4.24). Thus we conclude that the dominant contribution of wave effects to the generalized Rosenbluth potentials is due to the interaction of electrons with ion sound waves.

5. DERIVATION OF A SIMPLE USABLE MAGNETIZED KINETIC EQUATION (generalized magnetic Rosenbluth potentials)

We start with the magnetic Balescu-Lenard equation derived in Section 4:

$$\frac{\partial f_a}{\partial t} = \sum_{\beta} R_{a\beta} \frac{\partial}{\partial v} \int \int \int \frac{e^{i\alpha_{a\beta} + i\beta_{a\beta}}}{|\epsilon(k, \omega)|^2}$$

$$\times \left\{ \frac{O_a}{m_a} - \frac{O_{\beta}}{m_{\beta}} \right\} f_a(v) f_{\beta}(v') \; ds \; dt \; d\omega \; d^3k \; d^3v'$$

(5.1)

where $\epsilon(k, \omega)$ is the longitudinal plasma dielectric function

$$\epsilon(k, \omega) = 1 - \frac{i}{k^2} \sum_{\beta} \omega_{\beta}^2 \int_0^\infty dt \; e^{i\omega_{\beta}} O_{\beta f}(v') \; d^3v'$$

(5.2)
PLASMA KINETIC EQUATIONS

\[
O_\alpha = \left\{k_t \frac{\partial}{\partial v_i} + k_\perp \cos(\nu + \Omega_\alpha s) \frac{\partial}{\partial v_\perp}\right\}
\]

\[
O_\beta = \left\{k_t \frac{\partial}{\partial v_i} + k_\perp \cos(\nu' + \Omega_\beta t) \frac{\partial}{\partial v_\perp}\right\}
\]

\[
Y_\alpha = \frac{k_\perp v_\perp}{\Omega_\alpha} \left\{\sin \nu - \sin(\nu + \Omega_\alpha s)\right\} + (\omega - k_\perp v_\perp) s
\]

\[
Y_\beta = \frac{k_\perp v_\perp'}{\Omega_\beta} \left\{\sin \nu' - \sin(\nu' + \Omega_\beta t)\right\} + (\omega - k_\perp v_\perp') t
\]  

(5.3)

Cylindrical polar coordinates \((v_\perp, \phi, v_\parallel), (v_\perp', \psi, v_\parallel')\) and \((k_\parallel, \chi, k_\parallel)\) are used, and

\[
R_{\alpha\beta} = \frac{e_\parallel^2 e_\perp^2}{2e_\parallel^2(2\pi)^4} \frac{n_\beta}{m_\alpha}
\]

We shall consider a two-component plasma and restrict our attention to the ion equation \((\alpha = i)\). We shall also assume that ions are unmagnetized, in which case we may replace \(Y_\alpha\) and \(O_\alpha\) by their unmagnetized counterparts \(Y_\alpha = (\omega - k \cdot \mathbf{v}) s\), \(O_\alpha = k \cdot \partial / \partial \mathbf{v}\). It is then readily observed that (5.1) can be written in the standard Fokker-Planck form:

\[
\frac{\partial f_\alpha}{\partial t} = \frac{\partial}{\partial v_i} \left(f_\alpha \frac{\partial h}{\partial v_i} + \frac{1}{2} \frac{\partial}{\partial v_i} \left(\frac{\partial f_\alpha}{\partial v_\perp} \frac{\partial^2 g}{\partial v_\perp \partial v_\perp}\right)\right)
\]  

(5.4)

where the coefficients of friction and diffusion are written in terms of two scalar potentials:

\[
A = \frac{\partial h}{\partial v} = \sum_{\beta - i, e} \frac{R_{\beta \alpha}}{m_\beta} \int \frac{k}{k^3} d^3k
\]

\[
\times \int \int \int \exp\left[i s(\omega - k \cdot v) + i \omega t + i Y_\beta\right] O_{\beta\beta} ds dt d\omega d^3v'
\]  

(5.5)

\[
D = \frac{\partial^2 g}{\partial v \partial v}\bigg|_{\beta} = 2 \sum_{\beta} \frac{R_{\beta \alpha}}{m_\beta} \int \int \frac{kk'}{k^3} d^3k
\]

\[
\times \int \int \int \exp\left[i s(\omega - k \cdot v) + i \omega t + i Y_\beta\right] f_\beta ds dt d\omega d^3\omega'
\]  

(5.6)
The generalized magnetic Rosenbluth potentials can easily be derived from (5.5) and (5.6) by integrating over the velocity $\mathbf{v}$:

$$h = i \sum_{\beta=1,e} \frac{R_{\beta}}{m_{\beta}} \int \frac{d^3k}{k^4}$$

$$\times \int \int_{-\infty}^{\infty} \int \frac{\exp[i(s(\omega - k \cdot \mathbf{v}) + i\omega t + iY_{\beta})]}{s} \frac{O_{\beta f_{\beta}}}{|\epsilon(k, \omega)|^2} \, ds \, dt \, d\omega \, d^3v' \quad (5.7)$$

$$g = 2 \sum_{\beta=1,e} \frac{R_{\beta}}{m_i} \int \frac{d^3k}{k^4}$$

$$\times \int \int_{-\infty}^{\infty} \int \left\{1 - \frac{\exp[i(s(\omega - k \cdot \mathbf{v}))]}{s^2}\right\} \frac{\exp[it\omega + iY_{\beta}]}{|\epsilon(k, \omega)|^2} f_{\beta} \, ds \, dt \, d\omega \, d^3v' \quad (5.8)$$

The constant of integration in (5.8) has been selected in order to avoid a divergence for small values of $s$ (e.g. Ref.[16]). It is easily seen that in the limit $\Omega_{\beta} \rightarrow 0$ the potentials $h$ and $g$ reduce to the unmagnetized generalized Rosenbluth potentials derived in Section 4. Following Section 4, we use the equality

$$\frac{1}{|\epsilon(k, \omega)|^2} = \frac{1}{\epsilon_0^2} + \frac{\epsilon_0^2 - |\epsilon|^2}{\epsilon_0^2|\epsilon|^2} \quad \text{where} \quad \epsilon_0^2 = \left(1 + \frac{k_D^2}{k^2}\right) \quad (5.9)$$

to write (5.7) and (5.8) in the form:

$$h = h_c + h_w, \quad g = g_c + g_w \quad (5.10)$$

where

$$h_c = -i \sum_{\beta=1,e} \frac{R_{\beta}}{m_{\beta}} \int \int \frac{d^3k}{(k^2 + k_D^2)} \int \exp[itk \cdot \mathbf{v} + iY_{\beta}]O_{\beta f_{\beta}} \, dt \, d^3v' = \sum_{\beta=1,e} h_{c\beta} \quad (5.11)$$

$$g_c = 2 \sum_{\beta=1,e} \frac{R_{\beta}}{m_i} \int \int \frac{d^3k}{(k^2 + k_D^2)} \int \left\{1 - \frac{\exp[itk \cdot \mathbf{v}}}{i^2}\right\} \exp(iY_{\beta})f_{\beta} \, dt \, d^3v' = \sum_{\beta=1,e} g_{c\beta} \quad (5.12)$$
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\[ h_w = i \sum_{\beta = i, e} \frac{R_{\beta}}{m_{\beta}} \iint \frac{d^3k}{k^4} \times \iint_{-\infty}^{\infty} \exp[i(s(\omega - k \cdot v) + i \omega t + i Y_{\beta})W(k, \omega)O_{\beta}f_{\beta} \, ds \, dt \, d\omega \, d^3v'] \]

\[ = \sum_{\beta = i, e} h_{w}^{\beta} \quad (5.13) \]

\[ g_w = 2 \sum_{\beta = i, e} \frac{R_{\beta}}{m_{\beta}} \iint \frac{d^3k}{k^4} \times \iint_{-\infty}^{\infty} \left\{ \frac{1 - \exp[i(s(\omega - k \cdot v))]}{s^2} \right\} W(k, \omega) \exp[i\omega t + i Y_{\beta}]f_{\beta} \, ds \, dt \, d\omega \, d^3v' \]

\[ = \sum_{\beta = i, e} g_{w}^{\beta} \quad (5.14) \]

where

\[ W(k, \omega) = \frac{\epsilon_0^2 - |\epsilon|^2}{\epsilon_0^2|\epsilon|^2} \quad (5.15) \]

5.1. Calculation of \( h_c \) and \( g_c \)

The integration over \( dt \) and \( d^3k \) in Eqs (5.11) and (5.12) can be carried out to give a closed form of the expressions for \( h_c \) and \( g_c \). We first observe that, since the ions are unmagnetized, the ion terms \( h_c^i \) and \( g_c^i \) simply reduce to the ion components of the standard Rosenbluth potentials:

\[ h_c^i = (2\pi)^3 \frac{R_{\parallel}}{m_i} \ln \Lambda \int \frac{f_i(v')}{|v - v'|} \, d^3v' \quad (5.16) \]

\[ g_c^i = (2\pi)^3 \frac{R_{\parallel}}{m_i} \ln \Lambda \int |v - v'|f_i(v')d^3v' \quad (5.17) \]

As regards \( h_c^e \) and \( g_c^e \), we write

\[ i k \cdot v + i Y_e = i k_{\parallel} U_\parallel + i(A \cos v + B \sin v) \]
where

\[ U_\parallel = v_\parallel - v'_\parallel \]

\[ A = k_\perp \left\{ v_\perp - 2 \frac{v'_\perp}{\Omega_e} \sin \frac{\Omega_e t}{2} \cos \left( \mu + \frac{\Omega_e t}{2} \right) \right\} \]

\[ B = \frac{2k_\perp}{\Omega_e} v_\perp \sin \left( \frac{\Omega_e t}{2} \right) \sin \left( \mu + \frac{\Omega_e t}{2} \right) \]

and make use of the integrals

\[ \int_0^{2\pi} \exp \left[ iA \cos \nu + iB \sin \nu \right] d\nu = 2\pi J_0(\sqrt{A^2 + B^2}) \]

\[ \int_0^{2\pi} \exp \left[ ia \cos \theta \right] J_0(b \sin \theta) \sin \theta d\theta = \frac{2}{\sqrt{a^2 + b^2}} \sin(\sqrt{a^2 + b^2}) \]

to evaluate the two angle integrals in \( f_3^d \). After some algebra, we obtain

\[ h_\parallel = 4\pi R_e \frac{R_{in}}{e} \int \frac{k^4 dk}{(k^2 + k_d^2)} \int \frac{dt}{s^2} \left( \frac{\sin s}{s} - \cos s \right) \left\{ U_\parallel \frac{\partial f_\parallel}{\partial v_\parallel} + r \frac{\partial f_\parallel}{\partial v'_\perp} \right\} d^3 v' \]

\[ h'_\parallel = 4\pi^2 R_{in} \frac{R_{in}}{e} \int \left\{ \frac{1}{U} \left[ \ln \Lambda - E_\parallel(\alpha) + E_\parallel(\beta) \right] + \frac{v^2}{2p} \left[ E_\parallel(\alpha) - E_\parallel(\beta) \right] \right\} f_\parallel d^3 v' \] (5.21)

where

\[ s = kt \left\{ \frac{v^2}{2} + \frac{v'_\parallel^2 \sin^2(\Omega_e t/2)}{(\Omega_e t/2)^2} - \frac{2v'_\parallel v_\parallel \sin(\Omega_e t/2) \cos(\mu + \Omega_e t/2) + U_\parallel^2}{(\Omega_e t/2)^2} \right\} \]

\[ r = \left\{ v_\parallel \cos(\mu + \Omega_e t) - \frac{2v'_\parallel}{\Omega_e} \sin(\Omega_e t/2) \cos(\Omega_e t/2) \right\} \]

and \( \mu = \nu - \nu' \).

Exact analytic values of the remaining integrals are not possible. Approximate values can, however, be obtained if we make use of the approximations:
and confine ourselves to logarithmic accuracy. Using the modified potential 
\[ \phi_{\alpha\beta} = \frac{k_0^2}{(k^2 + k_0^2)} \phi_{\alpha\beta} \] (\phi_{\alpha\beta} is the Coulomb potential and \( k_0 \) is the distance of closest approach) instead of the cut-off at \( k = k_0 \) and splitting the region of \( t \) integration into \( 0 < t < \frac{1}{\Omega} \) and \( t > \frac{1}{\Omega} \), the \( k \) and \( t \) integrals can be evaluated in terms of the integral:

\[ \int_0^1 \frac{e^{-\alpha x} - e^{-\beta x}}{x} \, dx = \left\{ \ln \left( \frac{\beta}{\alpha} \right) - E_i(\alpha) + E_i(\beta) \right\} \]

\( E_i(x) \) being the exponential integral,

\[ E_i(x) = \int_0^x \left( \frac{e^{-t}}{t} \right) \, dt \]

After some lengthy analysis we arrive at

\[ h^e = \frac{4\pi^2 R_{se}}{m_e} \int \left\{ \frac{1}{U} \left[ \ln \Lambda - E_i(\alpha) + E_i(\beta) \right] + \frac{v_i^2}{2p} [E_i(\tilde{\alpha}) - E_i(\tilde{\beta})] \right\} f_e \, d^3 v' \]  
(5.24)

\[ g^e = \frac{4\pi^2 R_{se}}{m_i} \int \left\{ U \left[ \ln \Lambda - E_i(\alpha) + E_i(\beta) \right] + p [E_i(\tilde{\alpha}) - E_i(\tilde{\beta})] \right\} f_e \, d^3 v' \]  
(5.25)

where

\[ \alpha = \frac{k_D U}{\Omega_e}, \quad \beta = \frac{k_0 U}{\Omega_e}, \quad \tilde{\alpha} = \frac{k_D p}{\Omega_e}, \quad \tilde{\beta} = \frac{k_0 p}{\Omega_e} \]

\[ U = |v - v'|, \quad p = (v_i^2 + U_i^2)^{1/2} \quad \text{and} \quad \Lambda = k_0/k_D \]

(5.26)

It is readily confirmed that under conditions of weak magnetic field such that the strength parameter \( \eta \ll 1 \), these potentials reduce to the familiar unmagnetized Rosenbluth potentials (see Section 4).

In the case of strong magnetic field, such that \( \eta > 1 \), the potentials \( h^e \) and \( g^e \) contain, in addition to the terms proportional to \( \ln \Lambda_0 \), terms proportional to \( \ln \eta \) which for large values of \( \eta \) may give significant contributions. To illustrate this we evaluate the potentials explicitly, assuming the electron distribution function
f(v') to be Maxwellian. For values of \( \frac{v_i}{v_e} \ll 1 \), we find the following expressions for the potentials:

\[
\tilde{h} = \frac{m_i}{m_e} \Gamma^{ie} \left[ \frac{\Phi(v/v_e)}{v} \ln \left( \frac{\Lambda_0}{\sqrt{1 + \eta^2}} \right) + \frac{e^{-r^2 + s^2}}{\pi^{1/2} v_e} s^2 \ln \left( 1 + \frac{\eta^2}{2} \right)^{1/2} \right. 
\times \left. \{ K_1 - K_0 + 2 r^2 [K_0 + 2 s^2 (K_0 - K_1)] \} \right]
\]

\[
\tilde{g} = \Gamma^{ie} \left[ \left( \frac{v_e^2}{2v} \right) \Phi \left( \frac{v}{v_e} \right) + \frac{v_e}{\pi^{1/2}} \exp \left( -\frac{v^2}{v_e^2} \right) \ln \left( \frac{\Lambda_0}{\sqrt{1 + \eta^2}} \right) \right. 
\left. + \frac{e^{-r^2 + s^2}}{\pi^{1/2} v_e} s^2 \ln \left( 1 + \frac{\eta^2}{2} \right)^{1/2} (K_0 + K_1 + 2 r^2 K_1) \right]
\]

where \( K_0 \) and \( K_1 \) are modified Bessel functions of the second kind with arguments \( s^2 \), \( \Phi \) is the error function,

\[
r = \frac{v_i}{v_e} \quad \text{and} \quad s = \frac{v_i}{v_e \sqrt{2}}
\]

In the limits \( \frac{v_i}{v_e} \ll 1 \) and \( \frac{v_i}{v_e} \ll 1 \) the potentials (5.27) reduce, on retaining only the leading terms, to

\[
\tilde{h} = \frac{m_i}{m_e} \Gamma^{ie} \left[ \left( 1 - \frac{2v^2}{3v_e^2} \right) \ln \left( \frac{\Lambda_0}{\sqrt{1 + \eta^2}} \right) + \left( 1 + \frac{v_i^2}{v_e^2} \ln \frac{v_i}{v_e} \right) \ln \sqrt{1 + \eta^2} \right]
\]

and

\[
\tilde{g} = \frac{v_e}{\pi^{1/2}} \Gamma^{ie} \left[ \left( 1 + \frac{v_i^2}{3v_e^2} \right) \ln \left( \frac{\Lambda_0}{\sqrt{1 + \eta^2}} \right) + \left( 1 - \frac{v_i^2}{v_e^2} \ln \frac{v_i}{v_e} \right) \ln \sqrt{1 + \eta^2} \right]
\]
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MICROINSTABILITIES IN HOMOGENEOUS WARM MAGNETIZED PLASMAS

S. CUPERMAN
Department of Physics and Astronomy,
Tel Aviv University,
Ramat Aviv,
Israel

Abstract

MICROINSTABILITIES IN HOMOGENEOUS WARM MAGNETIZED PLASMAS.

CONTENTS: I. Introduction. II. General linearized dispersion equation: basic equation the dielectric tensor; calculation of the elements of the dielectric tensor; particular case-propagation along the magnetic field. III. Electrostatic instabilities: (A) Parallel propagation: the dispersion equation; stable cases; unstable cases: (a) Landau damping, (b) beam-plasma instability. (B) Oblique propagation: the dispersion relation; ion loss-cone instability. IV. Electromagnetic instabilities: electromagnetic ion-cyclotron instability; electromagnetic electron-cyclotron instability.

I. INTRODUCTION

The production, heating and confinement of a plasma results necessarily in a physical system out of thermodynamical equilibrium. In high-temperature plasmas the relaxation to equilibrium proceeds by collective means, rather than by binary collisions as in neutral systems or low-temperature plasmas. This is due to the electromagnetic properties of the plasma particles, which are able to produce accumulations of electrical charges and currents — sources of electromagnetic fields. Thus, free energy stored in certain degrees of freedom of the non-equilibrium plasma is transferred to electromagnetic fluctuations inherent in a plasma system. The amplitudes of certain waves grow to a significant level. At the same time, the waves act upon the plasma particles (eventually on others than those which supplied their energy), heat them and in this way transfer energy to deficient degrees of freedom. This process of collective relaxation via (enhanced) plasma waves is called plasma instability.
The plasma instabilities may be classified according to:

i) the nature of the free energy (reservoir) driving the system unstable; ii) the type of waves enhanced (electrostatic, electromagnetic or mixed e.s. and e.m.); iii) the direction of propagation of the unstable waves with respect to an applied magnetic field (parallel, transverse or oblique); iv) the range of frequencies of the excited waves (high or low frequencies); etc.

From the point of view of the nature of the free energy producing the instability, one may distinguish the following types of instabilities:

a. Streaming instabilities: They are produced by particle beams traversing a plasma or by currents driven through a plasma and producing relative drifts between different species in the plasma.

b. Rayleigh-Taylor instabilities: These occur whenever the plasma is non-uniform (has density gradients or sharp boundaries) and an external nonelectromagnetic force is applied to it. These instabilities are analogous to those occurring in hydrodynamics when a heavier fluid is supported by a lighter one.

c. Universal instabilities: They are due to the expansion energy (pressure) of confined plasma that is not in thermodynamical equilibrium.

d. Microscopic (kinetic) instabilities: Unlike types (a)-(c) of instabilities, which depend on the bulk properties of the plasma particles (and are describable by Maxwellians, for example), the microscopic instabilities depend on the details of particle distribution functions; here only a small group of resonant particles may be involved. Thus, plasmas which hydrodynamically (i.e. involving the whole plasma) are stable, may be microscopically (kinetically) unstable. Examples of such distributions are the bump on tail, loss-cone and bi-Maxwellian velocity distribution functions.

In these lectures we will discuss the microinstabilities which may occur in uniform warm and magnetized plasmas. We will limit ourselves
to the linear stage of evolution of such instabilities; that is, we will develop a linear theory. More specifically, in Sect. II we will develop a general formalism based on the dielectric tensor. For illustration, in Sects. III and IV we will treat in detail a number of selected specific instabilities. The complete plan of the lectures is given in the Contents.

Concerning references: for Sect. II the following selected works are recommended: Landau (1946), Bernstein (1958), Jackson (1960), Harris (1961), Stix (1962), Montgomery and Tidman (1964), Schmidt (1966), and Krall and Trivelpiece (1973). For Sects. III and IV, appropriate references will be indicated during the exposition.

II. GENERAL LINEARIZED DISPERSION EQUATION

1. Basic equations

The time and space behaviour of a warm, magnetized collisionless plasma is governed by the Vlasov-Maxwell system of equations. In gaussian cgs units these equations read, respectively,

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{r}} + q_\alpha \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_\alpha}{\partial p} = 0 \tag{II.1}$$

and

$$\nabla \times \mathbf{E} = -\frac{1}{c} (\partial \mathbf{B} / \partial t) \tag{II.2}$$

$$\nabla \times \mathbf{B} = 4\pi \mathbf{j} / c + \frac{1}{c} (\partial \mathbf{E} / \partial t) \tag{II.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{II.4}$$

$$\nabla \cdot \mathbf{E} = 4\pi \rho \tag{II.5}$$

In these equations, $f_\alpha(\mathbf{r}, \mathbf{p}, t)$ is the particle distribution function of species $\alpha$ normalized so that the probable number density $n(\mathbf{r}, t)$ is
\[ n = n_0 \int f \, d\vec{p}, \]  
where \( n_0 \) is the average number density; \( \vec{p} \) and \( \vec{v} \) are the relativistic momentum and the velocity, respectively, and are related as follows

\[ \vec{p} = \gamma m \vec{v} ; \quad \gamma = (1 - v^2/c^2)^{-1/2} ; \quad \vec{v} = cp/(m^2c^2 + p^2)^{1/2} \]  

The Vlasov and Maxwell equations are connected through the charge density, \( \rho \), and the current density, \( \vec{j} \), which are given by the following expressions:

\[ \rho = \sum_{\alpha} q_{\alpha} n_{0\alpha} \int f_{\alpha} \, d\vec{p} \]  

\[ \vec{j} = \sum_{\alpha} q_{\alpha} n_{0\alpha} \int \vec{v} f_{\alpha} \, d\vec{p} \]  

The other symbols used are standard.

The equations \( \text{II.1} - \text{II.8} \) represent a selfconsistent set of equations in the sense that the effective fields contain the contribution (reaction) of the plasma.

2. The dielectric tensor

We assume the initial (equilibrium) plasma to be of uniform density and immersed in a uniform magnetic field, \( \vec{B}_0 \), directed along the \( z \)-axis. Moreover, we assume that the distribution function \( f \), the electric field \( \vec{E} \) and the magnetic field \( \vec{B} \) are perturbed slightly about equilibrium values and that a Fourier-Laplace transform exists, i.e. the perturbed quantities all behave like

\[ \vec{E} = \vec{E}_{k,\omega} \exp(i\vec{k} \cdot \vec{r} - i\omega t) \]  

and therefore

\[ \partial/\partial t = -i\omega ; \quad \partial/\partial \vec{r} = i\vec{k} ; \quad \nabla x = i\vec{k}x \]  

\[ \text{(II.9')} \]
Under the stated circumstances, a dispersion relation governing the
behaviour of the collective fields, $\vec{E}$, $\vec{B}$ and therefore of the particle
distribution function $f$, can be derived.

A sketch of the derivation follows. First, it is convenient to
obtain from the Maxwell equations a general but implicit (formal)
dispersion relation for electromagnetic waves. Thus, using 11.9' in eqs 11.2 and 11.3 and eliminating $\vec{B}$ between the resulting equations, one obtains

$$\vec{k} \times (\vec{k} \times \vec{E}) + \frac{\omega^2}{c^2} \vec{K} \cdot \vec{E} = 0$$  \hspace{1cm} (II.10)

where we used the definition of the dielectric tensor, $\vec{K}$, namely

$$\vec{K} \cdot \vec{E} \equiv \vec{E} + \frac{4\pi i}{\omega} \vec{j}$$  \hspace{1cm} (II.11)

In terms of the refractive index

$$\vec{n} \equiv \frac{c}{\omega} \vec{k}$$  \hspace{1cm} (II.12)

eqq. II.10 provides the familiar wave equation

$$\vec{n} \times (\vec{n} \times \vec{E}) + \vec{K} \cdot \vec{E} = 0$$  \hspace{1cm} (II.13)

or symbolically

$$\vec{K} \cdot \vec{E} = 0$$  \hspace{1cm} (II.13')

Taking $\vec{k} = \vec{k} (k_x = k_{\perp}, 0, k_z = k_{\parallel})$, without loss of generality, for non-trivial solutions ($\vec{E} \neq 0$) eq. II.13' requires
This is the formal dispersion relation mentioned above. The relations between the elements of the tensor $R$ and those of the dielectric tensor $K$ are obvious.

3. Calculation of the elements of the dielectric tensor

The task now is to derive analytical expressions for the elements of the dielectric tensor, $K_{ij}$, in eq. 11.14. For this purpose we use eqs. 11.1 - 11.8. Also, it will appear convenient to use the wave equation 11.13 in a slightly different form. Thus, in terms of the conductivity tensor $\sigma$ related to the dielectric tensor by the relation

$$K = \gamma + \frac{4\pi i}{\omega} \sigma$$  

i.e.

$$\vec{K} \cdot \vec{E} = \vec{\omega} + \frac{4\pi i}{\omega} \sigma \cdot \vec{E}$$  

eq. 11.13 reads

$$c^2 k^2 \vec{E} - c^2 K (\vec{k} \cdot \vec{E}) - \omega^2 \left[ \gamma + \frac{4\pi i}{\omega} \sigma \right] \cdot \vec{E} = 0$$  

In obtaining this result we used the relation 11.12 and also expanded the double vector product appearing in 11.13.
We now summarize the steps leading to the derivation of the quantities $K_{ij}$:

i) **Initial equilibrium equations:** Assuming the plasma to be of uniform density (no electric fields present) and immersed in a uniform magnetic field, $\vec{B}_0$, one finds that the equilibrium distribution function, $f_0$, satisfies the equations

$$\frac{q}{c}(\vec{v} \times \vec{B}_0) \cdot \frac{\partial f_0}{\partial \vec{p}} = \frac{1}{m\gamma} \frac{q}{c} (\vec{p} \times \vec{B}_0) \cdot \frac{\partial f_0}{\partial \vec{p}} = 0 \quad (\text{II.1}')$$

and

$$\vec{j}_0 = 0 \quad (\text{II.3}')$$

Eq. II.1' indicates the existence of an azimuthal symmetry about the magnetic field, $\vec{B}_0$ (see Fig. 1). Indeed, carrying over the multiplication in II.1' and expressing the momentum $\vec{p}$ in cylindrical coordinates ($p_x = p_\perp \cos \varphi$, $p_y = p_\perp \sin \varphi$, $p_z = p_\parallel$) one finds

![FIG. 1. Geometry and notations used.](image)

1 Another, yet equivalent, method for the derivation of the elements $K_{ij}$ exists; it consists of the integration of the Vlasov equation along the particle trajectory (e.g., Stix (1962)).
\[
\frac{q}{c} (\mathbf{\nabla} \times \mathbf{B}_0) \cdot \frac{\partial \mathbf{f}}{\partial \mathbf{p}} = - \Omega \frac{\partial \mathbf{f}}{\partial \varphi} \tag{II.17}
\]

which equated to zero gives \( \frac{\partial \mathbf{f}_0}{\partial \varphi} = 0 \) or \( \mathbf{f}_0 = \mathbf{f}_0(p_i, p_t) \).

In II.17 \( \Omega = qB_0/mc^2 = \Omega_0/\gamma \) is the gyrofrequency.

ii) Linearization of the equations: For small perturbations,

\[
\mathbf{f} = \mathbf{f}_0(p_i, p_t) + \mathbf{f}^{(1)}(\mathbf{r}, \mathbf{p}, t) ; \quad \mathbf{E} = \mathbf{E}^{(1)} \tag{II.18}
\]

\[
\mathbf{B} = \mathbf{B}_0 + \mathbf{B}^{(1)} \quad \rho = \rho^{(1)} ; \quad \mathbf{j} = \mathbf{j}^{(1)}
\]

from II.1 - II.8 one obtains the following linearized equations:

\[
\frac{\partial \mathbf{f}^{(1)}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{f}^{(1)}}{\partial \mathbf{r}} + q \left( \mathbf{E}^{(1)} + \frac{\mathbf{v} \times \mathbf{B}^{(1)}}{c} \right) \frac{\partial \mathbf{f}_0}{\partial \mathbf{p}} + q \frac{\mathbf{v} \times \mathbf{B}_0}{c} \frac{\partial \mathbf{f}^{(1)}}{\partial \mathbf{p}} = 0 \tag{II.19}
\]

\[
\mathbf{v} \times \mathbf{E}^{(1)} = - \frac{1}{c} \frac{\partial \mathbf{B}^{(1)}}{\partial t} \tag{II.20}
\]

\[
\mathbf{v} \times \mathbf{B}^{(1)} = \frac{4\pi}{c} \sum_{\alpha} q_{\alpha} n_{\alpha} \int \mathbf{v} \mathbf{f}^{(1)}(\mathbf{r}, \mathbf{p}, t) d\mathbf{p} + \frac{1}{c} \frac{\partial \mathbf{E}^{(1)}}{\partial t} \tag{II.21}
\]

iii) Fourier-Laplace transforms of the linear equations:

The Fourier transform \( \mathbf{Q}^{(1)}(\mathbf{k}, \mathbf{p}, t) = \mathcal{F}(\mathbf{f}^{(1)}), \mathbf{B}^{(1)}, \mathbf{E}^{(1)}, \text{etc.} \) is defined as follows

\[
\mathbf{Q}^{(1)}(\mathbf{k}, \mathbf{p}, t) \equiv \int_{-\infty}^{+\infty} \frac{d\mathbf{r}}{(2\pi)^3} \mathbf{Q}^{(1)}(\mathbf{r}, \mathbf{p}, t) e^{-i\mathbf{k} \cdot \mathbf{r}'} = \mathcal{F}\{\mathbf{Q}^{(1)}(\mathbf{r}, \mathbf{p}, t)\} \tag{II.22}
\]

and leads to the results

\[
\mathcal{F} \left[ \frac{\partial \mathbf{Q}^{(1)}}{\partial \mathbf{r}} \right] = i\mathbf{k} \mathbf{Q}^{(1)} ; \quad \mathcal{F} \left[ \mathbf{v} \times \mathbf{Q}^{(1)} \right] = i\mathbf{k} \times \mathbf{Q}^{(1)} \tag{II.23}
\]
The inversion formula is

\[ Q^{(1)}(\vec{r}, \vec{p}, t) = \int_{-\infty}^{\infty} Q^{(1)}(k, \vec{p}, t) e^{ik \cdot \vec{r}} dk \]  

(II.22')

The Laplace transform \( \bar{Q}^{(1)} \) is defined (for \( \text{Re} s > 0 \)) as follows:

\[ \bar{Q}^{(1)}(k, \vec{p}, s) \equiv \int_{0}^{\infty} Q^{(1)}(k, \vec{p}, t) e^{-st} dt \equiv \mathcal{L}\{\bar{Q}^{(1)}(k, \vec{p}, t)\} \]  

(II.24)

and leads to the result:

\[ \mathcal{L} \left[ \frac{\partial \bar{Q}^{(1)}}{\partial t} \right] = - \bar{Q}^{(1)}(k, \vec{p}, 0) + s \bar{Q}^{(1)}(k, \vec{p}, s) \]  

(II.25)

The inversion formula is

\[ \bar{Q}^{(1)}(k, \vec{p}, t) = \frac{1}{2\pi i} \int_{\sigma - \infty}^{\sigma + \infty} \bar{Q}^{(1)}(k, \vec{p}, s) e^{st} ds \]  

(II.24')

where the integral converges to the right of some line \( \text{Re} s = \sigma = \text{const} \) (see Fig. 2).

FIG. 2. (a) Original path of integration; (b) deformed path of integration.
Application of the operators $\mathcal{F}$ and $\mathcal{L}$ to eqs 11.20 and 11.21 provides, respectively:

$$i\vec{k} \times \vec{E}^{(l)} = \frac{1}{c} \vec{B}^{(l)}(\vec{k}, 0) - \frac{s}{c} \vec{B}^{(l)}(\vec{k}, s)$$  \hspace{1em} (II.20')

and

$$i\vec{k} \times \vec{B}^{(l)} = \frac{4\pi}{c} \vec{j}^{(l)} - \frac{1}{c} \vec{E}^{(l)}(\vec{k}, 0) + \frac{s}{c} \vec{E}^{(l)}(\vec{k}, s)$$  \hspace{1em} (II.21')

Substitution of $\vec{E}^{(l)}$ from 11.20' into 11.21' gives

$$(c/s)[\vec{k} \times (\vec{k} \times \vec{E}^{(l)})] + (i/s)(\vec{k} \times \vec{B}^{(l)}(\vec{k}, 0)) = (4\pi/c)\vec{j}^{(l)} - (1/c)\vec{E}^{(l)}(\vec{k}, 0) + (s/c)\vec{E}^{(l)}$$  \hspace{1em} (II.26)

For simplicity in the following we use the notations:

$$\vec{E}^{(l)}(\vec{k}, 0) \equiv \vec{e} \ ; \ \vec{B}^{(l)}(\vec{k}, 0) \equiv \vec{b}$$  \hspace{1em} (II.27)

$$\vec{E}^{(l)}(\vec{k}, s) \equiv \vec{E} \ ; \ \vec{B}^{(l)}(\vec{k}, s) \equiv \vec{B} \ ; \ \vec{j}^{(l)}(\vec{k}, s) \equiv \vec{j}$$

After multiplication by sc and using the notations II.27, eq 26 becomes

$$(\vec{k} \times (\vec{k} \times \vec{E}) = \vec{k} (\vec{k} \cdot \vec{E}) - \vec{k}^2 \vec{E})$$

$$[s^2 + c^2 k^2] \vec{E} - c^2 k (\vec{k} \cdot \vec{E}) + 4\pi s \vec{j} = s \vec{e} + ic (\vec{k} \times \vec{b})$$  \hspace{1em} (II.26')

or, equivalently,

$$[s^2 + c^2 k^2] \vec{E} - c^2 k (\vec{k} \cdot \vec{E}) + 4\pi s \vec{e} \cdot \vec{E} = s \vec{e} + ic (\vec{k} \times \vec{b})$$  \hspace{1em} (II.26'')
Next, applying the operators $\mathcal{F}$ and $\mathcal{L}$ to eq. 11.19 and using

11.22 - 11.25, one obtains

$$\begin{align*}
\mathcal{F}^{(1)}(\mathbf{k}, \mathbf{p}, s)(s + i\mathbf{k} \cdot \mathbf{v}) + \frac{q}{c} (\mathbf{v} \times \mathbf{B}_0) \frac{\partial \mathcal{F}^{(1)}(\mathbf{k}, \mathbf{p}, s)}{\partial \mathbf{p}} \\
+ q \left[ \mathcal{F}^{(1)}(\mathbf{k}, s) + \frac{1}{c} \left( \mathbf{v} \times \mathcal{F}^{(1)}(\mathbf{k}, s) \right) \right] = \mathcal{F}^{(1)}(\mathbf{k}, \mathbf{p}, 0)
\end{align*}
$$

(II.28)

Substituting for $\mathcal{F}^{(1)}$ in eq. 11.28 its value from 11.20' and denoting

$$g \equiv \mathcal{F}^{(1)}(\mathbf{k}, \mathbf{p}, 0) \quad \text{and} \quad f \equiv \mathcal{F}^{(1)}(\mathbf{k}, \mathbf{p}, s)$$

(II.29)

one obtains the following expression for the Fourier-Laplace transform of the Vlasov equation:

$$(s + i\mathbf{k} \cdot \mathbf{v}) f + (q/c)(\mathbf{v} \times \mathbf{B}_0)(\partial f / \partial \mathbf{p}) + q \left[ \mathcal{E} - (i/s)(\mathbf{v} \times (\mathbf{k} \times \mathcal{E})) \right] (\partial f_0 / \partial \mathbf{p})$$

$$= g - (q/cs)(\mathbf{v} \times \mathbf{B}_0)(\partial f_0 / \partial \mathbf{p}).$$

(II.28')

Thus, the dispersion relation 11.26' requires the knowledge of the perturbed current $\mathbf{j} \equiv \mathcal{F}^{(1)}(\mathbf{k}, \mathbf{p}, s)$. On the other hand, the equation 11.28', which will provide a solution for $f \equiv \mathcal{F}^{(1)}$ (and consequently for the perturbed current $\mathbf{j}$), contains the perturbed field $\mathcal{E}$, which obeys eq. 11.26'. One is faced with a system of two equations for the unknowns $f$ and $\mathcal{E}$.

iv) A formal integration of the (transformed) Vlasov equation:

When the relation 11.17 (which is general) is used, eq. 11.28' can be rewritten as follows:

$$\frac{\partial f}{\partial \phi} - \frac{s + i\mathbf{k} \cdot \mathbf{v}}{\Omega} - \frac{\Phi}{\Omega} = 0$$

(II.30)
with
\[ \Phi \equiv q \left[ \vec{E} - \frac{i}{s} \vec{v} \times (\vec{k} \times \vec{E}) \right] \frac{\partial f_0}{\partial p} - g + \frac{q}{sc} (\vec{v} \times \vec{b}) \frac{\partial f_0}{\partial p} \]  

Eq. 11.30 is an ordinary differential equation of the type
\[ \frac{dy}{dx} + P(x)y + Q(x) = 0 \]  

having the solution (C is an integration constant)
\[ y(x) = \exp \left[ - \int_0^x P(x') dx' \right] \cdot \left[ C - \int_0^x Q(x') \exp \left[ \int_0^{x'} P(x'') dx'' \right] dx' \right] \]  

Thus, with the identification
\[ f = y ; \quad \varphi = x ; \quad -(s + i\vec{k} \cdot \vec{v})/\Omega \equiv P(x) ; \quad -\Phi/\Omega \equiv Q(x) \]  

the solution of eq. 11.30 is, by inspection,
\[ f = \exp \left[ - \int_0^\varphi \left[ -(s + i\vec{k} \cdot \vec{v}')/\Omega \right] d\varphi' \right] \cdot \left[ C - \int_0^\varphi \left[ -\Phi(\varphi')/\Omega \right] \exp \left[ \int_0^{\varphi'} \left[ -(s + i\vec{k} \cdot \vec{v}'')/\Omega \right] d\varphi'' \right] d\varphi' \right] \]  

Since in our case \( \vec{k} = k(\vec{k}_x = k_x \perp 0, k_z = k_y) \), using \( \vec{v} = \vec{v}(v_x = v_{\perp} \cos \varphi, v_y = v_{\perp} \sin \varphi, v_z = v_\parallel) \), one obtains \( \vec{k} \cdot \vec{v} = k_{\parallel} v_{\parallel} \cos \varphi + k_{\parallel} v_{\parallel} \). Consequently
\[ \int_0^\varphi (s + i\vec{k}_\perp \cdot \vec{v}') d\varphi' = (s + ik_\parallel v_\parallel) \int_0^\varphi d\varphi' + ik_\parallel v_\parallel \int_0^\varphi \cos \varphi' d\varphi' \]  

\[ = (s + ik_\parallel v_\parallel) \varphi + ik_\parallel v_{\parallel} \sin \varphi \]
and eq. 11.34 reads
\[
f = \exp \left\{ \left[ (s + ik_1 v_1) \varphi + ik_2 v_2 \sin \varphi \right] / \Omega \right\} \left[ C + \Omega^{-1} \int_0^\varphi \Phi(\varphi') \cdot \exp \{-\Omega^{-1} \cdot \left[ (s + ik_1 v_1) \varphi' + ik_2 v_2 \sin \varphi' \right] \} d\varphi' \right\}
\]  
(II.35)

The integration constant C can be determined by requiring f to be periodic in \( \varphi \) with a period of 2\( \pi \) (i.e. to be a single-valued function). Thus, requiring
\[
f(\varphi) = f(\varphi + 2\pi)
\]  
(II.36)

one obtains
\[
f(\varphi) = \left[ \Omega \left[ \exp \left\{ -\frac{2\pi}{\Omega} (s + ik_1 v_1) \right\} - 1 \right] \right]^{-1} \int_0^\varphi \Phi(\varphi') G(\varphi') d\varphi'
\]  
(II.37)

where
\[
G(\varphi') = \exp \left\{ \Omega^{-1} \left[ (s + ik_1 v_1)(\varphi - \varphi') - ik_2 v_2 (\sin \varphi' - \sin \varphi) \right] \right\}
\]  
(II.38)

and \( \Phi(\varphi) \) is given by 11.31.

An equivalent solution of the eq. 11.30 has been proposed by Bernstein (1958), namely
\[
f(\varphi) = (1/\Omega) \int_{-\infty}^\varphi \Phi(\varphi') G(\varphi') d\varphi'
\]  
(II.37′)

where the + (-) sign corresponds to positively (negatively) charged particles. The author indicated that the solution 11.37′ can be verified by substitution. The periodicity of the solution is apparent if one introduces in the integrand the new variable \( \alpha = \varphi - \varphi' \). The limits of integration are then independent of \( \varphi \).
v) **Combined Solution of the Wave Equation and Vlasov Equation**

Using the solution (11.37') in the expression for \( j \) (given by eq. 11.8) and substituting the result into the wave equation (11.26') lead to the following result

\[
(s^2 + c^2 k^2) \mathbf{E} - c^2 k (\mathbf{k} \cdot \mathbf{E}) + 4\pi s \sigma \cdot \mathbf{E} = I \tag{11.39'}
\]

where the vector \( \mathbf{I} \) collects the Fourier transforms of the initial values \( e, b \) and \( g \). Finally, (11.39) can be rewritten as (\( \sigma \) is the conductivity tensor defined in 11.15 and 11.26'')

\[
(s^2 + c^2 k^2) \mathbf{E} - c^2 k (\mathbf{k} \cdot \mathbf{E}) + 4\pi s \sigma \cdot \mathbf{E} = I \tag{11.39''}
\]

and symbolically

\[
\mathbf{R} \cdot \mathbf{E} = I \tag{11.39'''}
\]

In (11.39') we used the relation

\[
\sigma \cdot \mathbf{E} \equiv \sum_{\alpha} \frac{q_{0 \alpha}^2 \eta_{0 \alpha}}{m_{\alpha} \Omega_{0 \alpha}} \int_{-\infty}^{+\infty} dp \cdot \mathbf{p} \int_{-\infty}^{+\infty} dp' \int_{-\infty}^{+\infty} dp' G(\varphi') \left[ \mathbf{E} - \frac{i}{s} \mathbf{v} \times (\mathbf{k} \times \mathbf{E}) \right] \frac{\partial f_{0 \alpha}}{\partial \mathbf{p}'},
\]
vi) The elements of the dielectric tensor

First, we bring the quantity appearing in the square brackets in eq. II.40 into the following form:

\[ W = \left( \vec{E} - \frac{i}{s} \vec{\nabla} \times (\vec{k} \times \vec{E}) \right) \frac{\partial f_0}{\partial p} = AE_x + BE_y + CE_z \]

\[ = a \cos \varphi E_x + a \sin \varphi E_y + dE_z \quad (I I .41) \]

where

\[ a = \frac{k}{\omega} \left( \left( \frac{\omega}{k} - n_0 \right) \frac{\partial f_0}{\partial p_\perp} + n_1 \frac{\partial f_0}{\partial p_\parallel} \right) \]

\[ d = b + c \cos \varphi = \frac{\partial f_0}{\partial p_\parallel} + \cos \varphi \left( \frac{k}{\omega} \left( n_1 \frac{\partial f_0}{\partial p_\perp} - n_1 \frac{\partial f_0}{\partial p_\parallel} \right) \right) \quad (I I .42) \]

To obtain these results we have only to perform the multiplications indicated in the expression defining \( W \); thus, for products involving quantities other than \( \vec{E} \) we used cylindrical coordinates; products involving \( \vec{E} \) were calculated in cartesian coordinates and in the result the quantities other than \( \vec{E} \) were transformed to cylindrical coordinates.

Next, we perform the angular integrations in II.40. (Remember that \( d\varphi = p_\perp dp_\parallel dp_\varphi d\varphi \) and \( p = \rho \left( \rho_\varphi \cos \varphi , \rho_\varphi \sin \varphi , p_z \right) \). It is convenient to change the variable \( \varphi \) to \( \varphi' = \varphi + \alpha \) in which case

\[ \int_{+\infty}^{\varphi'} d\varphi' = -\int_{-\infty}^{0} d\alpha \]
Then, using also eq. 11.42, eq. 11.40 reads

\[
4\pi \vec{\sigma} \cdot \vec{E} = -\sum_{\alpha} \frac{\omega_{p\alpha}^2}{\Omega_{0\alpha}} \int_{-\infty}^{+\infty} dp_1 \int_{0}^{\infty} p_1 dp_1 \int_{0}^{2\pi} (p_1 \cos \varphi, p_1 \sin \varphi, p_1) d\varphi
\]

\[
\times \int_{-\infty}^{0} G(\varphi - \alpha) \{ a \cos(\varphi - \alpha) E_x + a \sin(\varphi - \alpha) E_y + [b + c \cos(\varphi - \alpha)] E_z \} d\alpha
\]

(II.43)

with

\[
G(\varphi - \alpha) = \exp \{ S + iz \sin \varphi - iz \sin(\varphi - \alpha) \}
\]

\[
S \equiv (s + ik_1 v_1)/\Omega ; \quad z \equiv k_1 v_1/\Omega
\]

(II.44)

Here we used the definition \( \omega_{p\alpha}^2 = 4\pi n_{p\alpha} q_{\alpha}^2/m_\alpha \) and considered the case \( q_{\alpha} > 0 \) (i.e. took \(+\infty\) as the lower integration limit in 11.37').

Having in mind the final task - explicit expressions for the elements of the dielectric tensor to be used in the general dispersion relation, eq. 11.14 - and remembering the relation 11.15, we rewrite 11.15' as

\[(K^{-1}) \cdot \vec{E} = \frac{i}{\omega} 4\pi \vec{\sigma} \cdot \vec{E}\]

(II.15'')

where \( ^{-1} \) is the unitary dyadic having only diagonal components different from zero. The components of the tensor \( K^{-1} \) are

\[
\begin{pmatrix}
K_{xx} - 1 \\
K_{yy} \\
K_{zz} 
\end{pmatrix} = -\frac{i}{\omega} \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\Omega_{0\alpha}} \int_{-\infty}^{+\infty} dp_1 \int_{0}^{\infty} p_1 dp_1 \int_{0}^{2\pi} \left( \begin{array}{c}
p_1 \cos \varphi \\
p_1 \sin \varphi \\
0
\end{array} \right) a \left( \cos(\varphi - \alpha) G(\varphi - \alpha) \right) d\alpha
\]

\[
\begin{pmatrix}
K_{xx} - 1 \\
K_{yy} \\
K_{zz} 
\end{pmatrix} = -\frac{i}{\omega} \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\Omega_{0\alpha}} \int_{-\infty}^{+\infty} dp_1 \int_{0}^{\infty} p_1 dp_1 \int_{0}^{2\pi} \left( \begin{array}{c}
p_1 \cos \varphi \\
p_1 \sin \varphi \\
0
\end{array} \right) a \left( \cos(\varphi - \alpha) G(\varphi - \alpha) \right) d\alpha
\]

\[
\begin{pmatrix}
K_{xx} - 1 \\
K_{yy} \\
K_{zz} 
\end{pmatrix} = -\frac{i}{\omega} \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\Omega_{0\alpha}} \int_{-\infty}^{+\infty} dp_1 \int_{0}^{\infty} p_1 dp_1 \int_{0}^{2\pi} \left( \begin{array}{c}
p_1 \cos \varphi \\
p_1 \sin \varphi \\
0
\end{array} \right) a \left( \cos(\varphi - \alpha) G(\varphi - \alpha) \right) d\alpha
\]

\[
\begin{pmatrix}
K_{xx} - 1 \\
K_{yy} \\
K_{zz} 
\end{pmatrix} = -\frac{i}{\omega} \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\Omega_{0\alpha}} \int_{-\infty}^{+\infty} dp_1 \int_{0}^{\infty} p_1 dp_1 \int_{0}^{2\pi} \left( \begin{array}{c}
p_1 \cos \varphi \\
p_1 \sin \varphi \\
0
\end{array} \right) a \left( \cos(\varphi - \alpha) G(\varphi - \alpha) \right) d\alpha
\]
with \( G(\varphi - \alpha) \) given by 11.44 and \( a, b, c \) given by 11.42.

The next step is to carry out the angular integrations in 11.45.

For this we make use of the Bessel-function identities:

\[
\begin{align*}
\int_{-\infty}^{0} e^{iz \sin \varphi} & = \sum_{n=-\infty}^{+\infty} e^{in\varphi} J_n(z) & (II.46) \\
\int_{-\infty}^{0} e^{-iz \sin(\varphi - \alpha)} & = \sum_{m=-\infty}^{+\infty} e^{-im(\varphi - \alpha)} J_m(z) & (II.46')
\end{align*}
\]

and express \( G(\varphi - \alpha) \) as (see 11.42)

\[
G = e^{S\alpha} e^{iz \sin \varphi} e^{-iz \sin(\varphi - \alpha)}
\]

\[
= e^{S\alpha} \left[ \sum_{n=-\infty}^{+\infty} e^{in\varphi} J_n(z) \right] \left[ \sum_{m=-\infty}^{+\infty} e^{-im(\varphi - \alpha)} J_m(z) \right]
\]

(II.47)
As an example, we carry over the detailed angular integration for a simple case, namely the contribution due to the "b" term in the element $K_{xz}$. Thus,

$$p_{ll} b \int_0^{2\pi} d\psi \int_0^0 d\alpha \cdot e^{i \delta \alpha} \left[ \sum_{n=0}^{+\infty} e^{-in\psi} J_n(z) \right] \left[ \sum_{m=-\infty}^{+\infty} e^{-im(\psi-\alpha)} J_m(z) \right]$$

$$= p_{ll} b \sum_n J_n J_m \int_0^\infty d\alpha \cdot e^{i \delta \alpha} \int_0^{2\pi} d\psi \cdot e^{i \psi (n-m)}$$

$$= \left\{ p_{ll} b \sum_{n=0}^{+\infty} J_n^2(z) \sum_{n=-\infty}^{+\infty} \frac{n}{n + i k_{ll} \eta + i n \pi} \right\}$$

$$= 2\pi p_{ll} b \sum_{n=-\infty}^{+\infty} J_n^2(z) \frac{\Omega}{\rho + i k_{ll} \eta + i n \pi}$$

(II. 48)

[Notice: $\int_0^{2\pi} d\psi \cdot \exp \{ i \psi (n-m) \} = 2\pi \delta(n) \text{ if } n = m \text{ (} n \neq m \text{).}]

As a second example we carry out the angular integrations for another element, namely $K_{yy}$. Since by taking $\partial / \partial z$ of 11.46 and 11.46', one obtains

$$i z \sin \psi e^{i \psi} = \sum_{n=-\infty}^{+\infty} e^{i \psi} J_n'(z)$$

(II. 49)

$$-i z \sin(\psi-\alpha) e^{-i \alpha} = \sum_{m=-\infty}^{+\infty} e^{-im(\psi-\alpha)} J_m'(z)$$

(II. 49')
one has
\[
\begin{align*}
&\pi a \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} d\omega \text{e}^{i \varphi} \text{e}^{-i \varphi} \left\{ \text{Si} \mu \right\} \left\{ \text{Si} (\mu - \varphi) \right\} \\
&= \pi a \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} d\omega \text{e}^{i \varphi} \left\{ \text{Si} \mu \right\} \left\{ \text{Si} (\mu - \varphi) \right\} \\
&= \pi a \sum_{n=1}^{\infty} \left[ J_n(z) \right]^2 \int_{-\infty}^{\infty} d\omega \left\{ \text{Si} \mu \right\} \left\{ \text{Si} (\mu - \varphi) \right\} \\
&= \frac{2 \pi a \sum_{n=1}^{\infty} \left[ J_n(z) \right]^2 \text{Si} \mu \left\{ \text{Si} (\mu - \varphi) \right\}}{\lambda + i k_{\|} v_{\|} + i n J_2} \quad (II.50)
\end{align*}
\]

After the other angular integrations in (II.45) are carried out as indicated above, one obtains the following expressions for $K_{ij}$:

\[
\begin{bmatrix}
K_{xx} & K_{xy} & K_{xz} & K_{x2} \\
K_{yx} & K_{yy} & K_{yz} & K_{y2} \\
K_{zx} & K_{zy} & K_{zz} & 1
\end{bmatrix}
\]

\[
\begin{align*}
K_{xy} &= \frac{\sum_{p=0}^{\infty} \frac{1}{\lambda \omega^2} \int d^3 p \cdot p_\perp \sum_{n=1}^{\infty} J_n(z)}{2 \pi} \\
K_{yx} &= -K_{xy} \\
K_{xz} &= \frac{\sum_{p=0}^{\infty} \frac{1}{\lambda \omega^2} \int d^3 p \cdot p_\perp \sum_{n=1}^{\infty} J_n(z)}{2 \pi} \\
K_{yz} &= -K_{yz} \\
K_{x2} &= \frac{\sum_{p=0}^{\infty} \frac{1}{\lambda \omega^2} \int d^3 p \cdot p_\perp \sum_{n=1}^{\infty} J_n(z)}{2 \pi} \\
K_{y2} &= -K_{y2} \\
K_{z2} &= -K_{z2}
\end{align*}
\]

\[
\begin{align*}
\frac{n^2}{\lambda^2} \frac{\partial J_n}{\partial p_\perp} \\
\frac{n}{\lambda} \frac{J_n}{\partial p_\perp} + \frac{k_{\|} v_{\|}}{\omega} \frac{\partial J_n}{\partial p_\perp} \\
\frac{(n - \frac{\lambda}{\omega}) J_n}{\omega - k_{\|} v_{\|} - n J_2} \\
\frac{n \pi v_{\|} \frac{\partial J_n}{\partial p_\perp} + \left( -\frac{n \pi}{\omega} \right) \frac{\partial J_n}{\partial p_\perp}}{\omega - k_{\|} v_{\|} - n J_2}
\end{align*}
\]

(II.51)
We recall that $J_n(z)$ and $J'_n(z)$ are the Bessel functions and their derivatives, $f_0$ is the equilibrium particle distribution function, and $\mathbf{p}$ and $m$ are the relativistic momentum and mass, respectively. Also, we use the definitions $z \equiv k_{11} v_{11}/\Omega$, $\Omega \equiv q B_0/m c$ and $\omega_p^2 \equiv 4\pi n_0 q^2/m$. (All quantities should carry the subscript $\mathcal{O}$, $\mathcal{O} = e, p,$ etc.). The only assumption used in deriving the above equations is that $f_0 = f_0(p_{11}, p_{1\parallel})$, $\partial f_0/\partial p = 0$. If we now make the additional assumption that $f_0$ is an even function of $v_{1\parallel}$ or $p_{1\parallel}$, then the $K_{xz}$ and $K_{yz}$ elements may be simplified. This is done by adding to $K_{xz}$ or $K_{yz}$ a term identical to the above expression for $K_{xz}$ or $K_{yz}$ except that the last factor, in the square bracket, is replaced by

\[
\frac{1}{\omega} \left( \frac{v_{1\parallel}}{v_{11}}, \frac{\partial f_0}{\partial p_{1\parallel}} - \frac{\partial f_0}{\partial p_{1\parallel}} \right)
\]

This new expression is identically zero, because it is odd in $p_{1\parallel} (p_{1\parallel} = \gamma m_0 v_{1\parallel}$ and $p_{1\parallel}/p_{1\parallel} = v_{1\parallel}/v_{11}$). One then finds that

$K_{xz} = K_{zx}$, $K_{yz} = -K_{zy}$

The nonrelativistic case is recovered by setting $\gamma \to 1$ in 11.51. Since it is preferable to deal with $v$ only, one replaces $\mathbf{p} \to \mathbf{v}$ and changes the normalization of $f_0$ so that $m^3 f_0 \to f_0$.

4. Particular Case: propagation along the magnetic field

For disturbances having their wave vectors $\mathbf{k}$ along $\mathbf{B}_0$, i.e. $k_{\perp}, k_{1\parallel} = 0$, the general results obtained in Sect. 11.3 greatly simplify. Thus, expanding the Bessel functions in the small argument $z \equiv k_{11} v_{11}/\Omega$ and taking the limit $z \to 0$ in the results give
\[ J_n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}; \quad J'_n = \begin{cases} \pm 1/2 & \text{if } n = \pm 1 \\ 0 & \text{if } n \neq \pm 1 \end{cases} \]

\[ nJ_n = 0 \text{ for all } n; \quad n \frac{J_n}{z} = \begin{cases} 0 & \text{if } n \neq \pm 1 \\ 1/2 & \text{if } n = \pm 1 \end{cases} \]

and consequently

\[ \left( n \frac{J_n}{z} \right) (J_n) = 0 \quad \text{for all } n \]

\[ \left( n \frac{J_n}{z} \right) (J'_n) = \begin{cases} \pm 1/4 & \text{if } n = \pm 1 \\ 0 & \text{if } n \neq \pm 1 \end{cases} \]

and \( (J_n) (J'_n) = 0 \) for all \( n \).

Thus, for the case of the constant magnetic field lying along the \( z \)-axis (i.e. \( \mathbf{B}_0 = \mathbf{B}_{0z} \)), by using \( 11.52 \) in \( 11.14 \) and \( 11.51 \) one obtains

\[ K_{zx} = K_{zy} = K_{xz} = K_{yz} = 0 \]

and

\[
\begin{bmatrix}
A & -iB & 0 \\
B & A & 0 \\
0 & 0 & C
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix} =
\begin{bmatrix}
l_x \\
l_y \\
l_z
\end{bmatrix}
\]

(II.55)

where

\[
A = \Delta^2 + c^2 k^2 - \Delta \sum_{\omega} \sum_{\omega_0} \int_{\omega}^{\infty} \int_{\omega_0}^{\infty} \frac{d\rho_0 \rho_1 \rho_2 \rho_3}{\Omega (\Delta + i k_{NH})} \frac{\Omega (\Delta + i k_{NH})}{\Omega^2 + (\Delta + i k_{NH})^2} \chi
\]

(II.56)
Notice that, if \( J \) is initially zero, it will remain so for all time in the linear approximation. Assuming that this is the case, from eq. 11.55 one obtains the dispersion relation

\[(A^2 - \beta^2) C = 0 \quad (II. 60)\]

It indicates the possible existence of transverse (i.e. electromagnetic) modes given by the dispersion equation

\[A^2 - B^2 = R_{xx} R_{yy} - R_{xy} R_{yx} = 0 \quad (II. 61)\]

and one longitudinal (i.e. electrostatic) mode given by the dispersion equation

\[C = R_{zz} = 0 \quad (II. 62)\]

### III. ELECTROSTATIC INSTABILITIES

#### A. PARALLEL PROPAGATION

1. The dispersion equation

The nonrelativistic dispersion relation for the electrostatic case and longitudinal propagation (\( k = k_n = k_z \), \( B_0 = B_{0z} \)) is given by eq.11.62 and eq.11.58.
Inspection of these equations indicates that the dispersion properties are unaffected by the magnetic field, since the two components of the velocity transverse to $\vec{k}$ do not participate in the motion. Thus, it is convenient to define a "reduced" distribution function $F_0(\nu_\parallel \equiv u)$ such that

$$F_0(u) \equiv \int \delta \left( u - \vec{k} \cdot \frac{\vec{v}}{k} \right) f_0(\vec{v}) \, d\vec{v}$$

Then, using also the notation $k_\parallel = k$, one obtains the dispersion relation

$$D \equiv 1 - i \sum_{\alpha} \frac{\omega_\alpha^2}{k} \int_{-\infty}^{+\infty} \frac{F'_0(u) \, du}{s + iku} = 0$$

or, using the notation $z \equiv is/k$,

$$D \equiv 1 - \sum_{\alpha} \frac{\omega_\alpha^2}{k^2} \int_{-\infty}^{+\infty} \frac{F'_0(u) \, du}{u - z} = 0$$

In obtaining eq. III.2, we carried out an integration by parts which introduced the factor $1/k$ and removed the term $p_\parallel$. Notice that eq. III.2 can be integrated to obtain

$$D(k,s) \equiv 1 + \sum_{\alpha} \frac{\omega_\alpha^2}{s^2} \int_{-\infty}^{+\infty} \frac{F'_0(u) \, du}{(s + iku)^2} = 0$$

2. Stable cases

Before proceeding with the solution of eq. III.2 in the general case, we consider two particular cases of electron oscillations:

1) Zero temperature limit, i.e. $F_0(u) = \delta(u)$ (and $\int_{-\infty}^{+\infty} F_0(u) \, du = 1$). Thus, substituting for $F_0(u)$ in eq. III.2$'$ and carrying out the integration, one obtains

$$1 + \omega_p^2/s^2 = 0$$

---

2 We assume that the ions are massive and fixed.
having the solutions

\[ s = -i\omega = \pm i\omega_p \]

or

\[ \omega = \pm \omega_p \]

Therefore, any initial perturbation leads to local oscillations at \( \omega = \omega_p \)

\[ \sim \exp \pm i\omega_p t. \]

ii) Single-humped distribution functions. Consider an equilibrium distribution function satisfying the conditions (see Fig. 3)

\[ \int_{-\infty}^{\infty} F_0(v) dv = 1 \]

\[ \frac{dF_0}{dv} \mid _{v=v_0} = 0 \]

\[ \frac{d^2F_0}{dv^2} \mid _{v=v_0} > 0 \]

**FIG. 3.** Examples of velocity distribution functions: (a) single-humped; (b) double-humped (bump on tail); (c) loss-cone.
To investigate the stability of such a single-humped distribution function, we rewrite eq. III.2' as follows:

\[ D = \frac{\omega_p^2}{k^2} \left[ \frac{k^2}{\omega_p^2} - \int_{-\infty}^{+\infty} \frac{F_0'(u) \, du}{u - z} \right] \equiv \frac{\omega_p^2}{k^2} \cdot H = 0 \quad \text{III.6} \]

Eq. III.6 is satisfied if

\[ \frac{k^2}{\omega_p^2} D = H = \text{Re } H + i \text{ Im } H = 0 \quad \text{III.7} \]

or, equivalently,

\[ \text{Re } H = 0, \quad \text{Im } H = 0 \quad \text{III.7'} \]

We will now prove that for \( F_0(u) \) defined by conditions III.5, requirements III.7' cannot be satisfied simultaneously. Denoting \( z = \text{is}/k = x + iy \), one has

\[ H = \frac{k^2}{\omega_p^2} - \int \frac{F_0'(u) \, du}{u - z} = \frac{k^2}{\omega_p^2} - \int \frac{F_0'(u) \, du}{u - x - iy} \]

\[ = \frac{k^2}{\omega_p^2} \int \frac{(u-x+iy)F_0'(u) \, du}{(u-x+iy)(u-x-iy)} = \frac{k^2}{\omega_p^2} - \frac{(u-x)F_0'(u) \, du}{(u-x)^2+y^2} - iy \frac{\int \frac{F_0'(u) \, du}{(u-x)^2+y^2}}{2} \quad \text{III.8} \]

Assume \( \text{Im } H = 0 \), i.e. write

\[ \text{Re } H = \text{Re } H - \frac{U-x}{y} \text{ Im } H \]

\[ = \left\{ \frac{k^2}{\omega_p^2} - \int \frac{(u-x)F_0'(u) \, du}{(u-x)^2+y^2} \right\} - \frac{U-x}{y} \left\{ \int \frac{F_0'(u) \, du}{(u-x)^2+y^2} \right\} \quad \text{III.9} \]
Using the transformation \( u' = u - U \) (\( du = du' \)), one may rewrite III.9 as

\[
\text{Re } H = \frac{k^2}{\omega_p^2} - \left( \frac{u' - U - x}{(u' + U - x)^2 + y^2} \right) F_0'(u') \left( u' \right) du' + (U - x)
\]

Since the integral in III.10 can be written as

\[
\int_{-\infty}^{+\infty} \frac{u' F_0'(u') \left( u' \right) du'}{(u' + U - x)^2 + y^2} = \frac{k^2}{\omega_p^2} - \int_{-\infty}^{+\infty} \frac{u' F_0'(u') \left( u' \right) du'}{(u' + U - x)^2 + y^2}
\]

by III.5, one sees that in each half of the integration range \( u' F_0'(u') < 0 \). Thus, replacing \( u' F_0'(u') \) by \( -|u' F_0'(u')| \) in III.10, one obtains

\[
\text{Re } H = \frac{k^2}{\omega_p^2} + \int_{-\infty}^{+\infty} \frac{|u' F_0'(u')| \left( u' \right) du'}{(u' + U - x)^2 + y^2} > 0
\]

because the integrand is always positive. Thus, a single-humped distribution function cannot produce complex solutions with \( \text{Re } p = \text{Im } \omega > 0 \), as required for instability.

3. Unstable cases

Landau damping

We now treat the case of arbitrary unstable particle distribution functions.

Since the region of definition of the Laplace transform is \( \text{Re } s > 0 \) (i.e. \( \text{Im } z > 0 \), \( z \equiv \text{i} s/k \)), it follows that analyticity exists only for \( \text{Re } s > 0 \) (\( \text{Im } z > 0 \)). To remedy this, one has to perform analytical continuation.

If we restrict the problem to small imaginary parts of the frequency, \( \omega \) (or, equivalently, to values \( |y| \ll x \)), one may use the Plemelj formula (\( \text{P} = \text{principal part} \))

\[
\lim_{\epsilon \to 0} \frac{1}{a - (b \pm i \epsilon)} = \text{P} \left( \frac{1}{a - b} \right) \pm \pi i \delta(a - b)
\]

\[
\text{P} \int \frac{h(b) \, db}{a - b} = \lim_{\eta \to 0} \left\{ \int_{-\infty}^{b-\eta} + \int_{b+\eta}^{+\infty} \right\}
\]
Thus

\[
\lim_{y \to 0} \int_{-\infty}^{+\infty} \frac{h(u) du}{u - (x + iy)} = P \int_{-\infty}^{+\infty} \frac{h(u) du}{u - x} + \pi i \int_{-\infty}^{+\infty} h(u) \delta(u - x) du
\]

Next, inspection of the definitions 11.22' and 11.24' shows that the time-behavior (growth or damping) of the plasma under investigation depends on the values of the quantity "s" present in both \( \tilde{Q}^{(1)} \) and in the exponent. Since the dispersion relation 11.2 has been derived for the quantities \( \tilde{Q}^{(1)} \), the problem now reduces to finding the complex values \( s_j(k) \) which are zeros of these equations. For this purpose it is possible to deform the integration contour indicated by 11.24' as indicated in Fig. 2. Thus, by Cauchy's theorem, eq. 11.24' can be rewritten as

\[
\mathcal{E}^{(1)}(k,t) = (2\pi i)^{-1} \int \frac{\mathcal{E}(\omega)(k,\omega) e^{\omega t} d\omega}{\omega - i\infty} - \sum_j e^{s_j(k)t} \text{Residue} \left[ \frac{1}{D(k,\omega)} \right] + (2\pi i)^{-1} \int \frac{\mathcal{E}(\omega)(k,\omega) e^{\omega t} d\omega}{\omega - i\infty}
\]

Then, provided that \( \mathcal{E}^{(1)}(k,s) \) satisfies rather weak integrability conditions, as \( t \to \infty \) the last term in eq. 11.13 is exponentially damped, and only the sum over the exponentials \( e^{s_j t} \) survives; thus, the problem reduces to solving the dispersion relation for the complex roots \( s_j(k) \).

For arbitrary values of \( \text{Im} \omega = \omega_1 \) and \( k \) the dispersion relation 11.2 can only be solved numerically. An approximate analytical solution can be obtained for \( |\omega_1| < \omega_r \) and \( \text{Im} \omega / \omega_r \ll 1 \).

Thus, consider a function \( \mathcal{I}(z) = \int_{-\infty}^{+\infty} h(u) du / (u - z) \), where \( h(u) \) is a function integrable along the real axis and \( \mathcal{I}(z) = x + iy \) is analytic in the upper half plane, \( \text{Im} z > 0 \) (Re \( \omega > 0 \)). This integral can be expanded in powers of \( y \) around a point \( z_0 = x_0 + iy_0 \) to obtain
\[ I(z) = I(z_0) + I^{(1)}(z_0)(z-z_0) + \frac{1}{2} I^{(2)}(z_0)(z-z_0)^2 \ldots \]

where \[ I^{(n)}(z_0) = (\partial^n I/\partial z^{(n)})_{z_0} \] Notice that

\[ I^{(1)}(z_0) = \int_{-\infty}^{+\infty} h^{(1)}(u) \frac{du}{(u-z_0)^2} = \int_{-\infty}^{+\infty} h^{(1)}(u) \frac{du}{(u-z_0)} \]

(after integration by parts) and, generally

\[ I^{(n)}(z_0) = \int_{-\infty}^{+\infty} h^{(n)}(u) \frac{du}{(u-z_0)} \]

For the case of \( z \) approaching the real axis from above one has \( y \equiv \text{Im} \, z \)

\[ \lim_{y \to 0^+} I(z) = \mathcal{P} \int_{-\infty}^{+\infty} h(u) \frac{du}{(u-x)} + i \pi \dot{h}(x) \]

and

\[ \lim_{y \to 0^+} I^{(n)}(z) = \mathcal{P} \int_{-\infty}^{+\infty} h^{(n)}(u) \frac{du}{(u-x)} + i \pi \ddot{h}(x) \]

Collecting the results 111.14 - 111.16, one obtains

\[ \lim_{y \to 0^+} I(z=x+iy) = \sum_{n=0}^{\infty} \frac{1}{n!} (iy)^n \left[ \mathcal{P} \int_{-\infty}^{+\infty} h^{(n)}(u) \frac{du}{u-x} + i \pi \ddot{h}(x) \right] \]

Applying the result 111.17 to eq. 111.2, with the identification \( h(u) = F_0'(u) \), to lowest order, one obtains

\[ \lim_{y \to 0^+} B = 1 - \frac{w_0^2}{k^2} \left\{ \left[ \int_{-\infty}^{+\infty} F_0'(u) \frac{du}{u-x} + i \pi F_0'(x) \right] \right\} \]

\[ + iy \left[ \mathcal{P} \int_{-\infty}^{+\infty} F_0''(u) \frac{du}{u-x} + i \pi F_0''(x) \right] = 0 \]
Equating to zero the real and imaginary parts of eq. 111.18, one obtains a set of
two coupled equations for $x$ and $y$, namely

\[ y \mathcal{P} \int_{-\infty}^{+\infty} \frac{F_0''(u) \, du}{u-x} + \pi y F_0'(x) = 0 \]  

and

\[ \frac{k^2}{\omega_p^2} - \mathcal{P} \int_{-\infty}^{+\infty} \frac{F_0'(u) \, du}{u-x} + \pi y F_0''(x) = 0 \]

To lowest order, i.e. neglecting the last term in 111.20, one obtains

\[ y = -\pi F_0'(x) \left/ \left[ \mathcal{P} \int_{-\infty}^{+\infty} \frac{F_0''(u) \, du}{u-x} \right] \right. \]

It is now possible to find an approximate expression for the denominator in eq.111.21

as $(z \equiv is/k = i(-i\omega)/k = \omega/k)$ \(^3\)

\[ \frac{k^2}{\omega_p^2} = \int \frac{F_0'(u) \, du}{u-z} = \int \frac{F_0'(u) \, du}{u-\omega/k} \]

and taking the derivative $\partial/\partial k$, one obtains

\[ \frac{2k}{\omega_p^2} = \frac{d}{dk} \left[ \frac{\omega(k)}{k} \right] \int \frac{F_0''(u) \, du}{u-z} \]

(after an integration by parts has been performed).

Since by 111.12b

\[ \lim_{y \to 0^+} \int \frac{F_0''(u) \, du}{u-z} = \mathcal{P} \int \frac{F_0''(u) \, du}{u-x} + i\pi F_0''(x) \]

\(^3\) The symbol $C$ indicates that analytical continuation has to be performed.
and

\[ \lim_{y \to 0^+} \frac{d}{dk} \left[ \frac{\omega(k)}{k} \right] = \lim_{y \to 0^+} \frac{d}{dk} \left[ \frac{\omega_r - i \omega_i}{k} \right] = \frac{d}{dk} \left[ \frac{\omega_r(k)}{k} \right] \]

one obtains for the limiting case \((y \to 0^+)\) of eq. 111.22 (to lowest order)

\[ P \int_{-\infty}^{\infty} \frac{F_0''(u)}{u-x} du = \frac{2k}{\omega_r} \int \frac{d}{dk} \left[ \frac{\omega_r(k)}{k} \right] \]

When the result 111.23 is inserted into eq. 111.21 one obtains \((y \equiv \omega_i/k, \omega \equiv \omega_r - i\omega_i)\)

\[ \text{Im } \omega \equiv -\omega_i = \frac{\pi}{2} \omega_r(k) \left( \frac{\omega_r}{k} \right)^2 F_0'(x \equiv \omega_r/k) \left[ 1 - \frac{k}{\omega_r} \frac{d\omega_r}{dk} \right] \]

The last step consists of obtaining an approximate analytical expression for \(\omega_r(k)\).

We choose to obtain it in the limit of small \(k\) \((|\kappa/\omega_r| \ll 1)\). Thus, in eq. 111.21 we take \(\omega = \omega_r\) and expand in the small parameter \(|\kappa u/s| = |\kappa u|/\omega < 1\).

One obtains

\[ D(k, \lambda) \approx 1 + \frac{\omega_r^2}{\lambda^2} \left\{ \int F_0(u) du \left[ 1 - 2 \frac{iku}{\lambda} + \frac{1}{2} \left( \frac{iku}{\lambda} \right)^2 \cdot 3 \cdots \right] \right\} = 1 + \frac{\omega_r^2}{\lambda^2} \left\{ \int F_0(u) du \cdot 2i \frac{k}{\lambda} \int F_0(u) u du - 3 \frac{k^2}{\lambda^2} \int F_0(u) u^2 du \cdots \right\} \]

Using the normalization condition \(\int F_0(u) du = 1\), denoting \(\langle u^2 \rangle = \int F_0(u) u^2 du\) (mean square velocity), and assuming the equilibrium distribution function to be isotropic (i.e. \(\int F_0(u) \cdot u \cdot du = 0\)), the last equation provides

\[ 1 + \frac{\omega_r^2}{s^2} \left[ 1 - \frac{3k^2}{s^2} \langle u^2 \rangle \right] = 0 \]

111.25

Since we assumed \(|\kappa u/s| \ll 1\), we expect \(s = \pm i\omega_p + i\omega^{(1)}\) with \(\omega^{(1)} < \omega_p\) (see Sec. 4). Thus, using \(s = \pm i\omega_p\) in the square brackets and solving for \(s\), one obtains \((s \equiv -i\omega)\)
\[ \omega_r(k) \equiv \pm \omega_p \left| 1 + 3k^2(u^2)/\omega_p^2 \right|^{1/2} \]

for the real part of the frequency.

Inspection of the results III.24 and III.26 shows that because
\[ s = -i \omega = -i(\omega_r - i\omega_i) = -i\omega_r - \omega_i \]
on one has \( e^{st} = e^{-t(\omega_r - \omega_i)} \), and therefore instability (growth) occurs if \( \omega_i < 0 \), i.e. if \( F'_0(u) = \omega_r/k > 0 \). Thus, for single-humped (and, in particular, a maxwellian) distribution functions, \( F'_0 < 0 \) and therefore \( \omega_i,k \) is negative, indicating damping, known as Landau damping\(^4\). However, if \( F_0 \) has the form shown in Fig. 3, i.e. it has a "bump on the tail", then \( F'_0 > 0 \) near the bump and a wave with a phase velocity equal to the particle velocity in the region where \( F'_0 > 0 \) will grow exponentially. This is called negative Landau damping.

The physical explanation of Landau damping is that particles with velocity slightly smaller than the phase velocity will be accelerated by the wave and absorb energy from it. Particles with velocities slightly greater than that of the wave will be decelerated and lose energy to the wave. If there are more particles that can gain than can lose energy (i.e. for \( F'_0 < 0 \)), then the wave is damped. If there are more particles that can lose than can gain energy (i.e. \( F'_0 > 0 \)), then the wave grows. For more discussion on Landau damping see Jackson (1960) and Dawson (1961).

At this point it should be mentioned that, for some particular velocity distribution functions the theory predicts steady-state plasma oscillations of arbitrary wave number and frequency (Van Kampen, 1955; Bernstein et al., 1957). For this, \( F_0(u,t=0) \) has to be constant along the particle trajectories; the relative number of trapped and untrapped particles has to be properly adjusted. The small

\(^4\) For a maxwellian, \( F_0(u) = (m/2\pi K) \exp(-mu^2/2KT) \) one obtains \( F'_0 = -(2\pi)^{-1/2} u v_{th}^2 \exp(-u^2/2v_{th}^2) \) and \( (\omega_i) = v_{th}^2 = KT/m \). Thus, to lowest order, (III.24) reads:
\[-\omega_i = -(\pi/8)^{1/2} \omega_p^2 (k^4/k)^3 \exp(-\omega_p^2/(2k^2v_{th}^2)) < 0 \]. (The quantity in the square brackets of Eq.(III.24) is positive). In terms of the Debye wave number \( k_D = \omega_p^2/v_{th}^2 \) one has \( \omega_i = (\pi/8)^{1/2} (kp/k)^3 \omega_p \exp(-k_D^2/2k^2), \) which shows that the damping constant vanishes exponentially for small wave numbers.
amplitude limit of the BGK (Bernstein, Green and Kruskal) mode is called a Van Kampen mode. A complete treatment of these modes is given in the two papers mentioned previously.

4. The beam-plasma instability

For non-isotropic streaming distributions and zero limit temperatures, i.e. for $f_{0\alpha}$ given by

$$f_{0\alpha} = \frac{1}{\pi v_{\perp}} \delta(v_{\perp}) \delta(v_{\parallel} - v_{\alpha})$$

eq 110

$$I_{11.27}$$

eq. 111.2 becomes

$$1 - \sum_{\alpha} \frac{\omega_{p\alpha}^{2}}{(\omega - kV_{\alpha})^{2}} = 0$$

If the thermal energy of the beam $W_{b,th}$ is initially small compared with the streaming energy of the beam, $W_{b,s}$, 111.28 represents a satisfactory approximation that affords a rather simple analysis of the physics involved. Thus, consider the case of a cold electron plasma traversed by a monochromatic energetic electron beam. For simplicity assume the ions to be at rest and therefore forming a fixed positive neutralizing background. Define

$$e = \frac{\omega_{pb}^{2}}{\omega_{pp}^{2}} = \frac{n_{b}}{n_{p}} ; \quad x = \frac{\omega}{\omega_{pp}} ; \quad y = kV_{b}/\omega_{pp}$$

$$I_{11.29}$$

where $n_{b}$ and $n_{p}$ are the particle densities in the beam and in the cold plasma, $\omega_{pb} = (4\pi n_{b}e^{2}/m_{e})^{1/2}$, $\omega_{pp} = (4\pi n_{p}e^{2}/m_{e})^{1/2}$ and $V_{b}$ is the streaming velocity of the beam. With the notations of 111.29, eq. 111.28 becomes a quartic for $x$:

$$x^{4} - 2yx^{3} + (y^{2} - 1 - e)x^{2} + 2yx - y^{2} = 0$$

$$I_{11.30}$$
At this point the following essential remarks should be made:

(i) For any $\xi \neq 0$, $I_{11.30}$ has complex roots indicating electrostatic instability, i.e. $\operatorname{Im} x > 0$. To see this, it is sufficient to consider solutions of $I_{11.30}$ for small values of $x$ and $y$. Therefore, second order values of $x$, $y$ yield a quadratic whose solution is

$$x = \left[\frac{y}{(1 + \epsilon)}\right][1 \pm ie^{1/2}] \quad I_{11.31}$$

(ii) The maximum growth rate of the instability, $\gamma_{\text{max}} \equiv (\operatorname{Im} x)_{\text{max}}$, and the corresponding real part of the frequency, $x_r (\gamma_{\text{max}})$, and normalized wave number, $y (\gamma_{\text{max}})$, are as follows

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\gamma_{\text{max}}$</th>
<th>$x_r (\gamma_{\text{max}})$</th>
<th>$y (\gamma_{\text{max}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ll 1$</td>
<td>$0.5 \sqrt{3} (0.5 \epsilon)^{1/3}$</td>
<td>$0.5$</td>
<td>$1 - 0.5 (0.5 \epsilon)^{1/3}$</td>
</tr>
<tr>
<td>$1$</td>
<td>$0.5 \sqrt{3}$</td>
<td>$0.5 \sqrt{3}$</td>
<td></td>
</tr>
</tbody>
</table>

The instability increases with $\xi$. These results are easily obtained from $I_{11.30}$ by calculating the maximum value from the relation $\operatorname{Im}(dx/dy) = 0$ and substituting the result back into $I_{11.30}$. Thus, significant growth rates are expected to occur.

(iii) The unstable wave spectrum ranges in (normalized) $k$-space between values $y_{\text{min}} \equiv (k V_b/\omega_{pp})_{\text{min}} = 0$ and $y_{\text{marg}} \equiv (k V_b/\omega_{pp})_{\text{marg}}$ which are given by the following expressions:

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$y_{\text{marg}}$</th>
<th>$x (y_{\text{marg}})$</th>
<th>$y (y_{\text{marg}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ll 1$</td>
<td>$(1 + \epsilon^{1/3})^{3/2}$</td>
<td>$(1 + \epsilon^{1/3})^{1/2}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$1$</td>
<td>$2^{3/2}$</td>
<td>$2^{1/2}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

The instability increases with $\xi$. These results are easily obtained from $I_{11.30}$ by calculating the maximum value from the relation $\operatorname{Im}(dx/dy) = 0$ and substituting the result back into $I_{11.30}$. Thus, significant growth rates are expected to occur.
FIG. 4. Numerical solution of the dispersion equation for electrostatic waves. The real and imaginary parts of the frequency, $\omega_r$ and $\omega_i$, are normalized to $\omega_{pp}: x_r \equiv \omega_r / \omega_{pp}$ and $\gamma \equiv \omega_i / \omega_{pp}$. For the wave number $k$, the following normalization is used: $y \equiv kV_b / \omega_{pp}$ where $V_b$ is the initial stream velocity of the beam and $\omega_{pp} \equiv (4\pi\eta e^2/m_e)^{1/2}$. Cases A, B and C correspond to values $\epsilon = 1, 10^{-1}$ and $10^2$, respectively. Full line $- \theta = 0^\circ$ (parallel propagation); dashed line $- \theta = 45^\circ$.
The complete numerical solution of \( \text{III.30} \) is given in figure 4, for three cases of interest, namely \( \xi = 10^{-2}, 10^{-1} \) and 1. For completeness we have also indicated the solutions obtained for the oblique propagation case in the absence of a static magnetic field (\( \theta = 45^\circ \)). Thus, for waves propagating with an angle \( \theta \) with respect to the streaming direction of the beam, both \( \gamma \equiv \text{Im}(\omega/\omega_{pp}) \) and \( x_r \equiv \text{Re}\omega/\omega_{pp} \) are displaced in \( y \equiv kV_b/\omega_{pp} \) space by a factor of \( (\cos \theta)^{-1} \); however, the maximum value of \( \gamma^r \), \( \gamma_{\text{max}}^r \) as well as the marginal value of \( x_r, x_{r,\text{marg}} \) remain unchanged.

As already mentioned, \( \text{III.30} \), which is based on expression \( \text{III.27} \), holds for beam-plasma systems with \( \xi_{\text{w}} \ll 1 \). When this condition does not hold, \( \text{III.24} \) should be used to obtain corresponding results for \( \gamma^r \) and \( x_r \). These calculations indicate that increasing the thermal spread of the beam leads to a decrease in the maximum growth rate of the instability. Consequently, in the case of an initial cold-beam-plasma system, we expect the system to behave according to the predictions of \( \text{III.30} \), so long as \( \xi \) remains much smaller than unity, and to evolve thereafter according to \( \text{III.24} \).

For completeness we show the results obtained in the computer simulation of three physical situations of interest, namely \( \xi = 1 \) (case A), \( \xi = 0.1 \) (case B) and \( \xi = 0.01 \) (case C) (see Fig. 5) (Caperman et al., 1976). Initially each of the simulation systems A, B and C consists of a relatively cold background electron plasma \( (kT_{e,p}^\parallel = 0.5kT_{e,p}^\perp = 1 \text{ eV}) \) penetrated by an electron beam whose directed (streaming) energy, \( W_{b,s}^\parallel \), is 1 keV and has a thermal spread in the direction of streaming, \( W_{b,th}^\parallel = 50 \text{ eV} \) (i.e. 5% of \( W_{b,s}^\parallel \)). The thermal spread of the beam transverse to the streaming is small, \( W_{b,th}^\perp = 2 \text{ eV} \). The three cases differ only by the relative beam concentration \( \xi \equiv n_b/n_p \), which is therefore the only independent parameter in our case; cases A, B and C correspond to values \( \xi = 1, 0.1 \) and 0.01, respectively. For simplicity the protons, in each case, were considered to be at rest and constitute a uniform neutralizing background.

Therefore, we are concerned with situations in which almost monoenergetic electron beams \( ((\Delta v_b)^2/v_b^2 = 5\%) \) interact with relatively cold plasmas and for which the linear theory of this section holds.
In simulation, the particle-in-cell method was used. Both electrostatic and electromagnetic interactions were simultaneously considered and periodic boundary conditions were applied. The beam-plasma system was described by 20,000 simulation particles uniformly distributed over 100 cells. The simulation length \( L \) was chosen to enable the development of electrostatic modes with wave numbers up to at least twice that of the largest unstable mode, \( k_{\text{max}} \), predicted by the linear theory (figure 4).

The time integration step used was \( \Delta T = 0.2 \omega_p^{-1} \) and the total energy was conserved within less than 0.1% over the complete run. As it is seen, a strong electrostatic instability is observed to occur at all values of \( \varepsilon \) used (see figure 6). This is easily seen from the exponential growth of the initially low electrostatic background, \( \tilde{W}_{\text{e.s.}}(0) \). The effective growth rate \( \tilde{\gamma} \) is largest for \( \varepsilon = 1 \) and decreases with \( \varepsilon \), as predicted by the linear theory. There is a reasonable overall agreement between experimental results and linear predictions.
As expected, all three systems are stable against electromagnetic instabilities; the electromagnetic wave activity, $\overline{W}_{e.m.}$ remains at about the initial level, which is several orders of magnitude lower than that of the electrostatic wave activity.

More details of the electrostatic activity developed in the beam-plasma systems A, B and C can be obtained by Fourier analysis of the total electrostatic wave energy, $\overline{W}_{e.s.}$. Thus, as seen in figure 7, in the linear stage there is good agreement between the behaviour of individual modes as observed in the experiments and predicted by the linear theory. Specifically, the most unstable modes for the cases A, B and C are those with $\tilde{k} \approx 6-7, 6-7$ and 6, respectively. The corresponding values of the growth rate are $\approx 0.27, 0.22$ and 0.21, which are to be compared with the linearly predicted ones, namely 0.35, 0.26 and 0.17 which should occur for the modes $\tilde{k} = 6.2, 5.8$ and 5.5, respectively. (A more convenient normalization for the wave-number $k$, namely $\tilde{k} = y/y_0$ with $y \equiv kV_b/\omega_{pp}$, $k_0 = 2\pi/L$, L being the simulation length has been used.)

Finally, Fig. 8 gives the time behavior of the particle distribution function. The instability quenches when the bump on the tail - represented by the electron beam - flattens such that $\omega_i \sim \partial f_0/\partial y = 0$ (see eq. III.24).

At this point we recall that the results III.3 - III.33 were obtained by assuming the ions to be massive and fixed. As a result we have obtained high-frequency (plasma electron) oscillations and corresponding Landau damping. However, for waves having slow enough phase velocities to match the thermal velocity of ions, ion Landau damping can occur.

B. OBLIQUE PROPAGATION

1. The dispersion equation

The dispersion relation for electrostatic waves (i.e. $E \parallel k$) for the case $\vec{k} = k_\parallel + k_\perp$ ($k_\parallel \neq 0, k_\perp \neq 0$) can be obtained from the general equation given in
FIG. 6. Time-behaviour of the (normalized) total electrostatic wave energy \( \overline{W}_{e.s.} \), total electromagnetic energy \( \overline{W}_{e.m.} \), total beam (streaming+thermal) energy \( \overline{W}_b \), total parallel thermal energy in the beam \( \overline{W}_{b,th} \), total parallel plasma energy and three-component plasma energy \( \overline{W}_p \) and \( \overline{W}_p^I \) respectively.

(i) \( \overline{W}_{e.s.} = \sum_k \frac{E_k^2}{8\pi} / W_{tot} \)

(ii) \( \overline{W}_{e.m.} = \sum_k \frac{B_k^2}{8\pi} / W_{tot} \)

(a) \( \epsilon = 1 \); (b) \( \epsilon = 0.1 \); (c) \( \epsilon = 0.01 \).
FIG. 7. Time behaviour of the various unstable electrostatic modes developed during the instability. Note that $k = y/y_0$ with $y = kV_b/\omega_{pp}$ and $k_0 = 2\pi/L$. 
FIG. 8. Velocity distribution functions of the beam-plasma systems at several times of evolution.
Sect. II by projecting the tensor given by 11.14 and 11.51 along the electric field. This procedure leads to

\[ I = \sum_{k} \frac{w_p^2}{k^2} \sum_{n=-\infty}^{+\infty} \int d^3V \frac{J_n^2 \left( k_n v_n / \Omega \right)}{k_n v_i - (\omega - n \Omega)} \left( \frac{\partial f_0}{\partial v_i} + \frac{n \Omega \partial f_0}{v_i} \right) \]

\[ \text{Eq. 111.34} \]

Without the last term, this is the original Harris dispersion relation for quasi-electrostatic waves (Harris, 1961). The additional contribution represents the lowest-order modification of the electrostatic waves, due to (electron) current-induced coupling to the transverse waves. As shown by Callen and Guest (1971), unless the condition \( \omega_{pe}^2/c^2k^2 \ll 1 \) is satisfied, this contribution should be considered.

Eq. 111.34 supports unstable electrostatic waves associated with anisotropic velocity distributions and has been thoroughly investigated. Besides the temperature anisotropy instabilities (Dnestrovsky et al., 1963; Hall et al., 1965) the research concentrated on unstable waves associated with non-monotonic perpendicular velocity distributions known as loss-cone instabilities (Krasovitskii & Stepanov, 1964; Rosenbluth & Post, 1965; Guest & Dory, 1965, 1968). Mirror machines have loss-cones, because particles with small \( v_i/V_{\parallel} \) are lost through mirrors. Recently it has been suggested that the injection spectrum of the magnetospheric charged particles can be modelled by loss-cone distribution.

2. **High-frequency electrostatic ion loss-cone instability**

As an example, we will consider here the stability of high-frequency electrostatic waves propagating almost perpendicular \( (k_{\perp} \gg k_{\parallel}, k_{\parallel} \neq 0) \) to the magnetic field in warm plasmas in which the ions are represented by loss-cone distribution functions. Also, we will investigate the effect of adding various amounts of cold plasma populations (e.g. Cuperman and Gomberoff, 1977). For the electrons, we assume an isotropic Maxwellian distribution given by
where \( b = (2K_e T_e/m_e)^{1/2} \) is the electron thermal speed and \( T_e \) the electron temperature.

For the ion-component of the plasma we assume the following population:

(i) A warm component characterized by a loss-cone distribution given by

\[
\frac{n_{p,w}(V_{\perp},V_{\parallel})}{n_{p,w}} = \frac{1}{\pi^{3/2} a^3} (\frac{V_{\perp}}{a})^{2m} \exp \left\{ -\frac{V_{\parallel}^2 + V_{\perp}^2}{a^2} \right\}
\]

where \( a = (2K_T p_{\parallel} / m_p)^{1/2} \) is the parallel thermal velocity and \( T_{p,\perp} = (m+1) \times x(m_{\perp} a^2/2) \) is the perpendicular temperature. The anisotropy factor is therefore

\( m = (T_{\perp}/T_{\parallel})_{p}^{-1}. \)

(ii) A cold isotropic component of density \( n_c \). Thus, the total density is

\( n_t = n_w + n_c. \)

Using standard procedures, the contribution to \( III.34 \) due to the electrons and the cold protons can be calculated. We obtain

\[
\varepsilon(k,\omega) = 1 + \frac{\omega_{pe}^2}{\Omega_e^2} \sin^2 \theta \left( 1 + \frac{\omega_{pe}^2}{k^2 c^2} \right) - \frac{\omega_{pe}^2}{k^2 b^2} Z^{(1)}(\omega/k_{\parallel} b)
\]

\[
- \frac{\omega_{p_i}^2}{\omega^2 - \Omega_i^2} \left( 1 - \frac{\Omega_i^2}{\omega^2} \cos^2 \theta \right) + \frac{2\pi \omega_{p_i}^2 (1 - \chi)}{k^2}
\]

\[
+ \sum_{n=-\infty}^{\infty} \int dV_{\parallel} \int dV_{\perp} \left\{ \frac{n_{w1} \partial \Phi_0}{V_{\perp}} + \frac{k_{1i} \partial \Phi_0}{V_{\parallel}} \right\} \frac{J_{n} (k_{1i} V_{\parallel}/b)}{\omega - k_{1i} V_{\parallel} - n\Omega_i}
\]

where \( Z^{(1)}(\omega/k_{\parallel} b) \) is the derivative of the plasma dispersion function, \( Z^{(1)}(\xi) = (\partial/\partial \xi) Z(\xi) \), and \( \sin \theta = k_{1\perp} / k. \ r \equiv n_{p,c} / (n_{p,c} + n_{p,w}) \).

Since we are concerned with high-frequency waves, \( \omega > \Omega_i \), the contribution of the warm proton branch can be simplified by using the 'weak magnetic field approximation' (Montgomery & Tidman, 1964; Rosenbluth & Post, 1965; Coroniti et al., 1972).
Thus III.37 can be written:

\[ \begin{align*}
\varepsilon(k, \omega) &= 1 + \frac{\omega_p^2}{\Omega_e^2} \sin^2 \theta \left( 1 + \frac{\omega_p^2}{k_e^2 c^2} \right) - \frac{\omega_p^2}{k_e^2 c^2} Z^{(1)}(\omega/k_e) \\
&\quad - \frac{\omega_p^2 r}{\omega^2} + \frac{2 \omega_p^2 (1-r)}{k_e^2} \int d^3 \nu \left( \frac{\partial f_p}{\partial \nu_x} + k_e v_x \frac{\partial f_e}{\partial \nu_e} \right) \\
&\quad \left[ (\omega - k_{ei} v_x - k_e v_x) \right] \tag{III.38}
\end{align*} \]

At this point, because of mathematical complexity, we have to specify the anisotropy factor \( m \) in expression III.38. We thus consider the case \( m = 2 \) corresponding to \((T_\perp/T_\parallel)_p = 3\).

Denoting the integral expression in III.38 by \( I \), and using expression III.36 for \( f_{p,w} \), one obtains

\[ I = L + M \tag{III.39} \]

where

\[ L = -\frac{1}{2a^2} \left\{ \begin{align*}
Z^{(1)}(x) + \frac{1}{2} Z^{(3)}(x) + \frac{1}{3!} Z^{(5)}(x) \\
+ \frac{1}{4} \left( \frac{k_{ei}}{k_e} \right)^2 Z^{(3)}(x) + \frac{1}{4} \left( \frac{k_{ei}}{k_e} \right)^2 Z^{(5)}(x) + \frac{3}{4!} \left( \frac{k_{ei}}{k_e} \right)^4 Z^{(3)}(x) + \ldots
\end{align*} \right\} \tag{III.40} \]

\[ M = -\frac{1}{2a^2} \left\{ \begin{align*}
\left( \frac{k_{ei}}{k_e} \right)^2 Z^{(1)}(x) + \frac{1}{4} \left( \frac{k_{ei}}{k_e} \right)^2 Z^{(3)}(x) + \frac{1}{4} \left( \frac{k_{ei}}{k_e} \right)^2 Z^{(5)}(x) \\
+ \frac{1}{3!} \left( \frac{k_{ei}}{k_e} \right)^2 Z^{(3)}(x) + \frac{3}{3!} \left( \frac{k_{ei}}{k_e} \right)^4 Z^{(3)}(x) + \ldots
\end{align*} \right\} \tag{III.41} \]

and \( x = \omega/k_e \). Adding expressions III.40 and III.41, and neglecting terms of order \((k_{ei}/k_e)^4\) and higher, III.38 reduces to

\[ \varepsilon(k, \omega) = 1 + \frac{\omega_p^2}{\Omega_e^2} \sin^2 \theta \left( 1 + \frac{\omega_p^2}{k_e^2 c^2} \right) - \frac{\omega_p^2}{k_e^2 c^2} Z^{(1)}(\omega/k_e) \\
- \frac{\omega_p^2 r}{\omega^2} - \omega_p^2 (1-r) \left[ Z^{(1)}(x) + \frac{1}{2} Z^{(3)}(x) + \frac{1}{3!} Z^{(5)}(x) \right] = 0 \tag{III.42} \]
Equation 111.42 can be brought to a more convenient form by using the identities

\[
Z^{(1)}(x) + \frac{1}{2} Z^{(3)}(x) + \frac{1}{32} Z^{(5)}(x) = -\frac{i}{8} \left[ x Z^{(2)}(x) - x^2 Z^{(3)}(x) \right]
\]

Inserting 111.43 into 111.42 yields

\[
\varepsilon(k, \omega) = 1 + \frac{\omega_p e^2}{\omega_c^2} \sin^2 \theta \left( 1 + \frac{\omega_p e^2}{k^2 e^2} \right) - \frac{\omega_p e^2}{k^2 b^2} Z^{(4)}(x)
\]

In order to determine the range of values of \( x \) for which the spectrum is unstable, we consider the marginal mode corresponding to \( \omega_i = 0 \).

Thus, using the expression \( Z(x) = i \pi^2 e^{-x^2} + \text{Re} Z(x) \) in 111.40, one obtains

\[
\frac{\omega_p e^2}{\omega_c^2} \frac{k z}{k z b^2} \exp \left(-\omega^2 / k z b^2\right) - (1-x) (1+4x^2-4x^4) \exp \left(-\omega^2 / k z b^2\right) = 0
\]

where we have used the approximations \( k \equiv k \) and \( \sin^2 \theta \equiv 1 \). Denoting

\[
\cos \theta = k_i / k \equiv \alpha^{1/2} (m_c / m_i)^{1/2}
\]

and

\[
m_i a^2 / m_c b^2 = \delta
\]

equation 111.45 becomes

\[
8 \delta^{3/2} \alpha^{-1} \exp(-\delta x^2 / \alpha c) - (1-x) (1+4x^2-4x^4) \exp(-x^2) = 0
\]

or equivalently
Equation 111.48 may also be written as

\[ x^2 = \ln \left[ 8 \left( \delta^3 / \Delta \right)^{1/2} / (1 + 4x^2 - 4x^4) \right] (1 - \alpha) / (\delta / \Delta - 1) \]  

The function \( g(x) = 1 + 4x^2 - 4x^4 \) in 111.49 is positive for values \( 1.1 > x \geq 0 \), and in the range \( 1 \geq x > 0 \) it varies between 2 and 1. Therefore, an approximate expression for the marginal stable value of \( x \) is

\[ x_{\text{marg}} \approx \frac{1}{2} \ln \left[ 8 \left( \delta^3 / \Delta \right)^{1/2} / (\delta / \Delta - 1) \right]^{1/2} \]  

The fact that the function \( g(x) \) vanishes for \( x = 1.1 \) indicates that for most of the unstable spectrum \( x < 1 \), and therefore the convergent series expansion of \( Z(x) \) for \( x < 1 \) can be used in 111.44. We can now calculate the growth rate and eventually the maximum growth rate by using the asymptotic expansion for \( Z(\xi_e) \) (\( \xi_e \equiv \omega / k \beta \gg 1 \)) and the convergent expansion for \( Z(x) \) (\( x \equiv \omega / k \beta < 1 \)).

Inserting the results into 111.44 and equating to zero the real and the imaginary parts of the resulting equation yields

\[ y = \frac{\omega_i}{\omega_p} = \frac{\pi^2 y x^3 \left[ (1 - \alpha)(1 + 4x^2 - 4x^4) \exp(-x^2) - \delta^3 / \Delta \exp(-\alpha x^2) \right]}{8 \left( \alpha C + 3 \alpha^2 \Delta \sin^{-2} \theta / \delta x^2 + \rho \right) -(1 - \alpha)x^3 B(x)} \]  

and

\[ y^2 = \frac{\omega_i^2}{\omega_p^2} = \frac{C + 3 \alpha^2 \Delta \sin^{-2} \theta / 2 \delta x^2 + \rho - \hat{\beta} x^2 + 0.25 (1 - \alpha)x^2 A(x)}{1 + (\hat{\omega}_p / \omega_p^2) \Delta \sin^2 \theta} \]  

where

\[ \hat{\beta} \equiv 8\pi n_e K T / B_0^2 \]  

The values of the constants \( a, b, \ldots \) and \( \bar{a}, \bar{b}, \ldots \) are indicated in Cuperman and Gomberoff, 1977).
FIG. 9. (1) Rates of growth, $\hat{\gamma}_{\text{e.s.}} \equiv \omega_1 / \omega_{pW}$, (2) real frequencies, $\tilde{\gamma}_{\text{e.s.}} \equiv \omega_r / \omega_{pW}$, (3) normalized group velocities, $\tilde{v}_g \equiv (\partial \omega_r / \partial k_i)^{-1} v_{pW}$, and (4) ratios $\tilde{S}_{\text{e.s.}} \equiv \gamma_{\text{e.s.}} / \tilde{v}_g$ as a function of $\tilde{k} \equiv k v_{pW} / \omega_{pW}$ for the quasi-electrostatic ion loss-cone mode. The external parameters are $\beta_p, \alpha = 0.4, Q^2 = 4$, $m = 2, \alpha = 1, \delta^2 = 25$. Here $Q^2 \equiv \omega_{pe}^2 / \Omega_e^2$ and $\omega_{pe} \equiv 4\pi n_{pW} e^2 / m_e$. The maximum values are indicated by arrows: (a) $\tau = 0$ and (b) $\tau = 0.8$. 

\[ \begin{align*} 
\hat{\gamma}_{\text{e.s.}} & \equiv \omega_1 / \omega_{pW} \\
\tilde{\gamma}_{\text{e.s.}} & \equiv \omega_r / \omega_{pW} \\
\tilde{v}_g & \equiv (\partial \omega_r / \partial k_i)^{-1} v_{pW} \\
\tilde{S}_{\text{e.s.}} & \equiv \gamma_{\text{e.s.}} / \tilde{v}_g 
\end{align*} \]
A(x) = -ax^4 + bx^6 - cx^8 + dx^{10} - ex^{12} + fx^{14} - gx^{16}

and

B(x) = -\tilde{a}x^3 + \tilde{b}x^5 - \tilde{c}x^7 + \tilde{d}x^9 - \tilde{e}x^{11} + \tilde{f}x^{13} - \tilde{g}x^{15}

As both x and y are functions of \( r \), 111.52 is a sixteenth-degree equation for \( \omega_r \).

**FIG. 10.** Maximum growth rates \((\text{Im } \omega)_{\text{max}}/\Omega\) as a function of the relative cold proton density \( r \equiv n_{p,c}/(n_{p,c} + n_{p,w}) \) for (i) the parallel propagating electromagnetic ion-cyclotron mode (e.m., \( k_1 = 0 \)), (ii) quasi-electrostatic ion loss-cone mode (e.s., \( k_1 \neq 0 \)), and (iii) flute-like mode (e.s., \( k_1 = 0 \)). The external parameters are \( \beta_{p,\perp} = 0.4 \), \( Q^2 = 4 \), \( m_{A_p} = 2 \), \( \delta^2 = 25 \) and \( \alpha = 1 \) (for e.s., \( k_1 \neq 0 \)).
Equations 111.51 and 111.52 have two advantages over 111.44: (i) they can be solved relatively simply, either numerically or graphically, with $x$ as the independent variable. These calculations are significantly less involved than that required for the solution of 111.44 for complex variable, $\xi$; (ii) they enable one to obtain simple and transparent expressions for the maximum growth rates. A simple approximate expression for maximum growth rate, $Y_{\text{max}}$, can be obtained as follows: to first order, the polynomials $A(x)$ and $B(x)$ in expressions 111.51 and 111.52 may be neglected, i.e.

$$Y = \frac{\frac{1}{2} y x^3 \left[(1-\zeta)(1+4x^2-4x^4)\exp(-x^2) - 8(\delta^3/\lambda)^{1/2}\exp(-\delta x^4/\lambda)\right]}{8(\zeta + 3\lambda^2 \sin^{-2}\theta/\delta x^2 + D)}$$  \hspace{1cm} 111.56

and

$$y^2 = \frac{\zeta + D + 3\lambda^2 \sin^{-2}\theta/2\delta x^2 - \beta x^2}{1 + \omega^2 \delta \sin^2 \theta/D}$$  \hspace{1cm} 111.57

The value of $x$ for which $Y$ is a maximum can be approximately obtained from

$$\frac{\partial Y}{\partial x} \approx (1-\zeta) \exp(-x^2) - 8(\delta^3/\lambda)^{1/2}\exp(-\delta x^4/\lambda) = 0$$  \hspace{1cm} 111.58

which gives

$$x_1^2 \approx \frac{\ln \left|8(\delta^3/\lambda)^{1/2}/(1-\zeta)\right|}{\delta/\lambda - 1}$$  \hspace{1cm} 111.59

Next, replacing this value for $x$ in 111.56 and 111.57, one obtains (neglecting the last term in the curly brackets)

$$Y_{\text{max}} \approx \frac{\frac{1}{2} y_1 x_1^3 \left[(1-\zeta)(1+4x_1^2-4x_1^4)\exp(-x_1^2)\right]}{8(\zeta + D + 3\lambda^2 \sin^{-2}\theta/\delta x^2)}$$  \hspace{1cm} 111.60

and

$$y^2(\delta_{\text{max}}) \approx y_1^2 \approx \frac{\zeta + D + 3\lambda^2 \sin^{-2}\theta/2\delta x^2 - \beta x^2}{1 + \omega^2 \delta \sin^2 \theta/D}$$  \hspace{1cm} 111.61
Thus, for given values of the parameters \( \beta \), \( \omega_p e / m_e \), \( \alpha \), \( \gamma \) and \( r \), the maximum growth rate is obtained from III.60, with \( x_1 \) and \( y_1 \) given by III.59 and III.61, respectively. Inspection of expressions III.60 and III.61 indicates that \( \gamma_{\text{max}} \) decreases with increasing \( r \) (cf. III.60) and \( \beta \) (cf. III.61). Thus, the largest values of \( \gamma_{\text{max}} \) occur in the limit \( r \to 0 \) and \( \beta \to 0 \) (i.e. vanishing stabilizing electromagnetic effects).

The results of this section are represented in Fig. 9, giving the rates of growth, real frequencies and group velocities for two relative cold proton populations \( n_{p,c}/n_{p,w} \) and in Fig. 10 (curve (ii)) giving the maximum growth rate as a function of the relative cold proton density.

IV. ELECTROMAGNETIC INSTABILITIES

We consider here the case of parallel propagation, i.e. \( k = k_{\parallel} \). Then eq. II.13 reduces to eq. II.61, with \( \alpha \) and \( \beta \) given by II.56 and II.57. With a slight change of notation (\( s \to -i\omega \), \( p \to \gamma \)) and normalization, eq. II.61 becomes

\[
F_L^R = -\omega^2 + c^2 k^2 - \pi w \sum_{\ell} \omega_p^2 \int_{-\infty}^{+\infty} dv_{\parallel} \int_{0}^{+\infty} v_\perp^2 dv_{\perp} \\
\left( \frac{\partial \rho_0}{\partial v_{\perp}} - \frac{k v_{\parallel}}{\omega} \frac{\partial \rho_0}{\partial v_{\parallel}} + \frac{k v_{\perp}}{\omega} \frac{\partial \rho_0}{\partial v_{\perp}} \right) \frac{1}{\omega - k v_{\parallel} \pm \gamma} = 0 \quad \text{IV.1}
\]

In eq. IV.1 the symbols \( R \) and \( L \) refer to the right- and left-hand polarizations and correspond to plus and minus signs, respectively, in the denominator when \( \omega > 0 \). The summation is over all plasma components.

1. Electromagnetic ion-cyclotron instability

Since the dispersion equation for this case is the same for an equilibrium distribution function represented by a bi-Maxwellian with thermal anisotropy

\[ A \equiv T_\perp / T_{\parallel} - 1 \]

or by a loss-cone distribution function with anisotropy exponent \( m = A \) (e.g. Cuperman and Gomberoff, 1977), we will assume a bi-Maxwellian distribution function, namely
\[ f_0 \equiv f_{II} = \frac{1}{2\pi} \exp\left(-\frac{\nu_+^2}{2\nu_+^2}\right) \cdot \frac{1}{\nu_{II}^2} \exp\left(-\frac{\nu_+^2}{2\nu_{II}^2}\right) \]  

where

\[ \nu_+ \equiv \left(\frac{K}{m}\right)^{\frac{1}{4}}, \quad \nu_{II} \equiv \left(\frac{K_{II}}{m}\right)^{\frac{1}{4}} \]

Upon substitution of eq. IV.2 into eq. IV.1, one obtains the following linearized dispersion relation for electromagnetic ion-cyclotron waves (left-hand polarization) propagating along the static and homogeneous magnetic field \( B_0 \):

\[ n^2 = c_0^2 \frac{k^2}{\omega^2} = 1 + \sum_{A} \frac{\omega_p^2}{\omega k} \left[ 1 + \frac{(k\nu_+ / \omega) A}{\nu_-} \right] d\nu_+ \]  

\[ A_\xi \equiv \left(\frac{T_1}{T_1}\right) - 1; \quad \xi_\xi \equiv (\omega - \Omega_\xi) / k \]

Expressing the integral part in IV.3 in terms of the plasma dispersion function

\[ Z(\xi) \equiv \left(\frac{1}{n\eta}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left[ \exp(-t^2)/(t-\xi) \right] d\xi \]

one may rewrite it as

\[ c_0^2 k^2 = \omega^2 + \sum_{\xi_\xi} \frac{\omega_p^2}{\xi_\xi} \left\{ A - \frac{1}{62} \nu_{II}^2 \right\} Z(\xi) \left[ (A+1)(\Omega - \omega) - \Omega \right] \]

\[ \xi_\xi \equiv (\omega - \Omega_\xi) / \nu_+ \cdot \nu_{II} \cdot \xi_\xi \cdot \xi \]

Assuming that 'warm' plasma components coexist together with 'cold' plasma components, and using for the 'cold' parts the asymptotic expansion of \( Z(\xi) \), eq. IV.6 becomes
where $\sum_w$ and $\sum_c$ indicate summations over warm (w) and cold (c) components, respectively.

For a plasma system consisting of warm and cold protons and cold electrons, eq. IV.7 can be rewritten in the following dimensionless form (Cuperman et al., 1975)

$$
2 \frac{\beta^2}{\rho^2} = \alpha \left\{ \frac{v_A \overline{\omega}}{c} \right\}^2 + \left\{ A - \frac{1}{\sqrt{2} R} \right\} \frac{Z}{R} \left[ \frac{\overline{\omega} - 1}{\sqrt{2} R} \right] \left[ (A + 1)(1 - \overline{\omega}) - 1 \right]
$$

where

$$
\overline{\omega} = \omega / \Omega_P ; \quad \overline{k} = k v_{th} / \Omega_P ; \quad v_A = \left[ B_0^2 / 4 \pi m_p (n_{p,w} + n_{p,c}) \right]^{1/2}
$$

$$
\beta = 8 \pi n_{p,w} K T_{p,w} / B_0^2
$$

($v_A$ - Alfvén velocity). We will now solve eq. IV.7 for complex frequency $\tilde{\omega} = \omega_r + i \omega_i$ and real wave number of the disturbance $k = E_k$.

### i) Marginal stability analysis

For a plasma system consisting of warm, anisotropic protons and cold protons, instability threshold values of $\omega_r$, and $k$ may be found in a rather simple way, by analysing the marginally stable wave for which the growth rate $\omega_i$ is zero. Thus, taking $\omega_i = 0$ and separating IV.7 into its real and imaginary parts, one finds
\[ c^2 k^2 = \omega_{ki}^2 + \omega_{pp}^2 \{ A_p - \frac{1}{\nu_{hi} k^2} [ (A_p + 1) (\Omega_p - \omega_r) - \Omega_p ] \} \]

\[ \cdot \left( \text{Re} \ Z (\xi_p) \right)^2 \omega_{pp}^2 \omega_{rr} - \omega_{pe}^2 \frac{\omega_r}{\omega_{rr} - \Omega_p} \]

\[ \text{Im} \ Z (\xi_p) = 0 \]

where \( \nu_{th} \equiv v_{th, p} \). Since \( \text{Im} \ Z \neq 0 \) for real argument, from IV.9 one obtains the maximum unstable frequency

\[ \omega_{r,m} = [ A_p / (A_p + 1) ] \Omega_p \]

Next, from IV.8 and IV.10, one obtains the maximum unstable wavenumber

\[ k^w_m = \frac{\omega_{pp} w A_p}{c \left( \frac{\Omega_p}{(A_p + 1)^{1/2}} \right)^2 \left( \frac{\Omega_p}{\omega_{pp,w}} \right)^2 \left( \frac{1}{A_p + 1} \right)^{1/2} } \]

The last term in the curly brackets represents the displacement current. In obtaining IV.11 we considered the system electrically neutral: \( n_e = n_{p,w} + n_{p,c} \). Also, in deriving IV.11 we neglected a term \( A_p \) as compared with \( (A_p + 1) (\Omega_e / \Omega_p) \) in the electron contribution, and used the notation

\[ \alpha_p = 1 + n_{p,c} / n_{p,w} \]

Inspection of IV.10 and IV.11 indicates the following general results:

(a) If only warm protons and cold electrons are present (i.e. \( n_{p,c} = 0 (\alpha_p = 1) \)) the maximum unstable frequency \( \omega_{r,m} \) is given by IV.10 while the maximum unstable wavenumber is

\[ k^w_m = \frac{\omega_{pp} w A_p}{c (A+1)^{1/2}} \left\{ 1 + \left[ \frac{\Omega_p}{\omega_{pp,w}} \right]^2 \right\}^{1/2} \]

IV.13
(b) If warm protons are added to the system, $\omega_{r,m}$ remains unchanged (still given by IV.10), while $k_{m}^{w+C}$ (given by IV.11) may be significantly different as compared with case (i). Thus, independently of the behaviour of the maximum growth rate, the addition of cold protons increases the instability range in $k$ space by a factor

$$\frac{k_{m}^{w+C}}{k_{m}^{w}} = \left[ \frac{\omega_{P}^{2} - \omega_{W}^{2}}{\omega_{W}^{2}} \right]^{1/2}$$

When the displacement current is small, $(\Omega_{p}/\omega_{W})^{2} / (A + 1) \ll 1$, one has

$$k_{m}^{w+C} / k_{m}^{w} \sim \omega_{p}^{1/2}$$

ii) Analytical solution of the dispersion relation

Assuming (i) $\omega_{i} \ll \omega_{r}$, (ii) $V_{r} \equiv [(\Omega_{p} - \omega)/k] \gg V_{th,p}$, (iii) $k^{2}c^{2}/\omega^{2} \gg 1$ and (iv) $k$ is given by the 'cold' plasma dispersion relation, from IV.7, one obtains the approximate analytical expressions (Cornwall and Schulz, 1971; Cuperman et al., 1975):

$$\delta \equiv \frac{\omega_{i}}{\Omega_{p}} = \frac{1}{L^{2}} \frac{(1-x)^{3/2}}{x^{2}(2-x)} \left[ \frac{(A+1)(1-x)}{(\beta/n)^{1/2}} \exp \left\{ \frac{(1-x)^{3}}{x^{2}p_{B}^{2}(\beta/n)} \right\} \right]$$

and

$$\left( kV_{A}/\Omega_{p} \right)^{2} = x^{2}(1-x)^{-1}, \quad x \equiv \omega_{r}/\Omega_{p}$$

To obtain these results we have i) used the asymptotic expansion for the plasma dispersion function $Z(\xi)$, ii) written $\omega \equiv \omega_{r} + i\omega_{i}$, iii) equated to zero the real and imaginary parts of the dispersion relation and iv) neglected the higher powers of the small quantity $\gamma \equiv \omega_{i}/\Omega_{p}$

The analytical results IV.14 - IV.15 are represented in Fig. 11 by dashed curves; the solid curves represent exact numerical results obtained from the
FIG. 11. Rates of growth $\gamma_k = \omega_{t,k}/\Omega_p$ and real frequencies $\omega_k = \omega_{r,k}/\Omega_p$ as a function of $\tilde{k} \equiv kv_{th,i}/\Omega_p$ for the parallel electromagnetic ion-cyclotron instability, for several values of the ratio of cold to warm plasma densities (indicated on each curve): $\beta_p = 0.1$; $A = 0.5$. Full line – numerical; dashed line – analytic.
solution of eq. IV.7'. As is seen, for low \( \beta \) values the agreement between the analytical and the numerical results is very good; the agreement is only satisfactory for higher \( \beta \)-values. Notice that, as already found from the marginal analysis stability, the addition of cold plasma has the important consequences (i) that it increases the instability range in \( k \) space, and (ii) that it increases the (relative) maximum rate of growth of the instability \( \gamma_{\text{max}} \) as expected.

iii) Absolute maximum growth rate

Fig. 11 also indicates that, for given \( \beta \) and \( A \) values, there is an 'optimum' value of the relative cold-plasma concentration \( n_{p,c}/n_{p,w} \), corresponding to an 'absolute' maximum rate of growth \( \gamma_{\text{abs max}} \). That is, further increase of \( n_{p,c}/n_{p,w} \) above the optimum value leads to a decrease of \( \gamma_{\text{max}} \) below \( \gamma_{\text{abs max}} \). We wish to obtain from IV.14 simple analytical expressions for the 'optimum' relative cold-plasma concentration, and for the corresponding (absolute) maximum growth rate, for given \( \beta \) and \( A \) values. These expressions should be used for qualitative analysis, and for quick estimates of expected effects.

As a first step, we calculate, for fixed \( x = \omega /\Omega_p \) mode, the maximum growth rate as a function of the relative cold plasma concentration

\[
\alpha = 1 + n_{p,c}/n_{p,w}
\]

Thus, from \( \partial \gamma / \partial \alpha = 0 \), one easily obtains

\[
\tilde{\alpha}_x = 2(1-x)^3/3x^2 \beta
\]

IV.16

where \( \tilde{\alpha}_x \) indicates the (optimum) \( \alpha \) value that maximizes the mode \( x \). Denoting the exponent in IV.14 by

\[
Q = (1-x)^3/\alpha \beta x^2
\]

IV.17
and, inserting IV.16 into IV.17, one obtains

\[ \tilde{Q} = \frac{3}{2} \]  

IV.18

i.e. for all \( x \) modes, at optimum \( \omega C \) values, the exponent is the same and equal to 1.5. This is a rather important result which will be very useful in the sequel.

Next we calculate, for fixed \( \omega C \), the maximum growth rate as a function of \( x \).

Thus, from \( \partial \gamma / \partial x = 0 \), after some algebra

\[ \frac{6Q - 5}{2(1-x)} + \frac{2(Q-1)}{x} - \frac{A+1}{A(1-x)} - \frac{1}{2-x} = 0 \]  

IV.19

By IV.17, \( Q \equiv (1-x)^{3/2} \omega C \beta x^2 \).

Now, for \( \omega C \equiv 1 + n_p c/n_p , w = 1 (n_p, c = 0) \), by solving IV.19 one obtains the value of \( x \), say \( x_1 \), which maximizes \( \gamma (\omega C = 1) \); then, by substituting in IV.14 the value found for \( x_1 \) one obtains \( \gamma_{\text{max}} (\omega C = 1) \). For \( \omega C = \omega C_{\text{opt}} \), one obtains the value of \( x \), say \( x_2 \), that optimizes \( \gamma (\omega C_{\text{opt}}) \) by solving IV.14 with \( Q = 1.5 \);

upon substitution of \( x_2 \) into IV.14 one finds \( \gamma_{\text{max}} (\omega C_{\text{opt}}) \). Both \( \gamma_{\text{max}} (\omega C = 1) \) and \( \gamma_{\text{max}} (\omega C_{\text{opt}}) \) have been calculated numerically by solving IV.19 in the manner described above (see figure 13 and below). But at this point we want to obtain analytical expressions for \( x_1, x_2, \omega C_{\text{opt}} \) and \( \gamma_{\text{max}} \).

By IV.19 and IV.18, the equation for \( \tilde{x}_2 \) reads

\[ (A+1) \tilde{x}_2^3 - (A-1) \tilde{x}_2^2 - 2(A+2) \tilde{x}_2 - 2A = 0 \]  

IV.20

Since \( \tilde{x}_2 < x_{\text{marg}} \equiv A/(A+1) < 1 \), and observing that the coefficient of \( \tilde{x}_2 \) is larger than those of \( \tilde{x}_2^3 \) and \( \tilde{x}_2^2 \), by successive approximations one obtains the following solutions for \( \tilde{x}_2 \), to first and second order:

\[ \tilde{x}_2^{(1)} \approx \frac{A}{A+2} \]  

and\[ \tilde{x}_2^{(2)} \approx \frac{A}{A+2} \left( 1 + \frac{2A}{A^3 + 11A + 20A + 16} \right) \]  

IV.21a,b
From IV.16 and IV.21a it follows that \( \zeta^{(1)} \approx \frac{16}{3} \beta (A + 2) \). For \( x_1 \) one may rewrite IV.19:

\[
(A+1)(2Q-1)x_1^3 - (2AQ-A-2)x_1^2 - 4(2AQ+2Q-2A-1)x_1 + 8A(Q-1) = 0
\]

For values \( Q \geq 1.2 \) (this corresponds to the coefficient of \( x_1 \) being larger than those of \( x_1^3 \) and \( x_1^2 \)), by successive approximations, one obtains the following solutions for \( x_1 \), to first, second, and third order \( (Q_m = \sqrt{3A(A+1)} \):

\[
X_1^{(1)} \approx \frac{A}{A+1} \left\{ 1 - \frac{1}{Q_m(3A+2)} \right\}
\]

\[
X_1^{(2)} \approx \frac{A}{A+1} \left\{ 1 - \frac{1}{Q_m(3A+2) - (7A/2 + 1)} \right\}
\]

and

\[
X_1^{(3)} \approx \frac{A}{A+1} \left\{ 1 - \frac{1}{Q_m(3A+2) + (1 - 7A/2)} \right\}
\]

Thus, substitution of \( x_1 \) (IV.23) into (IV.14) gives \( \gamma_{\text{max}} (\zeta_{\text{opt}} = 1) \), while substitution of \( \zeta \) (IV.16) , \( \tilde{x}_2 \) (IV.21b) and \( \tilde{Q} \) (IV.18) into IV.14 gives \( \gamma_{\text{max}} (\zeta_{\text{opt}}):

\[
\gamma_{\text{max}} (\zeta_{\text{opt}} = 1) = \left( \frac{\pi}{\beta} \right)^{1/2} \frac{A(1-x_1)}{x_1^2(2-x_1)} (1-x_1)^{5/2} \exp \left[ - \frac{(1-x_1)^3}{\beta x_1^2} \right]
\]

\[
\gamma_{\text{max}} (\zeta_{\text{opt}} = \zeta_{\text{opt}}) = \left( \frac{27\pi^3}{4} \right)^{1/2} \beta \frac{\tilde{x}_2^2}{(2-\tilde{x}_2)(1-\tilde{x}_2)^2} \cdot \left[ A(1-\tilde{x}_2) - \tilde{x}_2 \right] \exp (-1.5)
\]

The result IV.25 is represented in Fig. 12 by the dashed curves. For comparison, numerical solutions obtained by solving eq. IV.7' are also given. First,
it is seen again that for low \( \beta \)-values the agreement is very good. In fact, defining a parameter \( P \equiv \beta A_\rho (A_\rho +1) \), one finds that for \( P \ll 1 \) the agreement is very good and that it worsens with increasing \( P \).

iv) Relative enhancement due to addition of cold plasma

The relative enhancement due to the addition of cold plasma is easily obtained from IV.24 and IV.25 as the ratio

\[
F(\beta, A) = \frac{\delta_{\text{opt}} (\xi, \beta, A)}{\delta_{\text{max}} (\xi = 1, \beta, A)}
\]
As a last point, we look for a simple expression giving the relative enhancement \( F \) of the electromagnetic ion-cyclotron instability due to the addition of cold plasma. To first order in \( x_1 \) and \( x_2 \), \( IV.26 \) for \( F \) reads

\[
F^{(iv)}(p, A) \equiv F(A, p) \equiv \beta A^2 (A+1)
\]

\[
= \left[ \frac{27}{128} p^\frac{1}{2} \frac{A+2}{(A+1)^2} \frac{(1-P/A)^2}{(1+AP/A)^5/2} \right] \frac{3A^2 + 8A + 4 + AP}{A + 4} 
\cdot \exp \left\{ \frac{(1+AP/A)^3}{(1-P/A)^2} \right\}
\]

where \( \alpha = 3A + 2 \). Since the validity of our analytic expressions improves for small \( P \) values, and moreover it is also precisely in this (small) \( P \) range values that significant enhancement occurs, we consider this last case, and find, for \( P/\alpha \ll 1 \),

\[
F^{(iv)} \sim \left[ \frac{27}{128} \right] \exp \left\{ \frac{1}{1.5} \cdot \frac{3A+2}{A+4} \cdot \frac{(A+2)^2}{(A+1)^2} \right\} \cdot P^{1/2}
\]

In the exponential, \( P/\alpha \) should not be neglected. The simple expression \( I I I.28 \) indicates that, for given \( A \),

\[
F \propto P^{1/2} \exp \{P^{-1} f(P)\}
\]

where \( f(P, A) \) is a relatively slowly varying function of \( P \). Thus \( F \) is very large for \( P \ll 1 \) values, and decreases with increasing \( P \). These results are represented in Fig. 13. The strong enhancement of the electromagnetic ion-cyclotron instability, with the addition of cold plasma for small \( P \) is clearly demonstrated.

v) Summary

The main theoretical results obtained in this section are:

a) A plasma system consisting of warm and anisotropic protons (\( \beta_p = \frac{8\pi n_p}{\omega K} \cdot T_{\perp,p}/B_0^2 \neq 0, A_p \neq 0 \)) and electrons is unstable against the electromagnetic
ion-cyclotron (left-hand polarization waves) instability; the maximum growth rate increases with increasing $\beta_p$ and $A_p$;

b) Analytical expressions fit exact numerical results satisfactorily when

$$P_p \equiv \beta_p A_p^2 (A_p + 1) < 1;$$

\[ c) \text{When cold protons (density } n_{p,c} \text{) are added to a warm anisotropic plasma, the following effects are found:} \]

1. The growth rate of any mode $x = \omega_p / \Omega_p$ is maximized for a relative cold plasma concentration $n_{p,c} / n_{p,w} = [(1-x)/1.53]^{1/2} - 1$.

2. The absolute maximum growth rate occurs for an optimum $n_{p,c} / n_{p,w}$ value of about $[16/3 \beta_p (A_p + 2)]^{-1}$.
3. A significant enhancement occurs only if $P_p = P_p A_p^2 (A_p + 1) < 1$. The enhancement decreases exponentially with increasing $P_p$ values.

4. In all cases where $n_{p,c}/n_{p,w} \neq 0$ the instability range in $k$ space is increased by a factor of about $\left(1 + n_{p,c}/n_{p,w}\right)^{1/2}$. (Here, $n_{p,w}$ and $n_{p,c}$ are the warm and cold proton particle densities, respectively.)

2. Electromagnetic electron cyclotron instability

Results analogous to those obtained above can be derived for the electromagnetic electron cyclotron waves (i.e. whistler waves, having right-hand polarization) propagating along a static and homogeneous magnetic field. The reader is referred to the published works on this subject (e.g. Kennel and Petschek, 1966; Cuperman and Landau, 1974). As for the ion branch of the instability, the analytical predictions were confirmed by computer simulations (e.g. Cuperman and Salu, 1973). The more difficult oblique propagation case was also investigated by computer simulation experiments (e.g. Cuperman and Sternlieb, 1974).

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Figures 4—13 are reproduced from the Journal of Plasma Physics.

BIBLIOGRAPHY

PLASMA EQUILIBRIUM IN MAGNETIC CONFINEMENT DEVICES AND STABILITY OF MIRROR MACHINE PLASMAS

R.K. VARMA
Physical Research Laboratory,
Ahmedabad,
India

Abstract

PLASMA EQUILIBRIUM IN MAGNETIC CONFINEMENT DEVICES AND STABILITY OF MIRROR MACHINE PLASMAS.

Plasma confinement by magnetic fields; open and closed drift surfaces; remedies for 'open' drift surfaces; methods of introducing rotational transfer. Plasma equilibrium — criteria for closed drift surfaces: open-ended systems; toroidal systems with and without rotational transform; effect of particles on vacuum fields. Stability of plasmas in mirror machines: instability of a low-\(\beta\) plasma in a mirror machine; neutral injection mirror machines — the low-density regime.

1. INTRODUCTION

Confinement of charged particles by appropriate magnetic field configurations (magnetic bottles) has long been recognized as the most important means of confining high-temperature reactants of thermonuclear fusion reactions. It is at present the most widely adopted means of confinement. The 'appropriate' configuration, however, still eludes us and the search continues.

We present some basic general considerations for plasma equilibrium in magnetic confinement devices with reference to particle drifts and drift surfaces. Later we specialize to the case of the simple mirror machine and discuss its stability with respect to the most important fluid perturbations. Some means of stabilizing these 'flute' instabilities are also discussed.

Because of the manner of particle confinement in mirror machines, the plasma here is characterized by an anisotropic velocity distribution ('loss-cone' distribution), which displays a positive slope with respect to a certain component of the velocity and thus leads to a velocity space instability. These instabilities cannot be discussed here, for lack of space.
2. PLASMA CONFINEMENT BY MAGNETIC FIELDS

Two types of confinement geometry are considered: open-ended configurations, such as the mirror machine, and closed-ended configurations, such as the toroidal machines (tokamaks, stellarator, levitrons, etc.).

The mirror machine consists basically of a magnetic field region bounded on either side by regions of stronger magnetic field, like the one shown in Fig. 1(a). A charged particle is confined (adiabatically) in the weaker magnetic field region provided the total energy of the particle \( \mathcal{E} \) is less than the height of the adiabatic potential hump \( \mu B_{\text{max}} \), corresponding to the maximum of magnetic field \( B_{\text{max}} \):

\[
\mathcal{E} < \mu B_{\text{max}}
\]  

(Fig.1(b)) where \( \mu \) is the well-known adiabatic magnetic moment invariant:

\[
\mu = \mathcal{E}_\perp /B = \frac{1}{2} mv_\perp^2/B
\]  

and \( v_\perp \) is the velocity of the particle perpendicular to the magnetic field. If \( \theta_0 \) denotes the 'pitch angle' of the particle in the midplane of the mirror machine, where the magnetic field is \( B_0 \), so that \( v_\perp = v \sin \theta_0 \) and \( \mathcal{E}_\perp = \mathcal{E} \sin^2 \theta_0 \), then the inequality (2.1) reduces to

\[
\sin^2 \theta_0 > \frac{B_0}{B_{\text{max}}}
\]  

FIG.1. (a) Simple magnetic mirror field configuration, with weak magnetic field in the middle bounded by strong magnetic field on either side. (b) Schematic representation of the adiabatic potential as a function of the coordinate \( x \) along a field line of the mirror configuration of (a). The total energy of the particle \( \mathcal{E} < \mu B_{\text{max}} \) leads to adiabatic trapping of the particle.
All such plasma particles which satisfy the inequalities (2.1) or (2.3) are trapped adiabatically in the mirror machine. Conversely, particles that do not satisfy these inequalities will not be present in the trap. Obviously, the distributions of particles in such confining configurations will be anisotropic in the velocity space, and the confined plasmas will be subject to velocity-space instabilities.

While the mirror machines employ stronger magnetic fields at the two ends to reflect and confine the particles (through the invariance of the magnetic moment $\mu$) along the magnetic field lines, the closed-ended configurations seek to achieve this objective by joining the two ends, and thus having the plasma particles move along closed magnetic field lines.

### 2.1. Open and closed drift surfaces

While the particles move along the magnetic field lines to the lowest order in the smallness of the Larmor radius (in both the open-ended and closed-ended systems) they execute, in the next order, VB drifts, given by [1]

$$\vec{v}_D = \frac{m}{eB} \left( \frac{1}{2} \vec{v}_1^2 - \vec{v}_2^2 \right) \vec{B} \times \nabla \vec{B} \quad \text{B}^2$$

Since the drift depends on the sign of the charge, oppositely charged particles drift in opposite directions. This leads to charge separation and polarization electric fields for finite plasmas unless the drift surfaces close on themselves within the plasma volume. The electric fields so produced then cause the $E \times B$ drift of the plasma across the magnetic field and away from the intended region of confinement.

To see this, consider a plasma in a simple torus where Fig. 2 represents its cross-section. As is well known, the magnetic field which may be produced by currents going round the torus has a gradient pointing inwards — towards the axis, as shown. The VB drifts (Eq. (2.4)) are then downwards for positive ions and upwards for electrons, and the drift surfaces are cylindrical surfaces which intersect the boundaries of the toroid. This leads to charge separation and to polarization electric field, which in turn cause the $E \times B$ motion of the plasma outwards. One therefore finds that a simple torus is unable to hold a plasma in equilibrium. As will be seen later, this is simply an expression of the diamagnetism of the plasma whereby plasma is thrown from regions of stronger into regions of weaker magnetic field.

Consider, on the other hand, a simple axisymmetric mirror machine. Let Fig. 3 represent its cross-section. The VB in this case is directed radially inwards. The VB drift is then azimuthal, being opposite in sense for the different signs of the charges. The drift surfaces here are closed within the plasma volume and
FIG. 2. Cross-section of a simple torus (without rotational transform). $\nabla B$ points radially inwards towards the symmetry axis. With the given direction of the magnetic field (pointing outwards in the right-hand section and inwards in the left-hand section) the particles $\nabla B$ drift as shown, resulting in a polarization electric field $\vec{E}$. This leads to $\vec{E} \times \vec{B}$ drift of the plasma radially outwards.

FIG. 3. Cross-section of the axisymmetric magnetic field of the mirror machine. The magnetic field $B$ points out of the plane of the paper and $\nabla B$ points radially inwards. Arrows show the direction of the $\nabla B$ drift of the positive and negative charges.

do not intersect the plasma boundary but are parallel to it. No charge separation can result here provided the plasma density is axisymmetric. It therefore follows that a simple mirror machine can hold a plasma in equilibrium. This equilibrium may yet be unstable, but this will be considered later.

2.2. Remedies for ‘open’ drift surfaces

While a toroidal configuration has many advantages, it cannot be employed unless the open drift surfaces in such configurations are effectively closed and the polarization electric fields are eliminated. There are, broadly speaking, two ways in which the open drift surfaces of a simple toroidal device can be closed.

(a) Modification without rotational transform

The magnetic field configuration can be modified and additional drifts introduced so that the resulting drift surfaces are closed, while the magnetic field lines still close on themselves after one turn round the torus. Such a situation occurs, for instance, in a bumpy torus, which is essentially a large number of mirror machines joined end to end in a toroidal form (Fig. 4). For particles not trapped in any of the constituent mirror machines, the otherwise open drift
surfaces in a simple torus are closed owing to the azimuthal drifts induced by the inhomogeneity of the mirror machines. For the particles trapped in the mirror machines the drift surfaces are, of course, closed as in a simple mirror machine.

(b) **Modification with a rotational transform**

Another way to modify the magnetic field of a simple toroid is to superpose a magnetic field $B_\theta$ round the toroidal axis of the torus. A line of force then no longer closes on itself after a single turn (as in a simple torus) but rather winds round a toroidal surface (Fig. 5).

If $B_\theta = B_\theta(r)$ and $B_r = 0$ represent the additional superposed field, the equations of a line of force are given by
Equation (2.5a) gives \( r = \text{const.} \) along a line of force. If \( B_\theta \ll B_\phi \), integrating (2.5b) gives

\[
\Delta \theta = \frac{R B_\theta}{\gamma B_\phi} \Delta \varphi \tag{2.6}
\]

If one goes once round the torus, \( \Delta \varphi = 2\pi \), the line of force winds through an angle \( \Delta \theta = \iota \)

\[
\iota = \frac{2\pi R B_\theta}{\gamma B_\phi} \tag{2.7}
\]

The angle \( \iota \) is known as the *rotational transform*. If \( \iota = 2\pi/n \), a line of force winds round the toroidal surface \( r = \text{const.} \) once (i.e. \( \Delta \theta = 2\pi \)) as it goes round it \( n \) times (\( \Delta \varphi = 2\pi n \)). In general, one may have \( \iota = 2\pi m/n \). Such lines of force close on themselves and the magnetic surfaces they define are known as *rational*. If \( \iota \neq 2\pi m/n \), a line of force does not close on itself and winds round and round the particular surface \( r = \text{const.} \), filling it up densely. The magnetic surface so defined is *irrational* (Fig. 6). The quantity \( \epsilon = \iota/2\pi \) is often used.

The rotational transform \( \iota \) is, in general, a function of \( r \). Thus two neighbouring lines of force which are separated by an infinitesimal distance \( \Delta r \) at a particular point will, in general, move apart when followed in either direction. This amounts to the magnetic field having a shear. \( d\iota/dr \) is thus known as the shear of the magnetic field. The addition of a small \( B_\theta \) field to the major toroidal field \( B_\phi \) makes but a small correction to the \( \nabla B \) drift if \( B_\theta \ll B_\phi \).

Thus if (see Fig. 7)

\[
\vec{B} = B_\phi \hat{\varphi} + B_\theta \hat{\theta}
\]

the expression (2.4) for the drift is given by

\[
\vec{v}_d = \frac{m \left( \vec{v}_i^2 + \frac{1}{2} \vec{v}_c^2 \right)}{eB^3} \left\{ - \frac{B_\phi}{B} \frac{\hat{\varphi} \times \frac{\partial B_\phi}{\partial r}}{R^3} \hat{r} + \frac{B_\phi}{B} \frac{\partial B_\phi}{\partial r} + \frac{B_\theta}{B} \frac{\partial B_\phi}{\partial \varphi} \frac{\partial \varphi}{\partial r} \hat{\varphi} \right\}
\]

\[
\tag{2.8}
\]
One sees that the dominant drift is still the first term of the expression (2.8) which is in the z-direction and represents the contribution of the simple toroidal field. These drifts of electrons and ions along the z-direction would still lead to charge separation and polarization electric field as in a simple toroid. With the presence of the rotational transform, however, the portions of the torus above and below the plane of symmetry $z = 0$ are now connected with magnetic field lines. This allows the charges to flow along the field lines and thereby short-circuit the polarization electric field. The $E \times B$ expulsion of the plasma is thus prevented.

2.3. Methods of introducing rotational transform

Rotational transform may be introduced in any of the following ways, which lead to different forms of confinement schemes:

(a) Toroidal current carried by plasma
If the $B_\theta$ is produced by a toroidal current induced in plasma, what results is a tokamak type of machine. In such a scheme the field $B_\theta$ generally decreases away from the centre of the plasma discharge (Fig. 5).
FIG. 8. Sectional view of a stellarator. Broken lines are the lines of the toroidal field; solid field lines are the result of the helical windings. Adjacent windings carry opposite currents.

(b) Helical winding

The magnetic field $B_0$ and the rotational transform may be produced by a set of helical windings over the external surface of the toroid, carrying different signs of the current alternately. This gives rise to the stellarator type of scheme. The magnetic field $B_0$ here increases outwards from the centre of the plasma discharge (Fig. 8).

**Problem:** Find out the magnetic surfaces for a straight stellarator with helical winding of order $\ell$. Use the following magnetic potential to derive the magnetic field:

$$\chi = B_o \left[ z + \epsilon \left( \frac{r}{r_0} \right)^{\ell} \cos(\ell \theta - k z) \right]$$

3. PLASMA EQUILIBRIUM – CRITERIA FOR CLOSED DRIFT SURFACES

The question of the plasma equilibrium in a magnetic field configuration can most generally be considered through the steady-state solution $\partial f/\partial t = 0$ of the Boltzmann equation:

$$\mathbf{v} \cdot \nabla f + \frac{e}{m} \mathbf{v} \times \mathbf{B}_0 \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

(3.1)
with the equilibrium electric field zero. Since the magnetic fields are generally large, the Larmor radii of particles $a_L$ are small compared to the characteristic lengths $L$ of variation of the magnetic fields and other macroscopic quantities:

$$a_L \ll L, \quad L \approx \left[ \frac{1}{B |\nabla B|} \right]^{-1}, \quad \varepsilon \equiv \frac{a_i}{L} \ll 1 \quad (3.2)$$

The results of the adiabatic theory hold under the condition (3.2). In particular, the magnetic moment defined as

$$\mu = \frac{1}{B} \left[ \mathbf{v}^2 - \mathbf{(v \cdot \hat{n})}^2 \right] \quad (3.3)$$

(where $\hat{n}$ is a unit vector in the direction of the magnetic field) is a constant of motion in the adiabatic approximation. Since the condition (3.2) generally holds in confinement devices, it will be convenient to change the velocity variables from $\mathbf{v}$ to

$$\mathbf{E} = \mathbf{v}^2$$

$$\mu = \frac{1}{B} \left[ \mathbf{v}^2 - \mathbf{(v \cdot \hat{n})}^2 \right] \quad (3.4)$$

$$\phi = \text{Larmor phase angle}$$

The Boltzmann equation (3.1) then reduces to

$$\mathbf{v} \cdot \left[ \frac{\partial}{\partial x} \mathbf{f} + \frac{\partial \mathbf{E}}{\partial x} \frac{\partial}{\partial v} \mathbf{f} + \frac{\partial \mu}{\partial x} \frac{\partial}{\partial v} \mathbf{f} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \mathbf{f} \right] = - \Omega \frac{\partial \mathbf{f}}{\partial \phi}$$

$$\Omega = \frac{e B_0}{mc} \quad (3.5)$$

We may now carry out a small Larmor expansion of the solution to this equation:

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \quad (3.6)$$

Note that

$$\mathbf{v} \cdot \mathbf{v} \sim v/L \quad \text{and} \quad \Omega \frac{\partial}{\partial \phi} \sim \Omega$$
so that

\[ \vec{v} \cdot \nabla f \sim \frac{\Omega}{L} \left( \Omega \frac{\partial f}{\partial \phi} \right) \sim \frac{a_1}{L} \Delta \frac{\partial f}{\partial \phi} \sim \epsilon \Omega \frac{\partial f}{\partial \phi} \]

\[ \nabla \mu \frac{\partial f}{\partial \mu}, \nabla \phi \frac{\partial f}{\partial \phi} \sim \epsilon \Omega \frac{\partial f}{\partial \phi} \] \hspace{1cm} (3.7)

while \( \partial \xi / \partial x = 0 \). Making use of (3.7) and substituting the expansion (3.6) in Eq. (3.5), we get, to various orders

\[ \Omega \frac{\partial f_0}{\partial \phi} = 0 \] \hspace{1cm} (3.8a)

\[ \Omega \frac{\partial f_1}{\partial \phi} + \vec{v} \left[ \nabla + \nabla \mu \frac{\partial}{\partial \mu} + \nabla \phi \frac{\partial}{\partial \phi} \right] f_0 = 0 \] \hspace{1cm} (3.8b)

\[ \Omega \frac{\partial f_2}{\partial \phi} + \mathcal{L} f_1 = 0 \text{ etc.} \] \hspace{1cm} (3.8c)

From Eq. (3.8a) we obtain, to lowest order in \( \epsilon \), the solution

\[ f_0 = f_0 (\xi, \mu, x) \] \hspace{1cm} (3.9)

i.e. the solution \( f_0 \) is independent of the phase angle \( \phi \). Further, Eq. (3.8b) can be integrated with respect to \( \phi \) to give

\[ f_1 = -\frac{1}{\Omega} \int_0^\phi \vec{v} \left[ \nabla + \nabla \mu \frac{\partial}{\partial \mu} \right] f_0 + f_1 (\phi) \] \hspace{1cm} (3.10)

Since \( f_1 \) must be single-valued:

\[ f_1 (\phi = 0) = f_1 (\phi = 2\pi) \]

(3.10) gives the condition

\[ \int_0^{2\pi} \vec{v} \left[ \nabla f_0 + \nabla \mu \frac{\partial f_0}{\partial \mu} \right] = 0 \] \hspace{1cm} (3.11)
This is easily shown to yield

\[ \nabla \cdot \left( \mathbf{N} \nabla f_0 \right) = 0 \quad (3.12) \]

Thus, in addition to (3.9) we have, to the lowest order in Larmor radius, that the distribution function \( f_0 \) is constant following the field line, so that

\[ f_0 = f_0(\mathbf{E}, \mu, s) \quad (3.13) \]

where \( s \) is a label of the magnetic field line. According to (3.13), \( f_0 \) is an arbitrary function of \( \mathbf{E} \) and \( \mu \) and can vary arbitrarily from one field line to another labelled \( s \) (having been shown to remain constant along the field line).

The consequence of (3.13) is that, to the lowest order in Larmor radius, any arbitrary function of \( \mathbf{E} \), \( \mu \) and \( s \) is an equilibrium solution of the Boltzmann equation. There are no drifts in this lowest order — only the motion along the field line. To consider the nature of drifts and drift surfaces one has to determine the next and higher order parts \( f_1, f_2, \) etc., of the distribution function. Equations (3.8b) may thus be solved for the functions \( f_1, f_2, \) etc.

One may now consider only one species and group of particles and attempt to answer the following question:

Given particles with a certain energy \( \mathbf{E}' \) and the magnetic moment \( \mu' \), what is the condition for these particles to stay on a certain closed surface? If we accordingly write

\[ f = \xi(\mathbf{E} - \mathbf{E}') \xi(\mu - \mu') g(x, \phi) \quad (3.14) \]

then one may ask for the most general form of \( g \) for the drift surfaces to be closed.

As discussed above, the most general form of the lowest-order distribution function \( f_0 \) corresponding to (3.14) is

\[ f_0 = \xi(\mathbf{E} - \mathbf{E}') \xi(\mu - \mu') g(s) \quad (3.15a) \]

where \( g(s) \) is independent of \( \phi \) and the position coordinate along the field line, and depends only on the label \( s \) of the field line. The distribution function (3.15a) implies a particle density \( n \):

\[ n = \int d^3 \mathbf{r} f_0 = \int \frac{B d\mu d\mathbf{E}}{\mathbf{E} - \mu B} g(s) \xi(\mathbf{E} - \mathbf{E}') \xi(\mu - \mu') = \frac{B g(s)}{\mathbf{E} - \mu' B} \quad (3.15b) \]
This means that the density $n$:

(a) can vary arbitrarily from one field line to another;
(b) $n \sim B$ (varies proportional to $B$);
(c) $n \sim v_i^{-1}$ (varies inversely as the $v_i$).

Since one would be interested eventually in the moment quantities like currents and drifts in the higher order in the Larmor radius, one may work directly in terms of the exact moments of the Boltzmann equation (3.1) instead of solving Eq. (3.8b) to determine $f_1, f_2, \text{etc.}$ The zeroth and first moments of Eq. (3.1) are thus

\begin{equation}
\nabla \cdot (n \vec{u}) = 0 \quad , \quad \nabla \cdot \vec{P}' = \frac{e}{c} n \vec{u} \times \vec{B}
\end{equation}

(3.16a)

which give

\begin{equation}
\nabla \cdot \vec{j} = 0 \quad , \quad \nabla \cdot \vec{P}' = \frac{1}{c} \vec{j} \times \vec{B}
\end{equation}

(3.16b)

where the current $\vec{j} = ne \vec{u}$ has been introduced for a single species of particles, and the pressure tensor $\vec{P}'$ is

\begin{equation}
\vec{P}' = n m \vec{u} \vec{u} + \vec{P}
\end{equation}

(3.17)

$\vec{P}$ being the usual kinetic pressure tensor.

One may now consider the small Larmor radius expansion of the moment equations (3.16b) and write

\begin{equation}
\vec{j} = \vec{j}_0 + \epsilon \vec{j}_1 + \epsilon^2 \vec{j}_2 + \cdots
\end{equation}

\begin{equation}
\vec{P}' = \vec{P}_0' + \epsilon \vec{P}_1' + \epsilon^2 \vec{P}_2' + \cdots
\end{equation}

(3.18)

where $\vec{P}_0'$, $\vec{P}_1'$, etc., and $\vec{j}_0, \vec{j}_1$, etc., are the moments of the zeroth and first-order, etc., parts of the distribution function $f_0, f_1$, etc. Since

\[ \frac{1}{c} \vec{j} \times \vec{B} \sim mn u \Omega \]
PLASMA EQUILIBRIUM

\[ \nabla \cdot \mathbf{J} = n m v_n^2 / L = n m v_n \left( \frac{v_n}{\Omega_L} \right) \Omega = n m v_n e \Omega \]

the \( \mathbf{j} \times \mathbf{B} \) is a lower-order term in \( \epsilon \) because it is proportional to \( B \). Equations (3.16b) then give

\[ \nabla \cdot \mathbf{j}_o = 0 \quad , \quad \nabla \cdot \mathbf{j}_1 = 0 = \nabla \cdot \mathbf{j}_z = \ldots \quad (3.19) \]

\[ \frac{1}{c} \mathbf{j}_1 \times \mathbf{B} = \nabla \cdot \mathbf{P}_p' \quad , \quad \frac{1}{c} \mathbf{j}_2 \times \mathbf{B} = \nabla \cdot \mathbf{P}_p' \quad , \quad \text{etc.} \quad (3.20) \]

According to (3.19), the divergence of the currents in different orders must individually vanish, while according to (3.20) the first-order current \( \mathbf{j}_1 \) is determined in terms of the zero-order pressure \( P_0 \) and hence in terms of the zero-order distribution function \( f_0 \), as given by Eq. (3.15a).

If the \( \mathbf{n}_i \) denote unit vectors, then

\[ \mathbf{P}_{ij} = \mathbf{P}_{ij} + \mathbf{P}_{ij} - (\mathbf{P}_{ij} - \mathbf{P}_{ij}) \mathbf{n}_i \mathbf{n}_j \quad (3.21) \]

where \( p_{10} \) and \( p_{10} \), the perpendicular and parallel components of pressure, can be calculated using \( f_0 \) of Eq. (3.15a):

\[ p_{10} = \int d^3 \mathbf{v} \frac{1}{2} m v_n^2 f_0 = \int \frac{B d\mu d\xi d\phi}{\xi - \mu B} \frac{1}{2} m \xi B f_0 \]

\[ = \frac{\pi m \xi B}{\xi - \mu B} \zeta(s) \quad (3.21) \]

\[ p_{10} = \int d^3 \mathbf{v} v_n^2 f_0 = \frac{\pi m (\xi - \mu B)}{\xi - \mu B} g(s) \quad (3.22) \]

The first-order current \( \mathbf{j} \) is now determined in terms of the zero-order pressure, from Eq. (3.20), as

\[ \mathbf{j}_1 = \frac{\mathbf{B} \times \nabla \mathbf{P}_0}{B^2} + \mathbf{j}_1 \frac{\mathbf{B}}{B} \quad (3.23) \]
where \( j_1 \) is the ‘parallel’ current which is not determined from the momentum balance equation (3.20); \( \nabla \cdot j_1 = 0 \), finally gives

\[
\nabla \cdot j_1 = \nabla \left( \frac{\vec{B} \times \nabla \cdot \vec{P}}{B^2} \right) + B \frac{\partial}{\partial \ell} \left( \frac{J_{\|}}{B} \right) = 0
\]

(3.24)

where \( \ell \) is the coordinate along the field line. Equation (3.24) can be integrated to give

\[
J_{\|} = -B \int_{\ell_0}^{\ell} \frac{d\ell}{B} \nabla \left( \frac{\vec{B} \times \nabla \cdot \vec{P}}{B^2} \right) + J_{\|}(\ell_0)
\]

(3.25)

Using the expression (3.21) for \( \vec{P} \), we have now

\[
\nabla \cdot \vec{P} = \nabla \cdot \vec{P}_x + \vec{B} \cdot \nabla \left( \frac{p_u - p_\perp}{B^2} \right) + \frac{p_u - p_\perp}{B^2} \vec{B} \cdot \nabla \vec{B}
\]

Assuming the magnetic field \( \vec{B} \) to be curl-free (no significant plasma currents),

\[
\vec{B} \cdot \nabla \vec{B} = -\vec{B} \times (\nabla \times \vec{B}) + \frac{1}{2} \nabla B^2 \approx \frac{1}{2} \nabla B^2
\]

so that, dropping the subscript zero everywhere,

\[
\nabla \cdot \vec{P} = \nabla \cdot \vec{P}_x + \frac{1}{2} \left[ (p_u - p_\perp)/B^2 \right] \nabla B^2
\]

and

\[
\nabla \cdot (\vec{B} \times \nabla \cdot \vec{P}/B^2) = \nabla \cdot \left\{ \frac{\vec{B} \times \nabla \vec{P}_x}{B^2} + \frac{1}{2} \left( p_u - p_\perp \right) \frac{\vec{B} \times \nabla B^2}{B^2} \right\}
\]

\[
= \nabla \cdot \left\{ -\nabla \times \left( \vec{P}_x \vec{B}/B^2 \right) - \vec{P}_x \vec{B} \times \nabla \left( \vec{P}/B^2 \right) - \frac{1}{2} \left( p_u - p_\perp \right) \vec{B} \times \nabla \left( \vec{P}/B^2 \right) \right\}
\]

\[
= -\nabla \cdot \left\{ \frac{1}{2} \left( p_u + p_\perp \right) \vec{B} \times \nabla \left( \vec{P}/B^2 \right) \right\}
\]

(3.26)
It may be noticed that the $\nabla B$ drift has been recovered for a curl-free field. Using (3.26) in (3.25), we get

$$J_\parallel = - \frac{\kappa}{B} \nabla \cdot \left\{ \frac{1}{2} \left( |p_\parallel + p_\perp| \hat{B} \times \nabla \left( \frac{1}{B^2} \right) \right) \right\} + J_\parallel (L_0)$$  \hspace{1cm} (3.27)

We now consider the following cases: (a) open-ended systems (mirror machine, etc.); (b) toroidal systems without rotational transform (bumpy torus, etc.) and toroidal systems with rotational transform (tokamaks and stellarators).

3.1. Open-ended systems

For open-ended systems like the mirror machine, the current $j_B$ vanishes at the two ends where the plasma density vanishes. Hence Eq. (3.27) gives

$$\int_{\ell_1}^{\ell_2} \frac{d\ell}{B} \nabla \cdot \left\{ \left( |p_\parallel + p_\perp| \right) \hat{B} \times \nabla \left( \frac{1}{B^2} \right) \right\} = 0$$  \hspace{1cm} (3.28)

Using the expression (3.22) for $p_\parallel$ and $p_\perp$ corresponding to the distribution function (3.15a), we get

$$\int_{\ell_1}^{\ell_2} \frac{d\ell}{B} \nabla \cdot \left\{ \pi m \frac{B}{\epsilon - \mu B} \frac{q(s)}{\sqrt{\epsilon - \mu B}} \hat{B} \times \nabla \left( \frac{1}{B^2} \right) \right\} = 0$$  \hspace{1cm} (3.29)

as the condition for drift surfaces to be closed. As a particular case we see that, for an axisymmetric simple mirror machine, if $g(s)$ is fixed so that it is axisymmetric, and a function of only $r$ and $z$, then $\nabla g \times \hat{B}$ has only a non-zero $\theta$-component; it has a vanishing divergence since the whole system is axisymmetric. Equation (3.29) is thus trivially satisfied.
3.2. Toroidal systems with and without rotational transform

We next consider toroidal systems without rotational transform where the lines of force close on themselves after one circuit round the torus. Then from the single-valuedness of \( j_B \) we have, on integrating round the line of force,

\[
\oint \frac{dl}{B} \nabla \cdot \left\{ \left( |p_1| + |p_2| \right) \vec{B} \times \nabla \left( \frac{1}{B^2} \right) \right\} = 0
\]  

(3.30)

3.2.1. Systems without rotational transform

(a) Applying it to a simple torus, we find that \( \vec{B} \times \nabla B^{-2} \), which is proportional to the particle drift, is in the z-direction, and its divergence is everywhere non-zero with the same sign since \( p_1 \) and \( p_\perp \) vary with \( z \) in the same manner everywhere. Hence the condition (3.30) cannot be satisfied for a simple torus.

(b) One may apply this criterion to a bumpy torus — another toroidal system without a rotational transform. The treatment of the two particles — the 'trapped' and 'passing' particles — will be different. The trapped particles will behave as in a mirror machine, and the condition (3.29) will be trivially satisfied for them. The passing particles, on the other hand, will be subject to a combination of drifts: a vertical drift due to the toroidal curvature and an azimuthal drift due to the inhomogeneity of the magnetic mirror fields. The condition obtained from (3.30) on using expressions (3.22) for \( p_1 \) and \( p_\perp \) must then be satisfied for drift surfaces to be closed for such a system.

\[
\oint \frac{dl}{B} \nabla \cdot \left\{ \frac{1}{B} \left( \mathcal{E} - \mu B \right)^{\frac{1}{2}} \vec{B} \times \nabla g \right\} = 0
\]  

(3.31)

3.2.2. Systems with rotational transform

Equation (3.31) provides the condition for toroidal systems with rotational transform also.

Equations (3.29) and (3.31) may be expressed in an alternative form. If we sum them over a flux tube with the elemental flux \( d\Psi = \vec{B} \cdot d\vec{S} \), where \( d\vec{S} \) is the elemental area, we get
The integral may be transformed by using the Gauss theorem, yielding

\[ \oint d\mathbf{S} \cdot \mathbf{B} \times \nabla g(s) \left( \frac{1}{B} (\mathbf{E} - \mu \mathbf{B})^{1/2} \right) = 0 \]  

(3.33)

(the vanishing of a surface integral over the surface of the flux tube), as the required condition equivalent to (3.31).

Equation (3.33) may now be applied to a toroidal system with rotational transform. For such a system, since one or more lines of force define a magnetic surface, a flux tube is the flux enclosed between two magnetic surfaces, which are thus the only surfaces bounding the flux 'tube'. Since \( g \) is constant along a line of force (to the lowest order) \( g \) is constant in a magnetic surface. Consequently \( \nabla g \) is normal to the magnetic surfaces and hence parallel to \( d\mathbf{S} \). The integrand in (3.33) thus vanishes identically, and the condition (3.31) is trivially satisfied for systems with rotational transform.

**Problem 3.1:** Show that for a mirror machine or a bumpy torus the criteria (3.29) and (3.33) for closed drift surfaces can be expressed in the form

\[ \nabla J \times \nabla g = 0 \]  

(3.34)

when \( J \) is the action integral:

\[ J = \int d\xi (\mathbf{E} - \mu \mathbf{B})^{1/2} \]  

(3.35)

This implies that \( g = \text{const.} \) surfaces coincide with \( J = \text{const.} \) surfaces. The \( J = \text{const.} \) surfaces essentially define the drift surfaces. This result thus states that the drift surfaces must be the surfaces of constant \( g \).
Problem 3.2: If the plasma in a device is characterized by a scalar pressure $p$ so that the total current
\[ \vec{j} = \vec{B} \times \nabla p / B^2 + \int_{\Sigma} \vec{n} \]
then show that the criterion (3.34) corresponding to the problem 3.1 is replaced by
\[ \left| \nabla p \times \nabla K \right| = 0 \]  \hspace{1cm} (3.36)
where $K$ is now the integral
\[ K = \oint \frac{dl}{B} \]  \hspace{1cm} (3.37)
\[ \text{i.e. } K = \text{const. surfaces are parallel to and coincident with the } p = \text{const. surfaces.} \]

Interpretation of $K$: Let $V$ be the volume of a certain portion of plasma, so that $V = \int \ell \, d\Sigma$ (d$\Sigma$ being the elemental area). If $d\Psi$ denote the elemental flux, $d\Psi = Bd\Sigma$, then $V = \int \ell \, d\Psi / B$ and
\[ \frac{dV}{d\Psi} = \sqrt{'} = \int \frac{dl}{B} \equiv K \]  \hspace{1cm} (3.38)
Thus $K = dV/d\Psi$ denotes the volume of the plasma per unit magnetic flux.

The pressure cannot be a scalar for an open-ended system like a mirror machine, where the confined plasma has necessarily to be anisotropic. All toroidal systems can, however, admit scalar pressure.

Problem 3.3: Calculate $\int \ell / B = \text{const. surfaces}$ for (a) a simple torus and (b) a bumpy torus.

3.3. Effect of particles on vacuum fields

We have so far considered only one group of particles of a given $\ell$ and $p$ populating the magnetic or drift surfaces. More particles of the group can be added, as well as other particles of different $\ell$ and $\mu$. If the total number of particles is so small that they do not affect the vacuum fields, the various particles can simply be added independently of each other. The foregoing equations and expressions can be modified to take care of the addition of more particles if we
write the appropriate distribution function \( f(\varphi, \mu) \) in place of (3.15a) and obtain the corresponding expressions of pressure therefrom. The fields are affected by the particles in two ways:

(a) If \( \beta \sim 1 \), where \( \beta = \frac{8\pi nT}{B^2} \), then the total magnetic field in the system differs from the vacuum field \( B_0 \). In this case there is in principle nothing new, and the same equations remain valid in form except that \( B \) replaces \( B_0 \).

(b) If polarization electric fields arise which are ordered such that \( e\Phi \sim T \) (\( \Phi \) being the electrostatic potential) then all the foregoing equations are again valid, the action integral \( J \) now being defined as

\[
J = \int d\xi \left( E - \mu B - e\Phi \right)^{1/2}
\]

4. STABILITY OF PLASMAS IN MIRROR MACHINES

The plasma equilibrium in any confinement device is described completely by an appropriate distribution function \( f(x, v) \), which is a solution of the stationary Vlasov equation in the field geometry concerned. Such a distribution function possesses, in general, free energy with respect to macroscopic motion in coordinate \( x \)-space, and free energy with respect to the motion in velocity \( v \)-space. Correspondingly, one gets a coordinate space instability or a velocity space instability.

4.1. Instability of a low-\( \beta \) plasma in a mirror machine

We shall consider the stability of the equilibrium of a low-\( \beta \) plasma in a mirror machine \( (\beta = \frac{8\pi nT}{B^2}) \). For such a plasma the kinetic energy contained in the particle motion is insufficient to support a growing magnetic field perturbation. Hence \( \partial \vec{B} / \partial t = 0 \), which leads to

\[
\nabla \times \vec{E} = 0 \quad \text{or} \quad \vec{E} = -\nabla \Phi \quad (4.1)
\]

Thus only electrostatic instabilities can be sustained by such a low-\( \beta \) system.

We consider such an electrostatic instability with frequency \( \omega \ll \Omega_i \), the ion-cyclotron frequency. We also consider the magnetic field to be so strong that both the electron and ion Larmor radii \( a_e, a_i \) are small compared to the wavelength of the perturbation \( \lambda \), and compared to all characteristic lengths \( L \) over which the various macroscopic quantities vary: \( \lambda, L \gg a_i, a_e \).
In accordance with the conditions $L, \lambda \gg a_i, a_e$, we consider the plasma motion in the guiding-centre-fluid approximation and use the following expressions for the guiding-centre-fluid velocities:

\begin{align*}
\vec{V}_i &= \frac{c \vec{E} \times \vec{B}}{B^2} + \frac{c m_i}{e B} \left( \vec{V}_{ni} + \frac{1}{2} \vec{V}_{ni} \right) \frac{\vec{B} \times \nabla B}{B^2} + \frac{c m_i}{e} \frac{d \vec{V}_E}{dt} \times \frac{\vec{B}}{B^2} \\
\vec{V}_e &= \frac{c \vec{E} \times \vec{B}}{B^2} - \frac{c m_e}{e B} \left( \vec{V}_{ne} + \frac{1}{2} \vec{V}_{ne} \right) \frac{\vec{B} \times \nabla B}{B^2} \\
\vec{V}_E &= \frac{c \vec{E} \times \vec{B}}{B^2}
\end{align*}

(4.2)

(4.3)

where the first term in the expressions for $\vec{V}_i$ and $\vec{V}_e$ is the charge- and mass-independent $\vec{E} \times \vec{B}$ drift velocity, the second term gives both $\nabla B$ and curvature drift, while the third term (neglected in the expression for $\vec{V}_e$ owing to small electron inertia) is the polarization drift, in which $\vec{V}_E$ is the $\vec{E} \times \vec{B}$ drift velocity; $v_i^2$ and $v_e^2$ are mean squared averages of particle velocities perpendicular and parallel to the magnetic field. The other equations governing the dynamics of the system are the equations of continuity for the number densities of ions and electrons $n_i$ and $n_e$:

\begin{align*}
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{V}_i) &= 0 \\
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{V}_e) &= 0
\end{align*}

(4.4)

(4.5)

and the Poisson equation

\begin{equation}
\nabla^2 \phi = -4\pi e \left( n_i - n_e \right)
\end{equation}

(4.6)

4.1.1. Equilibrium

In equilibrium we have

\begin{equation}
\begin{aligned}
n_i^{(e)} &= n_e^{(e)} = n_0 \\
\vec{E} &= 0
\end{aligned}
\end{equation}

(4.7)
FIG. 9. Plane geometry for the Rayleigh-Taylor instability. The magnetic field points along the z-axis; the density gradient and $\nabla B$ force are both along the y-direction, while the perturbation has a propagation vector along the x-direction.

\begin{align}
\vec{v}_i^{(0)} &= \frac{c m_i}{e B} \left( \vec{v}_{i1} + \frac{1}{2} \vec{v}_{i2} \right) \frac{\vec{B} \times \nabla B}{B^2} \\
\vec{v}_e^{(0)} &= -\frac{c m_e}{e B} \left( \vec{v}_{e1} + \frac{1}{2} \vec{v}_{e2} \right) \frac{\vec{B} \times \nabla B}{B^2} \\
\nabla \cdot \left( \eta_0 \vec{v}_i^{(0)} \right) &= 0, \quad \nabla \cdot \left( \eta_0 \vec{v}_e^{(0)} \right) = 0
\end{align}

Instead of the actual mirror machine geometry we shall consider, for simplicity, a plane infinite geometry with the magnetic field $\vec{B}$ in the z-direction and with a gradient in the y-direction (Fig. 9), i.e.

$$\vec{B} = B_o \vec{e}_z \left( 1 + \sigma y \right)$$

Such a magnetic field is obviously not curl-free and is therefore not realistic, since it implies hidden currents in space. Nevertheless we assume this for simplicity of analysis and to demonstrate the most essential aspects of the calculation. The y-coordinate in the expressions (4.10) corresponds to the radial (or flux) coordinate in the mirror machine, while the x-coordinate corresponds to the azimuthal coordinate. The density is likewise assumed to be of the form:

$$n_o(y) = N_o \left( 1 + \sigma y \right)$$
Then

\[ \overrightarrow{V}_{(4.11)} = -\frac{n_i c}{e B} \left( \frac{1}{2} \overrightarrow{v}_{n_i} + \frac{1}{2} \overrightarrow{v}_{s_i} \right) \frac{\sigma B}{B} \overrightarrow{e}_x \]

\[ \overrightarrow{V}_{(4.12)} = +\frac{m_e c}{e B} \left( \frac{1}{2} \overrightarrow{v}_{n_e} + \frac{1}{2} \overrightarrow{v}_{s_e} \right) \frac{\sigma B}{B} \overrightarrow{e}_x \]

Since all quantities \( n_0, B, \) etc., are independent of \( x, \) the equilibrium equations of continuity (4.9) are trivially satisfied. \( V_{\| i} = V_{\| e} = 0 \) (where \( V_{\| i, e} \) are components of fluid velocities parallel to the magnetic field) is assumed, so that there are no parallel currents, in accordance with mirror-machine conditions.

4.1.2. Stability to linear perturbations

To investigate the stability of this equilibrium to linear perturbation, the set of equations (4.2) – (4.6) are linearized about the above equilibrium, giving:

\[ \frac{\partial \tilde{\eta}_i}{\partial x} + \nabla \cdot \left( \tilde{\eta}_0 \overrightarrow{V}_E + \tilde{\eta}_i \overrightarrow{V}_{(4.11)} \right) = 0 \]  

(4.12)

\[ \frac{\partial \tilde{\eta}_e}{\partial x} + \nabla \cdot \left( \tilde{\eta}_0 \overrightarrow{V}_E + \tilde{\eta}_e \overrightarrow{V}_{(4.12)} \right) = 0 \]  

(4.13)

\[ \overrightarrow{V}_i = \overrightarrow{V}_E = \frac{c \overrightarrow{e}_x \overrightarrow{B}}{B^2} \]  

(4.14)

\[ \overrightarrow{V}_e = \overrightarrow{V}_E = \frac{c \overrightarrow{e}_x \overrightarrow{B}}{B^2} \]  

(4.15)

\[ \nabla^2 \tilde{\phi} = -4 \pi e \left( \tilde{\eta}_i - \tilde{\eta}_e \right) \]  

(4.16)

where the tilde denotes perturbation quantities. Assume now that

\[ \left( \tilde{\eta}_i, \tilde{\phi}, \ldots \right) \sim \exp \left( -i \omega t + i k_x \right) \]  

(4.17)
i.e. we assume the $k$-vector of the perturbation to be in a direction perpendicular
to both $\vec{B}$ and $\nabla B$, and thus to be in the direction of $\nabla B$ drifts of the particles.

Then the above equations give

$$\tilde{n}_i = \frac{n_0 m_i e^2}{B^2} k^2 \Phi + \frac{c k}{B} \left( \frac{2 n_0 \sigma^2 - N_0 \sigma'}{\omega - k V_i^{(o)}} \right) \Phi$$

(4.18)

$$\tilde{n}_e = \frac{c k}{B} \left( \frac{2 n_0 \sigma^2 - N_0 \sigma'}{\omega - k V_e^{(o)}} \right) \Phi$$

(4.19)

where the following expressions for the various quantities (obtained on simplification) have been used in Eqs (4.12) and (4.13) to arrive at Eqs (4.18) and (4.19):

$$\vec{V}_e = \vec{V}_e = i c k \Phi \hat{\varepsilon}_\perp$$

$$\vec{V}_i = i c k \Phi \hat{\varepsilon}_\perp + \left( \frac{\omega - k V_i^{(o)}}{e B^2} \right) \frac{m_i e^2}{k^2} \Phi \hat{\varepsilon}_\perp$$

(4.20)

$$\text{div } \vec{V}_e = - \frac{2 i c k \Phi}{B} \sigma$$

$$\text{div } \vec{V}_i = - \frac{2 i k \Phi}{B} \sigma c + i \left( \frac{\omega - k V_i^{(o)}}{e B^2} \right) \frac{m_i e^2}{k^2} \Phi$$

(4.20)

The first term in the expression for $\tilde{n}_i$ (Eq.(4.18)) comes from the polarization

The expression for $\tilde{n}_e$ in (4.19) is the second term in the expression for $\text{div } \vec{V}_i$ (in the last of Eqs (4.20)).

Such a term is not present in the expression for $n_e$ because we neglected it owing
to small electron inertia. Using the expressions (4.18) and (4.19) for $\tilde{n}_i$ and $\tilde{n}_e$
in the Poisson equation (4.16), we get the following dispersion relation:

$$k^2 \left( 1 + \frac{4 \pi N_0 m_i e^2}{B^2} \right) = \frac{4 \pi N_0 c}{B} \sum_{j=z,e} q_j \left( \frac{2 \sigma^2 - \sigma'}{\omega - k V_j^{(o)}} \right)$$

(4.21)

$$\left( \Phi = e, \quad \Phi_e = -e \right)$$
It may be noted that the quantity

$$\varepsilon = \left(1 + \frac{4\pi N_0 m_i c^2}{B^2}\right)$$

(4.22)

occurring in Eq.(4.21) is the low-frequency dielectric constant of the plasma in a magnetic field, and that the part $4\pi N_0 m_i c^2/B^2$ comes from the polarization drift term. Two density regimes can be defined with respect to this parameter:

A high-density regime: $4\pi N_0 m_i c^2/B^2 \gg 1$ (4.23a)

A low-density regime: $4\pi N_0 m_i c^2/B^2 \ll 1$ (4.23b)

4.1.3. The high-density regime

If we consider $4\pi N_0 m_i c^2/B^2 \gg 1$, the dispersion relation (4.21) yields

$$1 = \sum_{j=1,e} \frac{q_j B}{m_i c} \frac{\omega}{k} \frac{\omega - k \nu^{(i)}(\omega)}{\left(\omega - k \nu^{(i)}(\omega)\right)^2}$$

(4.24)

This can be solved for $\omega$ to give

$$\omega^2 = \left(2\sigma - \sigma'^e\right) \frac{\omega B}{m_i B} \sum_j m_j \left(\nu^{(e)}_{nj} + \frac{1}{2} \nu^{(i)}_{nj}\right)$$

$$= \left(2\sigma - \sigma'^e\right) \frac{\omega B}{m_i} \sum_j \left(T_{nj} + T_{nj} \right)$$

(4.25)

where the expressions (4.11) for $V^{(0)}$ and $V^{(0)}_e$ have been used.

The result (4.25) may be compared with the growth rate for Rayleigh-Taylor instability of a plasma supported against gravity $g$:

$$\omega^2 = \sigma' g$$

(4.26)

If we consider $\sigma' > 2\sigma$, then the quantity $\left(-\sigma \Sigma_j (T_{nj} + T_{nj})/m_j\right)$ plays the role of gravity. If $\sigma$ be taken to be $1/R_c$ where $R_c$ is the radius of curvature of a curved magnetic field line, then the term $\sigma T_{nj}$ represents centrifugal force and the
\( \sigma T \) represents the \( \mu \nabla B \) force. The expression (4.25) predicts instability if \( \sigma' > 0 \) provided \( \sigma' > 2 \sigma \), very similar to the Rayleigh-Taylor case with constant gravity \( g \). However, the above treatment, using the actual magnetic forces in an inhomogeneous field, leads to the factor \( (2 \sigma - \sigma') \) instead of \( -\sigma' \). The instability is thus predicted if \( (2 \sigma - \sigma') \sigma < 0 \), while stability occurs for \( (2 \sigma - \sigma') \sigma > 0 \).

As this instability grows, the plasma moves into the regions of lower magnetic field. The instability is thus a manifestation of the diamagnetic nature of the plasma as a consequence of which it is expelled from regions of stronger magnetic field into regions of weaker magnetic field.

In the high-density regime, \( 4 \pi N_0 \mu_i c^2 / B^2 \gg 1 \), the contribution to the dispersion relation (4.21) from the \( \nabla^2 \tilde{\Phi} \) term of the Poisson equation (4.16) is small compared to that from the polarization drift through the ion density perturbation \( \tilde{n}_i \). Under this condition the dispersion relation (4.24) can as well be obtained by equating the expression (4.18) for the ion density perturbation \( \tilde{n}_i \) with the expression (4.19) for the electron density perturbation \( \tilde{n}_e \). The condition \( 4 \pi N_0 \mu_i c^2 / B^2 \gg 1 \) can thus be considered as one under which charge neutrality is a very good assumption. Not all plasmas satisfy this condition, however.

Problem 4.1: Carry out the above analysis using the one-fluid equations for the plasma.

Problem 4.2: Find out the growth rate in the low-density regime \( 4 \pi N_0 \mu_i c^2 / B^2 \ll 1 \).

4.1.4. An energy principle for flute instability in mirror machines

Recall that, for the flute instability discussed above, the following conditions were assumed to hold:

1. \( \beta = 8 \pi n T / B^2 \ll 1 \): so that only the electrostatic modes are considered.
2. \( E_y = 0 \): the potential is constant along the magnetic field lines — flute modes.
3. The frequency of the perturbation \( \omega \ll \Omega_i \), the ion-cyclotron frequency and the wavelength \( \lambda \gg a_i \), the ion Larmor radius.
4. The plasma fluid velocity is given by the hydromagnetic Ohm's law \( E + (1/c) \nabla \times B = 0 \), i.e. essentially given by the \( E \times B \) drift velocity.
5. \( 4 \pi n \mu_i c^2 / B^2 \gg 1 \).

When condition (3) holds, and when the ion Larmor radius is small compared to the characteristic length for the magnetic field variation \( L \equiv [ | \nabla B | / B ]^{-1} \), the particle fluid velocities can be expressed in terms of their drift velocities in the adiabatic or guiding-centre approximation. The charge- and mass-independent
E X B drift is the most dominant drift and, to the lowest order, both electrons and ions, and therefore the plasma as a whole, move with the velocity \( \mathbf{V} = c \mathbf{E} \times \mathbf{B}/B^2 \). As pointed out in condition (4), this relation is equivalent to Ohm's law in the hydromagnetic approximation: \( \mathbf{E} + (1/c) \mathbf{V} \times \mathbf{B} = 0 \). It can be shown easily [2] that the plasma motion governed by this form of Ohm's law conserves magnetic flux (linked by a given contour) following the plasma motion.

The characteristics of the guiding centre approximation, together with the foregoing, permit an energy principle to be constructed for the flute instability in mirror machines. We recall that the following adiabatic invariants exist:

The magnetic moment:

\[
\mu = \frac{\mathbf{\nu}_2 \cdot \mathbf{B}}{B}
\]  

(4.27)

The longitudinal action:

\[
\mathcal{J} = \int dt \left( E - \mu \mathbf{B} \right)^{1/2} = \int \frac{d\chi}{B} \left( E - \mu \mathbf{B} \right)^{1/2}
\]  

(4.28)

where \( \chi \) is the magnetic potential, such that \( \mathbf{B} = \nabla \chi \). An arbitrary function of these two adiabatic constants of motion, \( f(\mu, \mathcal{J}) \), is then a stationary solution of the Boltzmann-Vlasov equation in the adiabatic approximation.

Consider now a plasma in equilibrium in a mirror machine (or in general in any magnetic confinement device) in which the above two adiabatic invariants are defined. Consider an allowed variation of the system subject to the above conditions. The system is said to be stable or unstable according as the corresponding variation of the energy of the system \( \Delta W > 0 \) or \( \Delta W < 0 \).

The simplest possible variation of the system is a flute displacement, an interchange of the two flux tubes containing an equal amount of magnetic flux. It can be shown easily [3] that such an interchange (of flux tubes containing equal amounts of flux) leads to no change in the magnetic energy. In a low-\( \beta \) plasma, such as the one considered here, the magnetic field is nearly the vacuum field, and any distortion of the magnetic field must increase its energy. Thus the most dangerous displacements are those which leave the magnetic energy unchanged, namely, the interchange of flux tubes containing equal amounts of flux. This is also consistent with the constraint on the plasma motion defined by Ohm's law \( \mathbf{E} + (1/c) \mathbf{V} \times \mathbf{B} = 0 \), whereby the magnetic flux linked by a circuit in the plasma is conserved following the plasma motion.

\[1\] We follow Rosenbluth and Longmire [3] in the following construction.
We may now consider two flux tubes I and II (as shown in Fig. 10), containing equal flux $\Delta \Psi$. Let $f_I(J, \mu, s)$ and $f_{II}(J, \mu, s)$ be the corresponding distribution functions for the particles in flux tubes I and II respectively. We recall that $s$ is the label of the magnetic field line. The number of particles in the volume element $d^3x d^3v$ is

$$dN = \int f(J, \mu, s) d^3x d^3v$$

$$= \int f(J, \mu, s) \frac{d^3x}{B^2} \frac{B d\mu dJ}{\tau \sqrt{E - \mu B}} \frac{2\pi^2}{2\sqrt{E - \mu B}}$$  \hspace{1cm} (4.29)$$

Since

$$d^3x = 2\pi \frac{d^3x}{B^2}$$  \hspace{1cm} (4.30)$$

$$d^3v = 2\pi v_{\perp} dv_{\perp} dv_{\parallel} = \frac{\pi B d\mu dE}{2\sqrt{E - \mu B}}$$

and

$$\frac{dJ}{dE} = \frac{d}{dE} \int dl \sqrt{E - \mu B} = \frac{1}{2} \int \frac{dl}{\sqrt{E - \mu B}} = \frac{\tau}{2}$$  \hspace{1cm} (4.31)$$

$\tau$ being the bounce time between the turning points along the field lines.
Let $\delta \mathcal{E}$ be the change in the energy as a particle is displaced from flux tube I into flux tube II, subject to both $J$ and $\mu$ being conserved. Then the total change in the energy as all the particles in flux tube I are moved into flux tube II and those in flux tube II are displaced into flux tube I is

$$\Delta \mathcal{W} = -\Delta \Psi \int \frac{dj \mu d\chi}{B \sqrt{\mathcal{E} - \mu B}} \delta \mathcal{E}(J, \mu) \left\{ f_{\Pi}(J, \mu) - f_{I}(J, \mu) \right\}$$

$$= -\Delta \Psi \int \frac{dj \mu}{B \sqrt{\mathcal{E} - \mu B}} \int dJ d\mu \left[ f_{\Pi}(J, \mu) - f_{I}(J, \mu) \right] \delta \mathcal{E}(J, \mu)$$

$$= -\Delta \Psi \int dJ d\mu \left[ f_{\Pi}(J, \mu) - f_{I}(J, \mu) \right] \delta \mathcal{E}(J, \mu)$$

(4.32)

since the integral over $\chi$ is

$$\int \frac{d\chi}{B} (\mathcal{E} - \mu B)^{-1/2} = \gamma$$

To calculate $\delta \mathcal{E}(J, \mu)$, the change in the energy subject to the constancy of $J$ and $\mu$, we have from $\delta J = 0$

$$0 = \delta J = \delta \int \frac{d\chi}{B} (\mathcal{E} - \mu B)^{1/2}$$

$$= \int d\chi \left[ -\frac{\delta B}{B^2} (\mathcal{E} - \mu B)^{1/2} + \frac{1}{2} \frac{\delta \mathcal{E} - \mu \delta B}{B (\mathcal{E} - \mu B)^{1/2}} \right]$$

(4.33)

where $\delta B$ is the difference in the magnetic fields at the positions of the particle in flux tubes I and II: $\delta B = B_{\Pi} - B_{I}$.

From Eq.(4.33) we get

$$\delta \mathcal{E}(J, \mu) \int \frac{d\chi}{B \sqrt{\mathcal{E} - \mu B}} = \int \frac{d\chi}{B \sqrt{\mathcal{E} - \mu B}} \left[ \frac{2(\mathcal{E} - \mu B) \delta B}{B} + \mu \delta B \right]$$

or

$$\delta \mathcal{E} = \frac{1}{\tau} \int \frac{d\chi}{\sqrt{\mathcal{E} - \mu B}} \frac{\delta B}{B^2} \left( 2 \overline{v}_B^2 + \overline{v}_I^2 \right)$$

(4.34)
To calculate $\delta B$ (the change in the magnetic field in going from one flux tube to another) we note that $d$ is the normal distance between the two flux tubes at a point; then (see Fig. 11)

$$\delta B = d(\nabla B)_n = \frac{d B}{R_c}$$

(4.35)

$R_c$ being the radius of curvature of the field line ($R_c > 0$ if the centre of curvature of the field line is outside the mirror machine plasma, and vice versa). Furthermore, using the conservation of flux between the two flux surfaces (Fig. 11),

$$B r d \cdot \bar{\Psi} = \text{const.}$$

(4.36)

where $r$ is the radial coordinate of a point on the field line, we get

$$\frac{\delta B}{B} = \frac{d}{R_c} = \frac{\Psi}{B r R_c}$$

(4.37)

Using the expression (4.37) for $\delta B/B$ in (4.34) and the resulting expression for $\delta &$ in the expression (4.32) for $\Delta W$, we get

$$\Delta W = -\Psi \Delta \Psi \int dJ d\mu \left[ f_{II}(\mu, J) - f_I(\mu, J) \right]$$

$$+ \int \frac{d\mu}{\sqrt{\mu B}} \frac{\left(2 \overline{v}_n^2 + \overline{v}_l^2 \right)}{\tau B^2 R_c}$$

$$= -\Psi \Delta \Psi \int \frac{d\mu}{B^2 \gamma R_c} \left( \frac{1}{B} \int \frac{dJ d\mu}{\tau \sqrt{\mu B}} \left(2 \overline{v}_n^2 + \overline{v}_l^2 \right) \right)$$

$$\cdot \left[ f_{II}(\mu, J) - f_I(\mu, J) \right]$$

$$= -\Psi \Delta \Psi \int \frac{d\mu}{B^3 \gamma R_c} \left[ (p_{II} + p) - (p_{I} + p) \right]$$

$$\Delta W = -\Psi \Delta \Psi \int \frac{d\mu}{B^2 \gamma R_c} \delta \left( p_{II} + p \right)$$

(4.38)
FIG.11. $\Psi_1$ and $\Psi_\Pi$ represent flux surfaces in a mirror machine; $d$ is the normal distance between them at a point and $r$ is the radial distance from the symmetry axis.

If now the energy variation $\Delta W > 0$, the system is stable, and vice versa. This then is the energy principle for the flute instability in mirror machines. The stability criterion is thus

$$\int \frac{d\ell}{B^2 r R_c} S(\rho_\parallel + \rho_\perp) < 0$$

(4.39)

Since the pressure usually decreases as one moves outwards from the axis, $\delta (p_\parallel + p_\perp) < 0$ if we take the flux tube $\Pi$ to be situated in the outer region. In particular, if we assume $(p_\parallel + p_\perp)\vert_{\Pi} = 0$, the above criterion becomes

$$\int \frac{d\ell (\rho_\parallel + \rho_\perp)}{R_c r B^2} > 0$$

(4.40)

Note that, in a mirror machine, $R_c$ changes sign as one moves along a magnetic field line, being negative near the mid-plane and positive on either side. The stability is thus determined by the portions of the magnetic field lines having positive curvature, while the portions with negative curvature contribute to instability. If the average along the field line as given by (4.40) is positive, stability would result. The magnetic field increases outwards when the curvature is positive. The strongest way to satisfy stability is to require that the magnetic field increase outwards everywhere. This is the concept of a minimum-B configuration. The condition (4.39) (or (4.40)) is of course a weaker condition.

Ioffe, in Moscow, was the first to demonstrate, in 1962, the enhancement of the lifetime of plasma in a mirror machine by creating a minimum-B well. This was done by passing currents in bars (known as 'Ioffe bars' (Fig.12)) placed parallel to the axis of the machine so that adjacent bars carry opposite currents. The lifetime of the plasma in the machine was found to increase rapidly by about three orders of the magnitude from about 50 $\mu$s to about 50 ms as the current in the bars increased by about 10% above a certain value, which apparently corresponds to the point where a minimum B has just been reached [4]. In present-day
mirror machines the minimum-B is obtained by the use of the 'Yin-Yang coil', which is similar to a 'baseball coil' (shaped like the seam of a baseball).

4.1.5. Other stabilization mechanisms

The minimum-B stabilization as discussed above is a very effective way of stabilizing the hydromagnetic flute modes. Two other methods which are also possible are briefly described as follows:

(a) 'Line-tying'

If a conducting plate is placed across the magnetic field lines at one end of the mirror machine, it will allow the electric field that develops across the magnetic field lines during the growth of the instability to be short-circuited. As shown in Section 4.1, such electric fields are responsible for the expulsion of the plasma across the magnetic field through the $\mathbf{E} \times \mathbf{B}$ drift. Short-circuiting of the electric fields eliminates this instability. The mechanism is known as 'line-tying'. The name is derived from another view of the instability, in which the motion of the plasma is regarded as the motion of the magnetic field lines and of the material tied to them. The ends of the lines are supposed to be 'frozen' in the conducting plates. They are thus prevented from moving freely and from carrying the plasma along.

(b) Finite Larmor radius stabilization

In the zero Larmor radius limit, both electrons and ions move with the $\mathbf{E} \times \mathbf{B}$ drift velocity. When the ion temperature is high the Larmor radius of the
ions cannot be regarded as small. The ions then no longer see the same electric field as the electrons; they see an electric field, averaged over its gyro-orbit and which is therefore smaller by an amount \( \frac{1}{2} a_i^2 (\nabla^2 E) \), where \( a_i \) is the ion Larmor radius. A net current is associated with this difference, of the amount:

\[
\vec{J}_F = \frac{1}{2} \eta_e e c \left( a_i^2 \nabla^2 \left( \frac{E_x}{B} \right) \right) \approx \eta_e e \left( \frac{a_i}{r} \right) \left( \frac{c E_x}{B} \right)
\]

(4.41)

where \( r \) is the radius of the plasma. When this current can cancel the basic destabilization current:

\[
\vec{J}_D = \eta_e e \mu \nabla B \times \vec{B} \approx \frac{\eta_e e}{\omega r} \left( \frac{c E_x \vec{B}}{B} \right) \epsilon m_i c^2 \frac{R_c^2}{e B r_c^2}
\]

\[
\sim \eta_e e \left( \frac{r}{R_c} \right) \left( \frac{\Omega}{\omega} \right) \left( \frac{a_i}{r} \right)^2, \quad \Omega = \frac{e B}{m_i c}
\]

(4.42)

(\( v_{thi} \) is the ion thermal speed and \( R_c \) the radius of curvature of the field line), arising from gravity or \( \nabla B \) drift, stabilization of the instability would result. Since the finite Larmor radius current obviously cannot be large, only weakly unstable systems can be stabilized by this mechanism, as was first pointed out by Rosenbluth, Krall and Rostoker [4].

To see how weakly unstable the system must be for this stabilization mechanism to be effective, let us compare the two currents \( J_F \) and \( J_D \). We obtain from (4.41) and (4.42)

\[
n_e e \left( \frac{a_i}{r} \right)^2 \frac{c E_x}{B} = \eta_e e \left( \frac{a_i}{r} \right) \left( \frac{r}{R_c} \right) \left( \frac{\Omega}{\omega} \right) \frac{c E_x}{B}
\]

or

\[
\frac{\omega}{\Omega} = \frac{r}{R_c}
\]

(4.43)

as the condition for the finite Larmor radius current to compete with the destabilizing current. If we use the Rayleigh-Taylor growth rates for \( \omega \) we have

\[
\omega = \left( \frac{\Upsilon_{hi}}{\sqrt{\rho}} \right) \left( \frac{r}{R_c} \right) \left( \frac{\Omega}{\omega} \right) \left( \frac{c E_x}{B} \right)
\]

(4.44)
where

\[ \epsilon = \frac{\alpha_i}{\gamma} \ll 1 \]

The condition (4.43) for finite Larmor radius stabilization then says that we must have

\[ \frac{\omega}{\Omega} = \frac{\gamma}{R_c} = \epsilon^2 \] (4.45)

i.e. \( R_c \), the radius of curvature of the field line, must be large by two orders in \( \epsilon^{-1} \) compared to the radius of the plasma \( r \). It follows simultaneously that the growth rates of the instability \( \omega \) (in the absence of stabilization) are small by two orders in \( \epsilon \) compared to the ion cyclotron frequency \( \Omega \).

Equation (4.45) defines the finite Larmor radius ordering and specifies the extent of the weakly unstable system (the largeness of \( R_c \)) for the finite Larmor radius stabilization to be effective. For further details the original references [4–6] should be consulted.

4.2. Neutral injection mirror machine plasmas — the low-density regime

In neutral injection mirror machines the plasma is built up particle by particle from almost zero density, and the condition (4.23b) — \( 4\pi n m_i c^2 / B^2 \ll 1 \) — rather than (4.23a) holds. Arbitrarily large charge separation can occur under these conditions, so it is appropriate to work with separate equations for the electrons and ions along with the Poisson equation. Therefore the ion inertia (or, equivalently, the ion polarization drift) is unimportant. Appropriately to the neutral injection, we also assume that the particles have a near \( \delta \)-function distribution in the 'perpendicular' velocity with zero 'parallel' velocity. There is only a \( \mu \nabla B \) drift in this case, and no curvature drift.

One very important way in which the present treatment differs from the earlier one is that we no longer assume an isothermal equation of state. Instead we close the system of equations by giving an equation of motion for the magnetic moment \( \mu \). The system of equations for the problem are then:

\[ \frac{\partial \eta_i}{\partial t} + \nabla \cdot \left( \eta_i \vec{V}_i \right) = 0 \] (4.46a)
\[
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{v}_e) = 0 \tag{4.46b}
\]

\[
\vec{v}_i = \frac{c \vec{E} \times \vec{B}}{B^2} + \frac{c \mu_i}{e B^2} \vec{B} \times \nabla B \tag{4.47a}
\]

\[
\vec{v}_e = \frac{c \vec{E} \times \vec{B}}{B^2} - \frac{c \mu_e}{|e| B^2} \vec{B} \times \nabla B \tag{4.47b}
\]

\[
\frac{\partial \mu_i}{\partial t} + \vec{v}_i \cdot \nabla \mu_i = 0 \tag{4.48a}
\]

\[
\frac{\partial \mu_e}{\partial t} + \vec{v}_e \cdot \nabla \mu_e = 0 \tag{4.48b}
\]

and

\[
\nabla^2 \Phi = -4\pi e \left( n_i - n_e \right) \tag{4.49a}
\]

with

\[
\vec{E} = -\nabla \Phi \tag{4.49b}
\]

Eqs (4.46a,b) are the equations of continuity for the electrons and ions, while (4.47a,b) give the velocities for the electrons and ions with their inertia neglected. (4.48a,b) are the equations of evolution for \( \mu_i \) and \( \mu_e \) and represent 'equations of state' which effect the closure of the system of equations. These sets of equations were first given by Varma [7] to describe the behaviour of low-density plasmas in neutral injection mirror machines.

We consider the same plane geometry as in Section 4.1 with the magnetic field in the z-direction and varying linearly with y as in Eq.(4.10a),

\[
\vec{B} = B_0 \hat{e}_z (1 + \sigma y) \tag{4.50a}
\]

and the equilibrium density \( n \) varying with \( y \) as

\[
n = n_0 (1 + \sigma'y) \tag{4.50b}
\]
The electric field \( E = 0 \) in equilibrium, so the equilibrium velocities are

\[
\vec{v}_{e}^{(0)} = \vec{v}_{i}^{(0)} = \pm \frac{c \mu_{i e}}{|e| B^{2}} \vec{B} \times \nabla \vec{B}
\]

\[
= \pm \frac{c \mu_{i e}}{|e| B} \sigma B_{0} \vec{\nabla} B_x
\]  

(4.51)

Since none of the quantities \( n, \vec{B} \) or \( \vec{v}_{i e} \) depends on \( x \), the equilibrium equations are trivially satisfied.

Equations (4.46)—(4.49) can now be linearized for perturbations around the equilibrium state, yielding

\[
\left( \frac{\partial}{\partial t} + \vec{v}_{i}^{p} \cdot \nabla + \nabla \cdot \vec{v}_{i}^{p} \right) \vec{\phi}_{i} + \vec{v}_{i}^{p} \cdot \nabla n + n \nabla \cdot \vec{v}_{i}^{p} = 0
\]  

(4.52a)

\[
\left( \frac{\partial}{\partial t} + \vec{v}_{i}^{p} \cdot \nabla + \nabla \cdot \vec{v}_{i}^{p} \right) \vec{n}_{i} + \vec{v}_{i}^{p} \cdot \nabla n + n \nabla \cdot \vec{v}_{i}^{p} = 0
\]  

(4.52b)

\[
\vec{v}_{i}^{p} = \frac{c E \times \vec{B}}{B^{2}} \pm \frac{\mu_{i e} B \times \nabla B}{|e| B^{2}}
\]  

(4.53a,b)

and

\[
\left( \frac{\partial}{\partial t} + \vec{v}_{i}^{p} \cdot \nabla \right) \vec{\mu}_{i} + \vec{v}_{i}^{p} \cdot \nabla \mu_{i} = 0
\]  

(4.54a)

\[
\left( \frac{\partial}{\partial t} + \vec{v}_{i}^{p} \cdot \nabla \right) \vec{\mu}_{e} + \vec{v}_{i}^{p} \cdot \nabla \mu_{e} = 0
\]  

(4.54b)

\[
\nabla^{2} \vec{\phi} = -4\pi e \left( \vec{n}_{i} - \vec{n}_{e} \right)
\]  

with

\[
\vec{E} = -\nabla \vec{\phi}
\]  

(4.55)
where, in the above equations, the tilde denotes the perturbation quantities while quantities without the tilde are the equilibrium quantities. Using the low-β assumption, one finds that

\[
\nabla \cdot \tilde{V}_{i,e} \approx \frac{2c}{B_0^2} \nabla \Phi \cdot \mathbf{B} \times \nabla \mathbf{B} \pm \frac{c}{|e|} \nabla \tilde{\mu}_{i,e} \mathbf{B} \times \nabla \mathbf{B} \quad (4.56)
\]

and

\[
\nabla \cdot \tilde{V}_{i,e}^p = 0 \quad (4.57)
\]

Then assuming all perturbation quantities \( \tilde{n}_{i,e}, \Phi, \tilde{\mu}_{i,e} \) to vary as

\[
\left( \tilde{n}_{i,e}, \Phi, \tilde{\mu}_{i,e} \right) \sim \exp(-i\omega t + ikx) \quad (4.58)
\]

we get

\[
\tilde{n}_{i,e} = \frac{ck}{B_0} \tilde{\Phi} \left[ -\frac{\eta e}{\sigma - 2\sigma} \frac{k V_{i,e}^p}{(\omega - k V_{i,e}^p)^2} \right] \quad (4.59)
\]

The Poisson equation then gives the dispersion relation:

\[
1 = \sum_{j=i,e} \frac{\omega_{f_j}}{\Omega_j} \sigma \left\{ -\frac{(\sigma'/\sigma - 2)}{(\omega - k V_j^p)^2} + \frac{k V_j^p}{(\omega - k V_j^p)^2} \right\} \quad (4.60)
\]

If we assume the electron temperature to be small, appropriate to the neutral injection mirror machine, so that \( \omega \gg kV_e^0 \), we get the dispersion relation:

\[
\gamma^3 - 2\gamma^2 + \gamma \left[ 1 - q (1 - \sigma'/\sigma) \right] - \frac{q}{2} (1 - 2\sigma'/\sigma) = 0 \quad (4.61)
\]

where

\[
\gamma = \omega / k V_i^p \quad , \quad q = \frac{\omega_{f_i}^2}{\Omega_i(k V_i^p)} \frac{\sigma'}{k} \quad (4.62)
\]

We find that the dispersion relation (4.61) is now a cubic, in contrast to the quadratic dispersion relation found in the earlier case when the magnetic moment
is not perturbed through Eq.(4.54) (see Problem 4.2). This dispersion relation thus describes an additional mode. In the limit $\sigma/\sigma' \to 0$, (4.61) factors as

$$\left(\nu - 1\right)\left(\nu^2 - \nu + \frac{1}{2}\right) = 0$$

which gives the three roots as

$$\nu = 0 \quad \text{and} \quad \nu = \frac{1}{2} \pm \frac{1}{2} \left(1 - 4\frac{Q}{\Omega}\right)^{1/2}$$

We thus get $\nu = 1$ as the new mode of oscillation, while the other roots furnish the stability criterion:

$$\frac{\omega_0^2}{\Omega_i \left(k V_i^2\right)} \frac{\sigma'}{k} < \frac{1}{4}$$

i.e. the system would become unstable as soon as the density increases beyond the limit given by (4.65). Below this limit the system would oscillate with a frequency which depends on the density. The root $\nu = 1$, on the other hand, describes a density-independent mode of oscillation. Both density-dependent and density-independent modes of oscillation have actually been observed in neutral injection mirror machines such as ALICE, PHOENIX, DCX-II, with the oscillations jumping from one mode to another at times (e.g. [8]).

4.2.1. Quasi-linear theory

A theory describing the evolution of these modes in the quasi-linear coupling with the source term (neutral injection) has been considered by Simon and Weng [9]. It was found that the explosive non-linear behaviour as, for instance, observed in the DCX-II machine, cannot be explained without including the equation for the magnetic moment $\mu$ (Eq.(4.48)). Further work along this line has been done by Abhayankar and Simon [10].

4.2.2. High-density limit

The preceding analysis can also be carried out in the high-density case, i.e. $4\pi n_i m_i c^2 / B^2 \gg 1$. This amounts to including the ion polarization drift in the expression for the ion velocity. But from Section 4.1 we see that this can be easily done by including the dielectric constant $\epsilon = \left(1 + 4\pi n m_i c^2 / B^2\right)$ in the Poisson equation (4.49) so that it reads:
$$\nabla^2 (\varepsilon \Phi) = -4\pi e \left( \bar{n}_e - \bar{n}_i \right)$$  \hspace{1cm} (4.66)

With this, the dispersion relation becomes (in the high-density limit)

$$1 = \sum_j \frac{\Omega_j \sigma}{k} \left\{ -\frac{\left(\sigma/\sigma' - 2\right)}{(\omega - kV_j^p)} + \frac{kV_j^p}{(\omega - kV_j^p)^2} \right\}$$  \hspace{1cm} (4.67)

If we again assume the electron temperature to be small, so that $\omega \gg kV_e^p$, we again get a cubic:

$$\nu^3 - 2\nu^2 + \nu \left[ 1 - \nu \left( 1 - \frac{\sigma}{\sigma'} \right) - \nu \left( 1 - 2\sigma/\sigma' \right) \right] = 0$$  \hspace{1cm} (4.68)

where

$$\nu = \omega/kV_e^p$$  \hspace{1cm} (4.69)

and

$$\nu' = \left( \frac{\Omega_j}{kV_e^p} \right) \frac{\sigma'}{k}$$  \hspace{1cm} (4.70)

Now $\nu \gg 1$. But again in the limit $\sigma/\sigma' \to 0$, the cubic (4.68) can be factored as

$$(\nu - 1)(\nu^2 - \nu + 1) = 0$$  \hspace{1cm} (4.71)

We again have the mode $\nu = 1$, but the other quadratic factor gives

$$\nu = \frac{1}{2} \pm \frac{1}{2} \left( 1 - 4\nu' \right)^{1/2}$$  \hspace{1cm} (4.72)

Unless $\sigma'/k$ is very small, $\nu \gg 1$, and this gives instability with

$$\nu \ll \pm i \sqrt{\nu'} \quad , \quad \omega = \pm i \left[ \Omega \sigma' V_e^p \right]^{1/2}$$  \hspace{1cm} (4.73)

These are essentially the Rayleigh-Taylor growth rates in the limit $\sigma/\sigma' \to 0$ (see Eq.(4.25)).
REFERENCES

LINEAR WAVES AND INSTABILITIES IN FLUID PLASMAS

A.A. SKORUPSKI
Institute of Nuclear Research, Warsaw, Poland

Abstract

LINEAR WAVES AND INSTABILITIES IN FLUID PLASMAS.
Basic properties of linear waves and instabilities in a homogeneous (or quasihomogeneous) multicomponent plasma are described within the framework of ideal fluid equations. The analysis includes electrostatic and electromagnetic waves in an isotropic plasma, waves in a cold magnetized plasma, waves in a warm magnetized plasma with beams (parallel propagation), and a two-stream instability. Energy transport associated with stable linear waves is discussed in detail. Formulas obtained from an exact (non-linear) energy conservation equation in the presence of beams are shown to differ from those following from the linearized fluid equations.

1. GENERAL TOPICS

1.1. Introduction

The theory of plasma waves is closely related to that of plasma stability. Both use the same basic physical concepts, and when looking for waves in plasma one often finds certain instabilities. The theoretical tools designed primarily for analysing unstable configurations, e.g. the well-known MHD energy principle \[1, 2\], can provide useful information about the waves in stable systems. Basic concepts in both theories are the plasma model and plasma equilibrium. By model we mean a set of equations that can describe the time evolution of the plasma system, as well as the plasma response to excitation caused by an external source (e.g. a pair of grids or an antenna). Such a set of equations must therefore be complete, i.e. contain exactly as many equations as there are unknown parameters. The present work is based on the fluid plasma model, which provides a reduced description of the system. This means that the amount of information about a plasma which can be obtained is less than that provided by the less reduced (statistical or kinetic) models, e.g. by the Vlasov equation. Nevertheless, information is obtained about the macroscopic plasma parameters, such as the number density \(n\), macroscopic velocity \(\bar{v}\), scalar pressure \(p\), and temperature \(T\). These parameters will be defined for each species of plasma, and the species will be distinguished by
In general our plasma will consist of one or a few groups of electrons \((a = e_1, e_2, \ldots)\) and one or a few kinds of ion \((a = i_1, i_2, \ldots)\). This implies an assumption of **full ionization** (no neutral atoms).

Fluid models are mathematically simpler than kinetic models and are therefore useful for providing insight into the behaviour of complicated plasma configurations. Very sophisticated fluid equations can be used, including such effects as anisotropic pressure and various dissipative processes (e.g. electrical and thermal conductivity, viscosity, etc.), and the equations can be solved numerically [3, 4]. In theoretical work the dissipative effects are often neglected or are modelled in a simple way. Certain phenomena, such as Landau damping or the bulk of velocity space instabilities, are completely beyond the scope of fluid description. Thus one cannot describe these effects (even approximately) by fluid equations, however complicated they might be.

**By equilibrium** we shall mean a steady state \((\partial / \partial t = 0)\) in which the macroscopic parameters are time-independent. This state can differ from thermodynamic equilibrium. In that case analysis will be physically meaningful only on a time scale short compared to the relaxation time (after which the system reaches a thermodynamic equilibrium). Another restriction on the validity of our results arises because only the linear theory of plasma waves and instabilities will be given, and therefore only small deviations of the system from its equilibrium will be described. Each macroscopic parameter \(p\) will be written:

\[
p(\vec{r}, t) = p_0(\vec{r}) + p_1(\vec{r}, t), \quad |p_1| \ll |p_0|
\]

where \(p_0\) denotes the equilibrium value, and \(p_1\) is a small perturbation. Equations for perturbations are obtained on substituting (1.1.1) into the model equations, making the Taylor expansions about equilibrium up to the terms linear in \(p_1\), and subtracting the equilibrium equations satisfied by \(p_0\). The **linearized equations** for perturbations are obtained as a result. The coefficients of these equations are functions of \(p_0\) and are therefore time-independent. Thus one can always assume a harmonic time dependence of the perturbations, i.e. \(p_1\) proportional to \(\exp(-i\omega t)\). Similarly, if all equilibrium parameters are independent of a spatial variable, say \(x\), then \(p_1\) can be assumed proportional to \(\exp(ik_x x)\). More general solutions of the linearized equations can be obtained by taking appropriate superpositions of these simple time and/or space harmonics. If the equilibrium is spatially uniform \((p_0 = \text{const})\), we can take

\[
p_1 = p \exp[i(k \cdot \vec{r} - \omega t)], \quad p = |p| \exp(i\alpha_p)
\]
where

\[ \vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \]

is a wave vector and \( p \) is a complex amplitude\(^1\). In general the perturbations of the harmonic time and/or space dependence are complex quantities. The corresponding physical quantities are obtained by taking the real parts; they will be denoted by a subscript \( r \), e.g. \( p_{1r} = \text{Re} \, p_1 \).

It will be shown later that \( \omega \) and \( \vec{k} \) in (1.1.2) are usually related to each other, and only one of them is a free parameter. If we choose \( \vec{k} \) as an independent parameter and take it to be real, \( \omega(\vec{k}) \) will in general be complex:

\[ \omega = \omega_r + i \omega_i \quad \text{for real } \vec{k} \]  \hspace{1cm} (1.1.3)

Choosing the \( z \) axis along \( \vec{k} \), we thus obtain

\[
p_{1r} = \exp(\omega_i t) \left| p \right| \cos \left[ k (z - \frac{\omega_r}{k} t) + \alpha_p \right] \]  \hspace{1cm} (1.1.4)

The presence of the factor \( \exp(\omega_i t) \) in (1.1.4) makes \( p_{1r} \) grow or decay exponentially in time (for \( \omega_i \neq 0 \)), whereas the oscillating part of \( p_{1r} \) is a plane wave propagating along \( \vec{k} \) with speed

\[ v_p = \frac{\omega_r}{k} \]  \hspace{1cm} (1.1.5)

called phase velocity. This wave oscillates harmonically, both as a function of \( t \) (\( z = \text{const} \)) and \( z \) (\( t = \text{const} \)), with the periods \( T \) and \( \lambda \) (wavelength), given by

\[ T = \frac{2\pi}{\omega_r}, \quad \lambda = \frac{2\pi}{k} \]  \hspace{1cm} (1.1.6)

Problem 1

Show that the time average, over one period, of the product of two real plane-wave perturbations of the form (1.1.2) can be written:

---

\(^1\) Note that an underline denotes either a unit vector (e.g. \( \hat{x} \)) or a complex amplitude.
\[ p_{1r} q_{1r} = \exp\left(2\omega_i t\right) \frac{1}{2} \text{Re}\left(p q^*\right) \]  

(1.1.7)

if \(|\omega_i| \leq |\omega_i|\) (star denotes complex conjugate).

The equilibrium is called **unstable** if one or more perturbations can be found which grow in time (unstable modes). For harmonic time dependence the instability means that in some parameter range \(\omega_i > 0\), and \(\omega_i\) is called in that case the **growth rate** of the unstable mode. The **growth time** \(\tau\) is defined as the e-folding time of the unstable mode, \(\tau = 1/\omega_i\). For times \(t > \tau\), the unstable perturbation \(p_{1r}\) reaches values much larger than \(p_{1r}(t = 0)\), and as soon as they become comparable to \(p_0\) the linearization ceases to be valid. Usually the further growth of \(p_{1r}\) is limited by the non-linear terms in the model equations or by other effects not included in the model. This can lead to a new steady state (new equilibrium).

It might seem easier to demonstrate instability (i.e. find one growing perturbation) than to prove stability. Both tasks become comparable, however, if in the unstable case the maximum growth rate \(\omega_i \max\) is asked for. The corresponding growth time, \(\tau_{\min} = 1/\omega_i \max\) defines the lifetime of the unstable equilibrium considered.

1.2. Macroscopic fluid equations

Each species \(\alpha\) (or group of particles, such as an electron beam) are described in our model by a separate set of fluid equations. These sets are coupled to each other by self-consistent fields, while collisional coupling is neglected. Our model equations are therefore:

**Continuity**

\[ \frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \vec{V}_\alpha) = 0 \]  

(1.2.1)

**Momentum transfer**

\[ n_\alpha \frac{d\vec{V}_\alpha}{dt} = -\nabla p_\alpha + q_\alpha n_\alpha (\vec{E} + \frac{i}{c} \vec{V}_\alpha \times \vec{B}) \]  

(1.2.2)
Pressure-density relation

\[ \frac{d}{dt} \left( \frac{p_\alpha}{n_\alpha} \right) = 0 \quad \text{equivalent to} \quad \frac{d}{dt} p_\alpha = \frac{\nabla \cdot p_\alpha}{n_\alpha} \frac{d}{dt} n_\alpha \quad (1.2.3) \]

where \( n_\alpha, \vec{v}_\alpha \) and \( p_\alpha \) are the number density, macroscopic fluid velocity and scalar pressure, respectively; \( m_\alpha \) and \( q_\alpha \) denote the mass and charge of the particle; \( \vec{E} \) is the electric field, and \( \vec{B} \) the magnetic induction (both macroscopic, i.e. averaged over 'physically infinitely small' volumes) produced by external sources as well as electric charges and currents in plasma. The time derivative \( d/dt \) in Eqs (1.2.2) and (1.2.3) denotes the fluid (or material) derivative, defined as \( (\vec{V} \equiv \nabla_\alpha) \):

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V}_x \frac{\partial}{\partial x} + \vec{V}_y \frac{\partial}{\partial y} + \vec{V}_z \frac{\partial}{\partial z} \equiv \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla} \quad (1.2.4) \]

where

\[ \vec{\nabla} = \frac{\partial}{\partial x} \vec{x} + \frac{\partial}{\partial y} \vec{y} + \frac{\partial}{\partial z} \vec{z} \quad (1.2.5) \]

in the known del (or nabla) gradient operator. The fluid derivative (1.2.4) acting on some quantity gives its rate of change in the reference frame moving with the local fluid velocity.

Equations (1.2.1) and (1.2.2) can be derived by taking moments of the kinetic equations for the distribution functions of the \( \alpha \)-type particles and adopting appropriate simplifying assumptions. For our purposes these equations will be postulated, and their physical meaning is quite clear. Equation (1.2.1) is the conservation equation for the number of \( \alpha \)-type particles; this conservation is a consequence of the assumed full ionization. Equation (1.2.2) can be interpreted as Newton's second law for the \( \alpha \)-type fluid element (see Problem 2, where Eqs (1.2.1) and (1.2.2) are integrated over a moving volume). It is also instructive to integrate Eq.(1.2.1) over a fixed volume \( \Omega \) bounded by a surface \( S \). Using Gauss's theorem, we obtain

\[ \frac{d}{dt} \int_{\Omega} n_\alpha \, d\vec{r} = -\int_{S} (n_\alpha \vec{v}_\alpha) \cdot \vec{n} \, ds \quad (1.2.6) \]
where \( \vec{n} \) is a unit vector normal to \( S \), directed outside \( \Omega \). The left-hand side of Eq. (1.2.6) gives the rate of change of the total number of \( \alpha \)-type particles contained in \( \Omega \), and the right-hand side is the net incoming flux of these particles through the boundary.

**Problem 2**

Write Eqs (1.2.1) and (1.2.2) in the following integral form:

\[
\frac{d}{dt} \int_{\Omega(t)} n_{\alpha} \, d\vec{r} = 0 
\]

(1.2.7)

\[
\frac{d}{dt} \int_{\Omega(t)} m_{\alpha} n_{\alpha} \vec{V}_{\alpha} \, d\vec{r} = -\int_{S(t)} p_{\alpha} \vec{n} \, ds + \int_{\Omega(t)} q_{\alpha} n_{\alpha} (\vec{E} + \frac{1}{c} \vec{V}_{\alpha} \times \vec{B}) \, d\vec{r}
\]

(1.2.8)

where \( \Omega(t) \) is the fluid region (it moves together with the fluid and is therefore time-dependent), bounded by a moving surface \( S(t) \). Give the physical meaning of the terms in Eq. (1.2.8).

**HINT:** Make use of the Leibnitz rule for differentiation of the volume integral with moving boundary, i.e.

\[
\frac{d}{dt} \int_{\Omega(t)} f(\vec{r}, t) \, d\vec{r} = \int_{\Omega(t)} \frac{\partial f}{\partial t} \, d\vec{r} + \int_{S(t)} \vec{V} \cdot \vec{n} \, f \, ds
\]

(1.2.9)

where \( \vec{V} \) is the local velocity of the point on surface \( S(t) \).

The pressure-density relation (1.2.3) is added here to close the set of equations. It means that a unique relation between \( p_{\alpha} \) and \( n_{\alpha} \) (of the form \( p_{\alpha}/n_{\alpha}^{\gamma_{\alpha}} = \text{const} \)) is assumed, as one follows the motion of the fluid element. \( \gamma_{\alpha} \) will be specified later, either on some physical grounds or in order to obtain agreement with the results following from the kinetic theory. The latter approach can produce some new results, for example when one starts analysing a simple configuration (e.g. an infinite plasma) and then goes to a more complicated case (e.g. the same sort of waves but in a bounded plasma). The kinetic approach can be too difficult for the more complicated case, while still being a good guide for the choice of \( \gamma_{\alpha} \). Instructive examples of this approach are given in Ref. [5].
In certain applications it may be convenient to work with the one-fluid macroscopic parameters, defined as

**Total mass density:**

\[ \rho = \sum_{\alpha} m_{\alpha} n_{\alpha} = \sum_{i} m_{i} n_{i} \]  \hspace{1cm} (1.2.10)

**Bulk velocity of plasma (centre-of-mass velocity):**

\[ \vec{V} = \sum_{\alpha} m_{\alpha} \vec{V}_{\alpha} / \rho \approx \sum_{i} m_{i} n_{i} \vec{V}_{i} / \rho \]  \hspace{1cm} (1.2.11)

**Total pressure:**

\[ p = \sum_{\alpha} p_{\alpha} \]  \hspace{1cm} (1.2.12)

**Total charge density:**

\[ \rho_{q} = \sum_{\alpha} q_{\alpha} n_{\alpha} \]  \hspace{1cm} (1.2.13)

**Total current density:**

\[ \vec{j} = \sum_{\alpha} q_{\alpha} n_{\alpha} \vec{V}_{\alpha} \]  \hspace{1cm} (1.2.14)

where in Eqs (1.2.10) and (1.2.11) the electron contributions can be neglected \((m_e \ll m_i, \ n_e \sim n_i)\). Multiplying Eq.(1.2.1) by \(m_{\alpha}\) or \(q_{\alpha}\) and summing over all species, we obtain the one-fluid mass and charge conservation equations:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \]  \hspace{1cm} (1.2.15)

\[ \frac{\partial \rho_{q}}{\partial t} + \nabla \cdot \vec{j} = 0 \]  \hspace{1cm} (1.2.16)
Similarly, summing up the momentum transfer equations (1.2.2) over all species and using the continuity equations (1.2.1) and (1.2.15), we obtain after some manipulations (see Problem 4)

\[
\rho \frac{d\vec{V}}{dt} = -\sum_i \nabla \cdot \left[ m_i n_i (\vec{V}_i - \vec{V}) (\vec{V}_i - \vec{V}) \right] - \nabla p + \rho \frac{\vec{E}}{c} + \frac{1}{c} \frac{\vec{J} \times \vec{B}}{\rho}
\]

(1.2.17)

where the electron contributions to the first term on the right-hand side have again been neglected. The ion contributions to this term are also often negligible, for example, when there is only one ion component or when all ion species move with approximately the same macroscopic velocity (so that \(\vec{V} = \vec{V}_i\)). In that case Eqs (1.2.15) and (1.2.17) have exactly the same form and the same physical meaning for the bulk of the plasma, as Eqs (1.2.1) and (1.2.2) for a given \(\alpha\)-type fluid.

**Problem 3**

*Derive the following useful identities following from Eqs (1.2.1) and (1.2.4):*

\[
\rho_\alpha \frac{d\vec{V}_\alpha}{dt} = \frac{2}{\gamma_t} \left( \rho_\alpha \vec{V}_\alpha \right) + \nabla \cdot \left( \rho_\alpha \vec{V}_\alpha \vec{V}_\alpha \right)
\]

(1.2.18)

\[
\vec{V}_\alpha \cdot \left( \rho_\alpha \frac{d\vec{V}_\alpha}{dt} \right) = \frac{2}{\gamma_t} \left( \frac{1}{2} \rho_\alpha \vec{V}_\alpha^2 \right) + \nabla \cdot \left( \frac{1}{2} \rho_\alpha \vec{V}_\alpha \vec{V}_\alpha \right)
\]

(1.2.19)

where \(\rho_\alpha = m_\alpha n_\alpha\) is the partial mass density.

**Problem 4**

*Derive Eq.(1.2.17). HINT: Make use of the identity (1.2.18) and its one-fluid counterpart (\(\rho_\alpha \rightarrow \rho, \vec{V}_\alpha \rightarrow \vec{V}\)).*

A closed set of model equations is obtained if we supplement Eqs (1.2.1)−(1.2.3) by the Maxwell equations relating the fields \(\vec{E}\) and \(\vec{B}\) to the charges and currents:

\[
\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}
\]

(1.2.20)
\[ \nabla \times \vec{B} = \frac{i}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} (\vec{j} + \vec{j}^{\text{ext}}) \]  
(1.2.21)

\[ \nabla \cdot \vec{E} = 4\pi (\xi_q + \xi_q^{\text{ext}}) \]  
(1.2.22)

\[ \nabla \cdot \vec{B} = 0 \]  
(1.2.23)

where \( \rho_q^{\text{ext}} \) and \( \vec{j}^{\text{ext}} \) represent the external charges and currents (introduced into the plasma by external sources). They are given functions describing the sources. The number of equations is now equal to the number of unknowns, because Eqs (1.2.22) and (1.2.23) play the role of initial conditions (see Problem 5).

**Problem 5**

*Show that the Maxwell equations (1.2.21) and (1.2.22) are solvable only when*

\[ \frac{\partial \rho_q^{\text{ext}}}{\partial t} + \nabla \cdot \vec{j}^{\text{ext}} = 0 \]  
(1.2.24)

*i.e. when the external charge is conserved (see also Eq.(1.2.16)). Assuming that condition (1.2.24) is fulfilled, show that Eqs (1.2.21) and (1.2.20) imply*

\[ \frac{\partial}{\partial t} \left[ \nabla \cdot \vec{E} - 4\pi (\xi_q + \xi_q^{\text{ext}}) \right] = 0 \]  
(1.2.25)

\[ \frac{\partial}{\partial t} \nabla \cdot \vec{B} = 0 \]

**1.3. Equilibrium**

It will be assumed throughout that there is no electric field in equilibrium, i.e. \( \vec{E}_o = 0 \), which implies charge neutrality (see Eq.(1.2.22) where we assume \( \rho_q^{\text{ext}} = 0 \)):

\[ \xi_o \rho_{q_o} = \sum_{\alpha} q_{\alpha} n_{\alpha} = 0 \quad \text{for} \quad \vec{E}_o = 0 \]  
(1.3.1)
The equilibrium magnetic field in general will be the sum of the vacuum field $\mathbf{B}_0^{\text{vac}}$, produced by currents flowing in the external windings, and the plasma field $\mathbf{B}_0^{\text{pl}}$, produced by the current in the plasma $j_0$, 

$$\mathbf{B}_0 = \mathbf{B}_0^{\text{vac}} + \mathbf{B}_0^{\text{pl}} \quad (1.3.2)$$

$$\nabla \times \mathbf{B}_0 = \nabla \times \mathbf{B}_0^{\text{pl}} = \frac{4\pi}{c} j_0, \quad j_0 = \sum_{\alpha} q_\alpha n_\alpha j_\alpha \quad (1.3.3)$$

The equilibrium fluid equations thus become

$$\nabla \cdot (n_\alpha \mathbf{V}_\alpha) = 0 \quad (1.3.4)$$

$$\mathbf{V}_\alpha \cdot \nabla \mathbf{V}_\alpha = -\frac{1}{m_\alpha n_\alpha} \nabla p_\alpha + \frac{q_\alpha}{m_\alpha c} \left( \mathbf{V}_\alpha \times \mathbf{B}_0^{\text{vac}} + \mathbf{V}_\alpha \times \mathbf{B}_0^{\text{pl}} \right) \quad (1.3.5)$$

$$\mathbf{V}_\alpha \cdot \nabla p_\alpha = \frac{e_\alpha p_\alpha}{n_\alpha} \mathbf{V}_\alpha \cdot \nabla n_\alpha \quad (1.3.6)$$

In our theory of plasma waves we shall try to get as close as possible to a uniform equilibrium, for which the plane-wave solutions can be looked for. We therefore assume that

$$\mathbf{B}_0^{\text{vac}}, \mathbf{V}_\alpha = \text{const} \quad \text{and} \quad \mathbf{V}_\alpha \times \mathbf{B}_0^{\text{vac}} = 0 \quad (1.3.7)$$

Thus $\mathbf{B}_0^{\text{vac}}$ is assumed to be homogeneous and $\mathbf{V}_\alpha = \text{const}$ to be parallel to $\mathbf{B}_0^{\text{vac}}$. For $\mathbf{B}_0^{\text{vac}} = 0$, $\mathbf{V}_\alpha$ could in principle have arbitrary directions, but for simplicity they are also assumed to be parallel to each other.

In the major part of our analysis there will be no equilibrium current ($j_0 = 0$, $\mathbf{B}_0^{\text{pl}} = 0$), and the equilibrium parameters will be strictly uniform (see Eqs (1.3.4) and (1.3.5)), i.e.

$$n_\alpha, \mathbf{V}_\alpha, P_\alpha, \mathbf{B}_0 = \text{const} \quad \text{for} \quad j_0 = 0 \quad (1.3.8)$$

For small equilibrium current $j_0 = \text{const} \neq 0$ a small inhomogeneity in $\mathbf{B}_0$ and $p_\alpha$ is introduced (in the direction perpendicular to $j_0$). Choosing the z-axis of the coordinate system along $j_0$, and the y-axis along $\mathbf{B}_0^{\text{pl}}$, we obtain ($n_\alpha, \mathbf{V}_\alpha, \mathbf{B}_0^{\text{vac}} = \text{const}$).
1.4. Linearized fluid equations for a magnetized plasma with beams

\((E_0 = 0, \vec{B}_0 \neq 0, \vec{V}_{a0} \neq 0)\)

The linearization procedure described in Section 1.1 is now applied to the fluid equations (1.2.1)—(1.2.3). Thus we write

\[
\begin{aligned}
\frac{a}{c} = n_{d0} + n_{d1}, \quad \vec{V}_d = \vec{V}_{d0} + \vec{V}_{d1}, \quad \vec{E} = \vec{E}_1 \\
\vec{B} = \vec{B}_0 + \vec{B}_1, \quad \vec{p}_d = \vec{p}_{d0}(x) + \vec{p}_{d1}
\end{aligned}
\]  

The continuity equation (1.2.1) now becomes

\[
\frac{\partial n_{d1}}{\partial t} + \nabla \cdot \left[ (n_{d0} + n_{d1})(\vec{V}_{d0} + \vec{V}_{d1}) \right] = 0
\]

(1.4.2)

Neglecting in Eq.(1.4.2) the product \(n_{d1} \vec{V}_{a1}\) as second order in the perturbations, and subtracting the equilibrium equation (1.3.4), we obtain

\[
\frac{\partial n_{d1}}{\partial t} + \nabla \cdot \left[ n_{d0} \vec{V}_{d1} + n_{d1} \vec{V}_{d0} \right] = 0
\]

(1.4.3)

Similarly, from Eqs (1.2.2) and (1.3.5) we get \((\vec{V}_{a0} = \text{const})\)

\[
\begin{aligned}
\frac{\partial \vec{V}_{d1}}{\partial t} + \vec{V}_{d0} \cdot \nabla \vec{V}_{d1} = - & \frac{1}{m_d n_{d0}} \nabla p_{d1} \\
& + \frac{q_d}{m_d} \left[ \vec{E}_1 + \frac{1}{c} (\vec{V}_{d0} \times \vec{B}_1 + \vec{V}_{d1} \times \vec{B}_0^{\text{vac}} + \vec{V}_{d1} \times \vec{B}_0^{\text{pl}}) \right]
\end{aligned}
\]

(1.4.4)
We consider first the most important case of uniform equilibrium (1.3.8), corresponding to \( J_0 = 0 \), in which \( B_0^{pl} = 0 \), and \( \nabla p_{\alpha 0} = 0 \). We assume all perturbations to be plane waves, e.g.

\[
n_{d1} = n_0 \exp \left[ i \left( \mathbf{k} \cdot \mathbf{r} - \omega t \right) \right]
\]

and the same for \( \mathbf{V}_{\alpha 1} \), .... For such functions we can replace

\[
\frac{\partial}{\partial t} \rightarrow -i \omega , \quad \nabla \rightarrow i \mathbf{k}
\]

This reduces Eqs (1.4.3)—(1.4.5), which in general are linear partial differential equations, to algebraic equations. Solving Eqs (1.4.3) and (1.4.5), we express \( n_{\alpha 1} \) and \( p_{\alpha 1} \) in terms of \( \mathbf{V}_{\alpha 1} \):

\[
\frac{n_{d1}}{n_{d0}} = \frac{p_{d1}}{T_{d}^* p_{d0}} = \frac{\mathbf{k} \cdot \mathbf{V}_{d1}}{\omega - \mathbf{k} \cdot \mathbf{V}_{a0}}
\]

The equation for \( \mathbf{V}_{\alpha 1} \), which follows from (1.4.4), can now be written:

\[
-i (\omega - \mathbf{k} \cdot \mathbf{V}_{a0}) \mathbf{V}_{\alpha 1} = -C_w^2 \frac{i \mathbf{k} \cdot \mathbf{k} \cdot \mathbf{V}_{\alpha 1}}{\omega - \mathbf{k} \cdot \mathbf{V}_{a0}} + \frac{q_{d}}{m_{\alpha}} \left( E_1 + \frac{i}{c} \mathbf{V}_{a0} \times \mathbf{B}_1 \right)
\]

\[
+ \mathbf{V}_{d1} \times \omega_{\alpha d}
\]

where we have introduced the \textit{cyclotron-frequency vector} (see Problem 6 below):

\[
\omega_{\alpha d} = \frac{q_{d} B_{0}^{vac}}{m_{\alpha} c}
\]
and

\[ C_\alpha^2 = \frac{T_\alpha}{m_\alpha} \frac{P_{d0}}{n_{d0}} \]  

(1.4.11)

Assuming in equilibrium the ideal gas equation of state \( T_\alpha = T_{\text{eq}} \)

\[ p_{d0} = n_{d0} \frac{K}{m_\alpha} T_\alpha \]  

(1.4.12)

where \( K \) is the Boltzmann constant, we obtain \( (\gamma_\alpha \sim 1) \)

\[ C_\alpha^2 = \gamma_\alpha \frac{K}{m_\alpha} \frac{T_\alpha}{m_\alpha} \sim \frac{K}{m_\alpha} \frac{T_\alpha}{m_\alpha} \equiv v_{\text{T}_\alpha}^2 \]  

(1.4.13)

Thus \( C_\alpha \) is of the order of magnitude of the thermal velocity \( v_{\text{T}_\alpha} \).

**Problem 6**

A single particle of species \( \alpha \), moving in a plane perpendicular to a homogeneous magnetic field \( \mathbf{B}_0 \text{ vac} \), performs a circular motion ('cyclotron gyration') with angular velocity \( \omega_\alpha = -\frac{e}{m_\alpha} \mathbf{B}_0 \), where \( \omega_\alpha \) is given by Eq.(1.4.10). Sketch the electron and proton orbits for the particles starting from a given point and in the same direction but with different velocities.

Equations (1.4.8) and (1.4.9) derived here for \( J_\circ = 0 \) can also be used in the presence of a small current, \( J_\circ \neq 0 \). The last term in Eq.(1.4.4), and \( V_{\alpha1} \cdot \nabla p_{\alpha0} \) in (1.4.5), are constant as one moves along \( J_\circ \) (\( x = \text{const} \)). Therefore for \( J_\circ \neq 0 \) one can safely consider the waves propagating along \( J_\circ \), for which both terms in question remain small in the course of propagation (furthermore \( p_{\alpha0} \) is then uniform along \( \mathbf{k} \)).

The linearization procedure applied to the Maxwell equations (1.2.20)—(1.2.23) yields

\[ \nabla \times \mathbf{E}_1 = -\frac{i}{c} \frac{\partial \mathbf{B}_1}{\partial t} \]  

(1.4.14)

\[ \nabla \times \mathbf{B}_1 = \frac{i}{c} \frac{\partial \mathbf{E}_1}{\partial t} + \frac{4\pi}{c} \left( J_j + J_j^{\text{ext}} \right) \]  

(1.4.15)

2 As in Section 4.
\[ \nabla \cdot \mathbf{E}_1 = \varepsilon_0 \mu_0 \left( \varepsilon_0 q_{\text{ext}} + \varepsilon_0 q_{\text{ext}} \right) \]  
(1.4.16)

\[ \nabla \cdot \mathbf{B}_1 = 0 \]  
(1.4.17)

where (see Eqs (1.2.13) and (1.2.14))

\[ \mathbf{e}^{\mathbf{q}_{1}} = \sum_{q} q_{n} \mathbf{n}_{q} \]  
(1.4.18)

\[ \mathbf{j}^{\mathbf{q}_{1}} = \sum_{q} q_{n} \left( n_{\mathbf{c}} \mathbf{V}_{\mathbf{a}_{1}} + n_{\mathbf{d}_{1}} \mathbf{V}_{\mathbf{a}_{0}} \right) \]  
(1.4.19)

**Problem 7**

Show that a weakly damped \((|\omega| < |\omega|)\) plane plasma wave is well described in the quasi-stationary approximation, which neglects the displacement current term \((1/c) \partial \mathbf{E}_1 / \partial t\) in Eq. (1.4.15) \(\mathbf{B}^{\text{ext}} = 0\), if and only if it is a slow wave, i.e. \(v_p = \omega / k \ll c\).

### 1.5. Energy conservation for waves in fluid plasmas

To describe the energy associated with linear waves in plasmas correctly we shall start from exact and not linearized fluid equations; otherwise significant contributions might be omitted and equations lose their correct structure (see Problem 8 below). Approximations related to linearization will thus be introduced after deriving the energy balance equation for the plasma waves.

We adopt the notation:

\[ \mathbf{p} = \mathbf{p}_0 + \delta \mathbf{p} \]  
(1.5.1)

where, as in Eq. (1.1.1), \(\mathbf{p}_0\) stands for the equilibrium value of the plasma parameter \(\mathbf{p}\), and \(\delta \mathbf{p}\) represents the exact (non-linear) wave. The Maxwell equations, relating \(\delta \mathbf{E}\) and \(\delta \mathbf{B}\) to \(\delta \rho\) and \(\delta \mathbf{j}\), are obviously the same as Eqs (1.4.14)–(1.4.17) (we assume \(\rho^{\text{ext}} = 0\) and \(\mathbf{j}^{\text{ext}} = 0\)). Calculating \(\nabla \times \delta \mathbf{E}\) and \(\nabla \times \delta \mathbf{B}\) from these Maxwell equations and inserting them into the vector identity

\[ \nabla \cdot \left( \delta \mathbf{E} \times \delta \mathbf{B} \right) = \delta \mathbf{B} \cdot \left( \nabla \times \delta \mathbf{E} \right) - \delta \mathbf{E} \cdot \left( \nabla \times \delta \mathbf{B} \right) \]
we obtain, after multiplication by $c/4\pi$,

$$\frac{\partial}{\partial t} (\delta W_E + \delta W_M) + \nabla \cdot \delta \vec{P} + \delta \vec{E} \cdot \delta \vec{j} = 0$$

(1.5.2)

where

$$\delta W_E = \frac{\delta \vec{E}^2}{8\pi}, \quad \text{electric energy density}$$

(1.5.3)

$$\delta W_M = \frac{\delta \vec{B}^2}{8\pi}, \quad \text{magnetic energy density}$$

$$\delta \vec{P} = \frac{c}{4\pi \sigma} \delta \vec{E} \times \delta \vec{B}, \quad \text{Poynting vector}$$

(1.5.4)

All the quantities defined in Eqs (1.5.3) and (1.5.4) contain only the time-dependent contributions pertinent to the wave.

We first consider the cold plasma ($p_\alpha = 0$), for which an effective argument can be made. Taking the scalar product of Eq.(1.2.2) with $\vec{V}_\alpha$, and using the identity (1.2.19), we obtain after summation over all species ($\vec{E} = \delta \vec{E}$, as $\vec{E}_0 = 0$; $\vec{V}_\alpha \cdot (\vec{V}_\alpha \times \vec{B}) = 0$)

$$\delta \vec{E} \cdot \delta \vec{j} = \frac{\sigma \omega \vec{w}_K}{\partial t} + \nabla \cdot \delta \vec{j}_K$$

(1.5.5)

where

$$\vec{w}_K = \sum \frac{1}{2} \xi_\alpha \vec{V}_\alpha^2, \quad \text{kinetic energy density}$$

(1.5.6)

$$\delta j_K = \sum \frac{1}{2} \xi_\alpha \vec{V}_\alpha \vec{V}_\alpha, \quad \text{kinetic energy current density}$$

In equilibrium Eq.(1.5.5) becomes

$$0 = \frac{\sigma \omega \vec{w}_{K0}}{\partial t} + \nabla \cdot \delta \vec{j}_{K0}$$
which, if subtracted from Eq.(1.5.5), results in the replacement \( W_K \to \delta W_K \) and \( J_K \to \delta J_K \) in Eq.(1.5.5). Calculating \( \delta \vec{E} \cdot \delta \vec{j} \) from Eq.(1.5.5) thus modified, and inserting the result into Eq.(1.5.2), we arrive at the wave energy balance equation, which can be written

\[
\frac{\partial}{\partial t} \delta W + \nabla \cdot \delta \vec{j} = \delta \vec{E} \cdot \vec{j}_0
\]  

(1.5.7)

where

\[
\delta W = \delta W_E + \delta W_M + \delta W_K \quad \text{total energy density}
\]

\[
\delta \vec{j} = \delta \vec{P} + \delta \vec{J}_K \quad \text{total energy current density}
\]

and

\[
\delta W_K = \sum_a \left( \frac{1}{2} \rho_a \vec{V}_a \cdot \vec{V}_a - \frac{i}{2} \rho_{do} \vec{V}_{do} \cdot \vec{V}_{do} \right)
\]

\[
\delta \vec{J}_K = \sum_a \left( \frac{1}{2} \rho_a \vec{V}_a \cdot \vec{V}_a - \frac{i}{2} \rho_{do} \vec{V}_{do} \cdot \vec{V}_{do} \right)
\]

(1.5.9)

Thus the total energy associated with the wave in a cold plasma consists of the electromagnetic (EM) energy \( \delta W_E + \delta W_M \), and the kinetic energy \( \delta W_K \). If \( \vec{j}_0 = 0 \), or \( \vec{E} \) is perpendicular to \( \vec{j}_0 \), then in view of Eq.(1.5.7) the total energy of the wave is conserved.

The total energy flux (i.e. flow in unit time) through a surface \( S \) is given by (see Eq.(1.2.6))

\[
\int_S \delta \vec{F} \cdot \vec{n} \, ds
\]

and \( |\delta \vec{F}| \) gives the energy flux through a unit surface normal to \( \delta \vec{F} \). The Poynting vector in (1.5.8) can thus be interpreted as the current density for EM energy.

To specify the exact expressions (1.5.8) to linear waves, consider now a real linear wave (see Eq.(1.1.2), \( \omega \) real),

\[
p_{tr} = |\vec{p}| \cos (\gamma + \alpha \gamma) \quad , \quad \gamma = \vec{k} \cdot \vec{r} - \omega t
\]

(1.5.10)
and a non-linear wave $\delta p$ which propagates at the same speed ($v_p = \omega/k$), and tends to (1.5.10) in the small-amplitude limit. Such a non-linear wave can be written:

$$
\delta p = p_{1r} + \delta p'_0 + \sum_{n=2}^{\infty} \delta p'_n \cos(n \psi) + \delta p''_n \sin(n \psi)
$$

(1.5.11)

(This follows from the fact that in the reference frame in which $p_{1r}$ is stationary, $\delta p$ is also stationary and periodic with period $\lambda$. Choosing, for example, $k = k \omega$, we obtain $\psi = (2\pi/\lambda) (z - v_p t)$. Thus $\delta p(\psi)$ is a periodic function with period $2\pi$, and Eq.(1.5.11) is its Fourier expansion.) The coefficients $\delta p'_n$ and $\delta p''_n$ can be expanded in powers of the linear amplitude, and are at least second order in $|p_1|$.

Note that $\delta W_{E1}, \delta W_{M}$ and $\delta F$ are quadratic in $\delta E$ and $\delta F$, whereas $\delta W_K$ and $\delta J_K$ in general ($\omega_0 \neq 0$) contain terms from first to fourth order in $\delta \rho_\alpha$ and $\delta V_\alpha$. Thus only first- and second-order contributions to $\delta W_K$ and $\delta J_K$ will be pertinent to linear waves. These contributions are

$$
\delta W_K^{(1)} = \sum_{\alpha} \varepsilon_{\alpha 0} \delta V_{\alpha} \cdot \delta V_{\alpha_0} + \frac{1}{2} \varepsilon_{\alpha 0} \delta \vec{V}_0 \cdot \delta \vec{V}_0
$$

$$
\delta W_K^{(2)} = \sum_{\alpha} \delta \vec{V}_\alpha \cdot \delta \vec{V}_\alpha + \frac{1}{2} \varepsilon_{\alpha 0} \delta \vec{V}_0 \cdot \delta \vec{V}_0
$$

$$
\delta J_K^{(1)} = \sum_{\alpha} \frac{1}{2} \varepsilon_{\alpha 0} (\delta \vec{V}_\alpha \cdot \delta \vec{V}_\alpha + 2 \delta \vec{V}_\alpha \cdot \delta \vec{V}_\alpha) + \frac{1}{2} \varepsilon_{\alpha 0} \delta \vec{V}_0 \cdot \delta \vec{V}_0
$$

$$
\delta J_K^{(2)} = \sum_{\alpha} \frac{1}{2} \varepsilon_{\alpha 0} (\delta \vec{V}_\alpha \cdot \delta \vec{V}_\alpha + 2 \delta \vec{V}_\alpha \cdot \delta \vec{V}_\alpha)
$$

(1.5.12)

All quantities in the linear approximation are denoted by a subscript 1. For $\vec{V}_{\alpha 0} = 0$ there is only one non-vanishing term in Eqs (1.5.12) $[(1/2)\rho_{\alpha 0} \delta \vec{V}_\alpha^2]$, and we immediately obtain (see Eq.(1.5.8))

$$
W_1 = W_{E1} + W_{M1} + W_{K1}
$$

$$
\vec{J}_1 = \vec{F}_1
$$

(1.5.13)
where (see also Eq.(1.1.7), for real $\omega$)

\begin{align}
W_{E1} &= \frac{\overrightarrow{E}_{1r}^2}{8\pi} \\
W_{W1} &= \frac{1}{8\pi} \frac{1}{2} \overrightarrow{E} \cdot \overrightarrow{F}^* \tag{1.5.14}
\end{align}

\begin{align}
W_{M1} &= \frac{\overrightarrow{B}_{1r}^2}{8\pi} \\
W_{W1} &= \frac{1}{8\pi} \frac{1}{2} \overrightarrow{B} \cdot \overrightarrow{F}^* \tag{1.5.15}
\end{align}

\begin{align}
W_{K1} &= \sum_{\alpha} \frac{1}{2} \rho_{\alpha 0} \overrightarrow{v}_{\alpha 1r}^2 \\
W_{K1} &= \sum_{\alpha} \frac{1}{2} \rho_{\alpha 0} \frac{1}{2} \overrightarrow{v}_{\alpha} \cdot \overrightarrow{v}_{\alpha}^* \tag{1.5.16}
\end{align}

\begin{align}
\overrightarrow{P}_1 &= \frac{c}{4\pi} \overrightarrow{E}_{1r} \times \overrightarrow{B}_{1r} \\
\overrightarrow{P}_1 &= \frac{c}{4\pi} \frac{1}{2} \text{Re} (\overrightarrow{E} \times \overrightarrow{B}^*) \tag{1.5.17}
\end{align}

For $\overrightarrow{v}_{\alpha 0} \neq 0$ the situation becomes more complicated. The first-order contributions in Eq.(1.5.12) are different from zero and they dominate over the really interesting second-order contributions to the wave energy and energy current such as $\delta W_{E}, \delta W_{M}$ and $\delta \overrightarrow{P}$. For the non-linear waves, these first-order contributions in (1.5.12) cannot be eliminated by time-averaging because, in general (see Eq.(1.5.11)),

\begin{align}
\overrightarrow{\delta p} = \delta \overrightarrow{p} & = 0 \langle |p|^2 \rangle \neq 0 \tag{1.5.18}
\end{align}

However, after time-averaging, these contributions become second order in the linear amplitudes and are thus comparable to other interesting contributions. The time-averaging can be performed effectively in the infinite ion-mass limit ($m_i \rightarrow \infty$), and this result can be generalized to other situations.

Notice first that, on average, the non-linear wave (1.5.11) leaves the equilibrium charge and current densities undisturbed, i.e.

\begin{align}
\overrightarrow{\delta q} = \sum_{\alpha} q_{\alpha} \delta \overrightarrow{n}_{\alpha} &= 0 \tag{1.5.19}
\end{align}

This follows immediately from the non-linear analogue of the Maxwell equations (1.4.15) and (1.4.16), after time-averaging (terms involving time or space differentiations will average to zero). In the limit $m_i \rightarrow \infty$ only electrons take part
in the dynamics ($\alpha = e$), and from Eqs (1.5.19) we obtain (after multiplication by $m_e / q_e$)

$$\delta \xi_e = 0 \quad (1.5.20)$$

$$\xi_{eo} \delta \mathbf{v}_e + \xi_e \delta \mathbf{v}_e = 0 \quad (1.5.21)$$

Summing up the first- and second-order contributions in Eqs (1.5.12) (averaged in time), we can group terms into pairs which, in view of Eq.(1.5.21), will be zero. Using (1.5.20) also, and specializing the results to the linear waves, we finally get

$$\overline{W_{K1}} = \frac{1}{2} \xi_{eo} \frac{1}{2} \mathbf{v}_e \cdot \mathbf{v}_e^* \quad (1.5.22)$$

$$\overline{\mathbf{j}_{K1}} = \frac{1}{2} \xi_{eo} \left[ \left( \frac{1}{2} \mathbf{v}_e \cdot \mathbf{v}_e^* \right) \mathbf{v}_{eo} + \text{Re} \left( \mathbf{v}_{eo} \cdot \mathbf{v}_e \cdot \mathbf{v}_e^* \right) \right] \quad (1.5.23)$$

The result (1.5.22) is independent of $\mathbf{v}_{eo}$. The previously derived formula for $\overline{W_{K1}}$, Eq.(1.5.16), should therefore be expected to be valid also for $\mathbf{v}_{ao} \neq 0$. Similarly, it can be expected that in general $\overline{\mathbf{j}_{K1}}$ will be a sum over all species of terms like (1.5.23). It follows that the mean kinetic energy current (which was higher-order for $\mathbf{v}_{ao} = 0$) now has a second-order contribution in linear amplitudes, which in the linear approximation is given by Eq.(1.5.23). This contribution should be added to the Poynting vector, i.e.

$$\overline{\mathbf{J}} = \overline{\mathbf{P}_1} + \overline{\mathbf{j}_{K1}} \quad \text{for } \mathbf{v}_{do} \neq 0 \quad (1.5.24)$$

where $\overline{\mathbf{P}_1}$ is given by Eq.(1.5.17). The first term in Eq.(1.5.23) represents the convective transport of the mean kinetic energy with mean macroscopic velocity.

The average energy transport velocity $\overline{\mathbf{U}}$, for a linear wave propagating in plasma, can be defined as

$$\overline{\mathbf{U}} = \overline{\mathbf{J}} / \overline{W_I} \quad (1.5.25)$$
Derive the energy balance equation for linear waves in cold, magnetized and uniform plasma with beams ($V_{ao} \neq 0$ and satisfying (1.3.7); $J_0 = 0$), starting from the linearized fluid equations (1.4.3) and (1.4.4) (real vectors, $B_0^{pl} = 0$):

$$\frac{\partial}{\partial t} \left[ W_{E1} + W_{M1} + \sum_a \left( \frac{\partial}{\partial t} V_{d1} \cdot V_{d1} + \frac{1}{2} \frac{\partial}{\partial a} (V_{d1}^2) \right) \right]$$

$$+ \nabla \cdot \left[ \sum_a \left( \frac{\partial}{\partial t} (V_{d0} \cdot V_{d1}) V_{d0} + \frac{1}{2} \frac{\partial}{\partial a} (V_{d1}^2 V_{d0} + 2 V_{d0} V_{d1} V_{d1}) \right) \right]$$

$$= \sum_a \left[ \frac{\partial}{\partial t} (V_{d0} \cdot (V_{d1} \cdot \nabla V_{d1})) - \frac{q_a}{c} n_a V_{d0} \cdot (V_{d1} \times B_1) \right]$$

(1.5.26)

**HINT:** Start from Eq.(1.5.2) for linear waves, take the scalar product of Eq.(1.4.4) with $m_\alpha (n_{ao} V_{a1} + n_{a1} V_{ao})$, and sum over $a$.

It is instructive to compare Eq.(1.5.26) with the exact equation (1.5.7), specialized to linear waves ($\delta W \rightarrow W_{11}$, $\delta J \rightarrow J_1$, and $J_0 = 0$ as in Problem 8). Both equations are in agreement for $V_{ao} = 0$; for $V_{ao} \neq 0$, however, they have serious discrepancies. Notice that the spurious source terms on the right-hand side of Eq.(1.5.26) contain the products which were neglected when linearizing the momentum transfer equation, i.e. $V_{a1} \cdot \nabla V_{a1}$ and $V_{a1} \times B_1$.

Situations with beams can be found in which Eq.(1.5.26) takes the form of a conservation equation, e.g. a pure longitudinal wave (see Sections 1.7 and 4.2). For this wave, $B_1 = 0$, and all other vectors ($\vec{E}_1$, $\vec{V}_{a1}$, $\vec{K}$, $\vec{V}_{ao}$, $B_0^{vac}$) are parallel to a fixed direction. The only non-vanishing source term in Eq.(1.5.26) can then be written ($K = Z$):

$$\frac{\partial}{\partial t} V_{ao} \cdot (V_{d1} \cdot \nabla V_{d1}) = \frac{1}{2} \frac{\partial}{\partial a} V_{d0} \frac{dV_{d1}^2}{dz}$$

which cancels with $\nabla \cdot (1/2 \rho_{ao} V_{a1}^2 V_{ao})$. Equation (1.5.26) in that case leads to the following expressions pertinent to a pure longitudinal wave (for $m_1 \rightarrow \infty$):

$$\frac{\partial}{\partial t} V_{K1} = \frac{1}{2} \frac{\partial}{\partial a} \left| V_e \right|^2 + \frac{1}{2} Re \left( \frac{\rho_e V_e^2}{z} \right) V_{eo}$$

(1.5.27)
This differs considerably from the results following from Eqs (1.5.22) and (1.5.23):

\[ \bar{W}_{K1} = \frac{1}{2} \varepsilon_{eo} \frac{1}{2} |V_e|^2, \quad \bar{J}_{K1} = 3 \bar{W}_{K1} V_{eo} \]  

(1.5.29)

In particular, \( \bar{W}^{lin}_{K1} \), given by Eq. (1.5.27), depends on \( V_{eo} \) and is not positive definite. This discrepancy is discussed in more detail in Section 4.2.

The analysis given in this section for a cold plasma can be generalized to warm plasmas. Using Eqs (1.2.3) and (1.2.1), an additional term

\[ \sum_\alpha \vec{V}_\alpha \cdot \nabla p_\alpha \]

which will appear on the right-hand side of Eq. (1.5.5) can be written in a 'conservative' form:

\[ \vec{V}_\alpha \cdot \nabla p_\alpha = \frac{2}{2} \frac{\partial}{\partial t} \frac{p_\alpha}{t_\alpha - 1} + \nabla \cdot \left( \frac{\delta p_\alpha}{t_\alpha - 1} \vec{V}_\alpha \right) \]  

(1.5.30)

This will introduce the following thermal contributions to the wave energy and to wave energy current densities:

\[ \delta W_T = \sum_\alpha \frac{\delta p_\alpha}{t_\alpha - 1}, \text{ thermal energy density} \]

(1.5.31)

\[ \delta \vec{J}_T = \sum_\alpha \frac{\delta \vec{V}_\alpha}{t_\alpha - 1} \left( \vec{p}_{\delta 0} \delta \vec{V}_\alpha + \delta \vec{p}_\alpha \vec{V}_{\delta 0} + \delta \vec{p}_\alpha \delta \vec{V}_\alpha \right), \text{ thermal energy current density} \]

It can be seen that \( \delta W_T \) and \( \delta \vec{J}_T \) contain terms linear in \( \delta p_\alpha \) and \( \delta \vec{V}_\alpha \) even for \( \vec{V}_{\delta 0} = 0 \), which in general will survive after time-averaging in the form of second-order contributions. This will introduce thermal corrections \( \bar{W}_{T1} \) and \( \bar{J}_{T1} \) to the energy and energy current densities \( \bar{W}_1 \) and \( \bar{J}_1 \). (More about \( \bar{W}_1 \) in a warm plasma in Section 2.1.)
1.6. Plasma dielectric tensor

A convenient macroscopic characteristic of a continuous medium (e.g. a dielectric or plasma) is the dielectric tensor. It enables certain facts related to the wave propagation to be formulated in a simple and model-independent way.

Note that the Maxwell equations (1.4.15) and (1.4.16) can be written in a form analogous to that in vacuum \( \hat{j}_1 = 0, \rho_\text{q1} = 0 \) if we introduce the electric displacement vector \( \hat{D}_1 \), which for the harmonic time dependence, \( \exp(-i\omega t) \), is defined as

\[
\hat{D}_1 = \hat{E}_1 + \frac{4\pi i}{\omega} \hat{j}_1
\]  

(1.6.1)

Using this definition and the charge-conservation equation (1.2.16), we can write the Maxwell equations as

\[
\nabla \times \hat{E}_1 = \frac{i\omega}{c} \hat{B}_1
\]  

(1.6.2)

\[
\nabla \times \hat{B}_1 = -\frac{i\omega}{c} \hat{D}_1 + \frac{4\pi}{c} \hat{j}_1 \text{ext}
\]  

(1.6.3)

\[
\nabla \cdot \hat{D}_1 = \frac{4\pi}{c} \delta_\text{q1} \text{ext}
\]  

(1.6.4)

\[
\nabla \cdot \hat{B}_1 = 0
\]  

(1.6.5)

Using Eq.(1.4.19) in (1.6.1), we obtain

\[
\hat{D}_1 = \hat{E}_1 + \frac{4\pi i}{\omega} \sum_q q_d (n_{d\text{e}} \hat{V}_{\text{e}1} + n_{d\text{i}} \hat{V}_{\text{i}d})
\]  

(1.6.6)

In the linear theory \( \hat{D}_1 \) should depend linearly on \( \hat{E}_1 \), i.e.

\[
\hat{D}_1 = \vec{\varepsilon} \cdot \hat{E}_1
\]  

(1.6.7)

where \( \vec{\varepsilon} \) is a linear operator. The detailed form of \( \vec{\varepsilon} \) depends on the model adopted, which should define \( \nabla\alpha_i \) and \( n_{\alpha1} \) in (1.6.6) in terms of \( \hat{E}_1 \). For the plane-wave perturbations of the form (1.4.6), \( \vec{\varepsilon} \) is an algebraic linear operator, i.e. a second-order tensor, or \( 3 \times 3 \) matrix (in a fixed coordinate system). The tensor \( \vec{\varepsilon}(\omega, \mathbf{k}) \) is called a plasma dielectric tensor. Calculating \( \hat{B}_1 \) from Eq.(1.6.2) (\( \nabla \rightarrow ik \)),

and, using this result in (1.6.3), we obtain an equation for \( \vec{E}_1 \) which can be written:

\[
\left( \frac{kc}{\omega} \right)^2 \left[ \vec{E}_1 - \vec{E}_1 \cdot \hat{k} \hat{k} \right] - \vec{E}_1 \cdot \vec{E}_1 = \frac{q^2 i}{\omega} \frac{\vec{E}_1}{j_1}^{\text{ext}} \tag{1.6.9}
\]

If \( \tilde{\varepsilon}(\omega, \vec{k}) \) is known for the equilibrium in question (for arbitrary \( \omega \) and \( \vec{k} \)), then Eq.(1.6.9) describes the plasma response to the excitation by a single \( (\omega, \vec{k}) \) harmonic (for \( j_1^{\text{ext}} \neq 0 \)), and the plasma eigenoscillations (if \( j_1^{\text{ext}} = 0 \)). Here we examine the eigenoscillations only, and therefore assume

\[
\vec{E}_1^{\text{ext}}, j_1^{\text{ext}} = 0 \tag{1.6.10}
\]

Equation (1.6.9) is in that case a set of three linear homogeneous algebraic equations (for \( E_{1x}, E_{1y}, E_{1z} \)). This set is solvable only if its characteristic determinant is zero, which leads to a relation between \( \omega \) and \( \vec{k} \) called the dispersion relation. Note that no such relation exists in the problem of excitation (if \( j_1^{\text{ext}} \neq 0 \)), where \( \omega \) and \( \vec{k} \) are independent parameters.

**Problem 9**

*Find the dispersion relation and a general form of waves in a dielectric characterized by a scalar dielectric constant \( \varepsilon > 0 \) \( (\tilde{\varepsilon} \cdot \vec{E}_1 = \varepsilon \vec{E}_1) \), by introducing the coordinate system with the z-axis taken along \( \vec{k} \),

\[
\omega = v_p k, \quad \text{where} \quad v_p = c/\sqrt{\varepsilon} \quad \text{(dispersion relation)} \tag{1.6.11}
\]

\[
\vec{E}_1 = \left( \vec{x} E_x + \vec{y} E_y \right) \exp \left[ ik \left( z - v_p t \right) \right]
\]

\[
\vec{B}_1 = \frac{c}{v_p} \left( \vec{y} E_x - \vec{x} E_y \right) \exp \left[ i k \left( z - v_p t \right) \right] \tag{1.6.12}
\]

where \( E_x \) and \( E_y \) are arbitrary complex constants. Sketch \( \vec{E}_{1z} \) and \( \vec{B}_{1z} \) as functions of \( t \left( z = \text{const} \right) \), and functions of \( z \left( t = \text{const} \right) \), for

(i) \( E_x = 0 \),  (ii) \( E_x = i E_y \),  (iii) \( E_x = i 2 E_y \)
FIG. 1. Directions of real fields $\vec{E}_1$ and $\vec{B}_1$ associated with a plane wave propagating in the plasma; definition of longitudinal and transverse components of $\vec{E}_1$ (real $\omega$).

In general, the electric field described by Eq.(1.6.9) can be decomposed into a longitudinal component $E_1^l$ ($\vec{E}_1^l \cdot \vec{k} = 0$), and a transverse component $\vec{E}_1^t$ ($\vec{E}_1^t \cdot \vec{k} = 0$), i.e.

$$E_1 = E_1^l + E_1^t \quad \text{and} \quad \vec{E}_1^l = \vec{k} \cdot \vec{E}_1$$

(1.6.13)

Note in this context that the magnetic field $\vec{B}_1$ is always transverse, as $\vec{B}_1 \cdot \vec{k} = 0$ (see Eqs (1.6.5), (1.6.8)). These notions (longitudinal and transverse) are also pertinent to the real fields, as $\vec{k}$ is real. Thus, for example, $\vec{B}_1^l \cdot \vec{k} = 0$, which means that the real magnetic field $\vec{B}_1^l$ is always orthogonal to $\vec{k}$. This field is also orthogonal to $\vec{E}_1^l$ if $\omega(\vec{k})$ is real (see Fig.1), as

$$\vec{B}_1^l = \frac{c}{\omega} \vec{k} \times \vec{E}_1^l \quad \text{for real } \omega$$

(1.6.14)

Usually one or more intervals of real $\omega$ can be found, which are referred to as the propagation bands. They have either natural bands ($\omega = 0$ or $\omega = \infty$) or are bounded by finite frequencies at which the wavelength $\lambda (= 2\pi/\kappa)$ becomes either infinite or zero. These situations are referred to as the cutoff ($k = 0$) or resonance ($k = \infty$). If $\vec{B}_1 \neq 0$, then $\vec{B}_1 \to 0$ at the cutoff (and $\vec{P}_1 \to 0$).

(More about resonances in Section 3.3.)
Problem 10

Assuming that \( \omega \) is real, express the magnetic energy (1.5.15) and the Poynting vector (1.5.17) in terms of the longitudinal and transverse components of \( \mathbf{E} \) (\( \nu_p = \omega/k, \ k = \mathbf{k}/k \)):

\[
\overrightarrow{W_M} = \frac{c^2}{\nu_p^2} \frac{|\mathbf{E} \times \mathbf{B}|^2}{4\pi}
\] (1.6.15)

\[
\overrightarrow{P_1} = \frac{c^2}{\nu_p} \left[ \frac{|\mathbf{E} \times \mathbf{B}|^2}{8\pi} \mathbf{k} - \Re \left( \frac{\mathbf{E} \cdot \mathbf{E}^*}{8\pi} \right) \right]
\] (1.6.16)

Give the physical meaning of Eqs (1.6.15) and (1.6.16) for waves propagating in vacuum (see Problem 9, \( \epsilon = 1 \)).

1.7. Electrostatic versus electromagnetic waves and instabilities

The electrostatic wave (or instability) is a generalization of the electrostatic field, which is produced by static electric charges. The charges at rest cannot produce any magnetic field because both the current and the displacement current in the Maxwell equation,\( \mathbf{j}_\text{d} = 0 \) are zero. For the electrostatic wave these two currents (in Eq.(1.4.15), \( \mathbf{j}_\text{ext} = 0 \)) are not zero, but they cancel each other, so that again no magnetic field is produced.

Thus a characteristic feature of electrostatic waves (or instabilities) is that they do not perturb the magnetic field, i.e. (see Eq.(1.4.14))

\[
\mathbf{B}_1 = 0 \rightarrow \nabla \times \mathbf{E}_1 = 0 \rightarrow \mathbf{E}_1 = -\nabla \phi_1 \text{, for electrostatic waves}
\] (1.7.1)

Inserting \( \mathbf{E}_1 \) from (1.7.1) into Poisson's equation (1.4.16) (\( \rho^{\text{ext}} = 0 \)), we obtain an equation for the (time-dependent) scalar potential \( \phi_1 \):

\[\text{Inserting } \mathbf{E}_1 \text{ from (1.7.1) into Poisson's equation (1.4.16) (} \rho^{\text{ext}} = 0 \text{), we obtain an equation for the (time-dependent) scalar potential } \phi_1 \text{:} \]
\( \nabla^2 \phi_1 = -4\pi \rho_{q1} \) \hspace{1cm} (1.7.2)

which fully describes the electrostatic wave.

A characteristic feature of electromagnetic waves (or instabilities) is that they do not perturb the electric charge density, i.e. (see Eq.(1.4.16))

\[ \rho_{q1} = 0 \rightarrow \nabla \cdot \vec{E}_1 = 0, \text{ for electromagnetic (EM) waves} \] \hspace{1cm} (1.7.3)

These notions acquire a simple meaning for the plane waves \((\nabla \rightarrow ik)\):

\[ \vec{k} \times \vec{E}_1 = 0 \quad \text{for electrostatic waves} \quad (\vec{E}_1 = \vec{E}_1^0) \] \hspace{1cm} (1.7.4)

\[ \vec{k} \cdot \vec{E}_1 = 0 \quad \text{for EM waves} \quad (\vec{E}_1 = \vec{E}_1^1) \] \hspace{1cm} (1.7.5)

Thus the plane electrostatic wave is pure \textit{longitudinal}, and the EM wave is pure \textit{transverse}.

We recall (Fig.1) that \( \vec{B}_{1r} \) is always perpendicular to \( \vec{k} \). Thus for EM waves both \( \vec{E}_{1r} \) and \( \vec{B}_{1r} \) are perpendicular to \( \vec{k} \), and therefore the Poynting vector \( \vec{P}_1 \) is directed along \( \vec{k} \) (see Eq.(1.5.17)). If \( \omega \) is real, then in addition \( \vec{E}_{1r} \) is orthogonal to \( \vec{B}_{1r} \) (see Eq.(1.6.14)). This is not generally the case for complex \( \omega \), except when the EM wave is \textit{linearly} polarized. Choosing, in that case, the x-axis along \( \vec{E}_{1r} \) \((\vec{E}_1 = \vec{x} E_1)\) and the z-axis along \( \vec{k} \) \((\vec{k} = \vec{z})\), we obtain

\[ \vec{E}_{1r} = \vec{x} \text{ Re} \vec{E}_1 \]
\[ \vec{B}_{1r} = \text{ Re} \frac{\vec{k}}{\omega} \vec{k} \times \vec{E}_1 = \vec{y} \text{ Re} \left( \vec{E}_1 / \omega \right) \] \hspace{1cm} (1.7.6)

In standard electrodynamics of continuous media the EM waves are of great importance. They are the only waves that can propagate in a vacuum, or in a dielectric characterized by \( \varepsilon > 0 \) (see Problem 9, where \( \vec{k} \cdot \vec{E}_1 = 0 \)). In plasma theory the electrostatic waves and instabilities are important and are extensively discussed in the literature. They are both simple (scalar potential) and related to important physical phenomena (e.g. Landau damping, large growth rates of instabilities, etc.).
2. WAVES IN ISOTROPIC PLASMAS ($\vec{E}_0 = \vec{B}_0 = 0; \vec{V}_{\alpha_0} = 0$)

2.1. General properties

We assume in this section that, in equilibrium, plasma is field-free ($\vec{E}_0 = \vec{B}_0 = 0$), and all macroscopic velocities are zero (no beams). With these assumptions the plasma equilibrium is strictly uniform (as $j_0 = 0$, see Section 1.3) and isotropic (there is no preferred direction in equilibrium). Waves in such plasma have certain general features, independent of the model. They will also be exhibited in our simple fluid analysis.

The linearized momentum transfer equation (1.4.9) now becomes

$$-i\omega \vec{V}_{\alpha l} = -C_{\alpha}^2 i \frac{1}{k} (\vec{k} \cdot \vec{V}_{\alpha l})/\omega + \left( \frac{q_{\alpha}}{m_{\alpha}} \right) \vec{E}_l$$

(2.1.1)

where the first term on the right-hand side represents the plasma thermal motion (the pressure force, see (1.4.13)); in cold plasma $C_{\alpha}^2 = 0$. Equation (2.1.1) separates into two equations for the longitudinal and transverse components of $\vec{E}_l$ and $\vec{V}_{\alpha l}$ (see (1.6.13)). This is evident for a cold plasma but is also true for $C_{\alpha}^2 \neq 0$, as $k (k \cdot \vec{V}_{\alpha l}) = k^2 \vec{V}_{\alpha l}$. Solving these two equations, we obtain

$$\vec{V}_{\alpha l} = i \frac{q_{\alpha}}{m_{\alpha}} \frac{1}{\omega} \vec{E}_l$$

(2.1.2)

We can now express the electric displacement $\vec{D}_l$, given by (1.6.6) ($\vec{V}_{\alpha_0} = 0$), in terms of $\vec{E}_l$ and determine the plasma dielectric tensor, i.e.

$$\vec{D}_l \equiv \vec{E} \cdot \vec{E}_l = \vec{E}_l - \sum_{\alpha} \frac{4\pi n_{\alpha_0} q_{\alpha}^2}{m_{\alpha}} \left( \frac{\vec{E}_l^2}{\omega^2 - C_{\alpha}^2 k^2} + \frac{\vec{E}_l^t}{\omega^t} \right)$$

(2.1.3)

This result can also be written (see (1.6.13))

$$\vec{E} \cdot \vec{E}_l = \epsilon^t \vec{E}_l^t + \epsilon^t \vec{E}_l^t \equiv \left[ \epsilon^t \vec{k} \vec{k} + \epsilon^t \left( \vec{I} - \vec{k} \vec{k} \right) \right] \cdot \vec{E}_l$$

(2.1.4)

leading to

$$\epsilon = \epsilon^t \vec{k} \vec{k} + \epsilon^t \left( \vec{I} - \vec{k} \vec{k} \right)$$

(2.1.5)

where $\vec{k} = k/k$, $\vec{I}$ is a unit tensor, and

$$\epsilon^t = 1 - \sum_{\alpha} \frac{\omega_{\alpha_0}^2}{\omega^2 - C_{\alpha}^2 k^2}$$

(2.1.6)
We have introduced here the plasma frequency for the \( \alpha \)-species \( \omega_{p\alpha} \) and for the whole plasma \( \omega_p \), defined by:

\[
\omega_{p\alpha}^2 = \frac{4\pi n_{\alpha} q_{\alpha}^2}{m_{\alpha}}, \quad \omega_p^2 = \sum_{\alpha} \omega_{p\alpha}^2 \approx \omega_{p\text{e}}^2
\]  

(2.1.8)

In the cold plasma \( (C^2_0=0) \)

\[
\varepsilon^l = \varepsilon^t = 1 - \frac{\omega_p^2}{\omega^2} \approx \varepsilon_0
\]  

(2.1.9)

and from Eq.(2.1.4) we have

\[
\varepsilon \cdot \vec{E}_1 = \varepsilon_0 (\varepsilon_1^t + \varepsilon_1^l) = \varepsilon_0 \vec{E}_1
\]  

(2.1.10)

Thus the dielectric tensor of a cold plasma reduces to a scalar \( \varepsilon_0 \), given by (2.1.9), called the plasma dielectric constant. Inserting (2.1.4) into the equation for \( \vec{E}_1 (1.6.9) \) \((\vec{j}_1 = 0)\) we obtain

\[
\left[ \left( \frac{k_c}{\omega} \right)^2 - \varepsilon^t \right] \varepsilon_1^t - \varepsilon^l \vec{E}_1^l = 0
\]  

(2.1.11)

Equation (2.1.11) obviously separates into two independent equations, \( \varepsilon^l \vec{E}_1^l = 0 \), with the dispersion relation

\[
\varepsilon^l (\omega, k) = 0
\]  

(2.1.12)

and \[ \left( \frac{k_c}{\omega} \right)^2 - \varepsilon^t \right] \varepsilon_1^t = 0 \] with the dispersion relation

\[
\left( \frac{k_c}{\omega} \right)^2 = \varepsilon^t (\omega, k)
\]  

(2.1.13)

Equations (2.1.4) and (2.1.5), derived here in the fluid model, are quite general. They follow from the fact that in the uniform and isotropic medium there is no preferred direction. This leads immediately to the first part of Eq.(2.1.4), where \( \varepsilon^l \) and \( \varepsilon^t \) are scalar functions of \( \omega \) and \( k \) (not \( \vec{k} \), owing to isotropy), their detailed form depending on the model; in the fluid model they
are given by Eqs (2.1.6) and (2.1.7). Equations (2.1.12) and (2.1.13), which are a consequence of (2.1.4), are therefore also general.

It follows that in an isotropic medium all perturbations are either pure longitudinal (electrostatic), with the dispersion relation (2.1.12), or pure transverse (EM), with the dispersion relation (2.1.13). If \( e^k \neq 0 \) for arbitrary \( \omega \) and \( k \), then the electrostatic waves are excluded (see Problem 9, where \( e^k = e^t = e \neq 0 \)). On the other hand, a necessary condition for existence of the (undamped) EM waves is \( e^t > 0 \). This condition also becomes sufficient if, as in our fluid plasma, \( e^t \) is only a function of \( \omega \) (see (2.1.7)). In that case \( k^2(\omega) = (\omega^2/c^2)e^t(\omega) \) is positive (for real \( \omega \)) if and only if

\[
e^t(\omega) > 0, \quad \text{real } \omega
\]  

Equation (2.1.14) defines the propagation bands (\( \omega \) and \( k \) both real) for EM waves in an isotropic medium. In plasma, both types of waves are admitted, and their properties are discussed in some detail in Sections 2.2 and 2.3.

The dielectric tensor (2.1.5) can be used to express in a simple way (assuming, as in Section 1.5, that \( \omega \) is real) the total-energy associated with the wave in isotropic plasma, in terms of its electric energy. First, the kinetic energy can be expressed in terms of electric energy by using Eqs (2.1.2). For longitudinal waves we obtain (see Eqs (1.5.14) and (1.5.16))

\[
\overline{W}_K^t = \sum \frac{\omega_p^2 \omega^2}{(\omega^2 - C^2_0 k^2)^2} \overline{W}_E^t
\]  

which can be written (see Eq.(2.1.6))

\[
\overline{W}_K^t = \frac{1}{2} \omega \frac{\partial}{\partial \omega} \overline{W}_E^t = \frac{1}{2} \omega \left( \omega \overline{\varepsilon}^t \right) \overline{W}_E^t
\]  

(2.1.15') (we used the fact that \( e^k = 0 \)). Similarly for the transverse waves,

\[
\overline{W}_K^t = \omega_p^2 \omega^2 \overline{W}_E^t = \left[ \frac{\partial}{\partial \omega} \left( \omega \overline{\varepsilon}^t \right) - 1 \right] \overline{W}_E^t
\]  

(2.1.16)

In cold plasma, using (2.1.9), we obtain

\[
\frac{\partial}{\partial \omega} \left( \omega \overline{\varepsilon}_0 \right) = 1 + \frac{\omega_p^2}{\omega^2}
\]  

(2.1.17)

where for the electrostatic waves (\( \overline{\varepsilon}_0 = 0 \)) \( \omega_p^2/\omega^2 = 1 \). With this in mind it can be seen that (2.1.15') and (2.1.16) lead to the same formula for the total energy of the wave, i.e.
where $W_{M1} = |\vec{B}|^2/16\pi$. Indeed, specializing Eq. (2.1.18) to longitudinal and transverse waves, we obtain

$$
\bar{W}_{E1} = \frac{1}{16\pi} \bar{E}^* \cdot \frac{\partial}{\partial \omega} \left( \omega \bar{\epsilon} \right) \cdot \bar{E} + \bar{W}_{M1}
$$

(2.1.18)

$$
\bar{W}_{M1} = \frac{1}{16\pi} \bar{E}^* \cdot \left( \frac{\partial}{\partial \omega} \left( \omega \bar{\epsilon} \right) \cdot \bar{E} + \bar{W}_{M1} \right)
$$

(2.1.19)

which is in agreement with Eqs (2.1.15') and (2.1.16).

Formula (2.1.18) for $\bar{W}_{E1}$, derived here for a cold isotropic plasma, is more general. This problem of the mean energy associated with linear waves in dispersive media was treated by von Laue [6], Stix [7], and Landau and Lifshitz [8]. The formula for $\bar{W}_{E1}$ derived by Stix coincides with Eq. (2.1.18) if $\bar{\epsilon}$ is Hermitian for real $\omega$ (i.e. $\epsilon_{ij}^* = \epsilon_{ji}$). That is the case for the dielectric tensor (2.1.5) and therefore, in our fluid model, Eq. (2.1.18) should be applicable also to a warm isotropic plasma.

Note that one can eliminate $\bar{B}$ from Eq. (2.1.18) by using Eqs (1.6.15) and (1.6.9), which leads to

$$
\bar{W}_{E1} = \bar{E}^* \cdot \left( \frac{\partial}{\partial \omega} \left( \omega \bar{\epsilon} \right) + \bar{E} \right) / 16\pi
$$

(2.1.20)

Using formulas (2.1.15') to (2.1.17), as well as (1.6.15) and (2.1.13), we easily arrive at the following wave-energy *equipartition relations* in the fluid model, valid for waves in isotropic plasma:

$$
\bar{W}_{K1} = \bar{W}_{E1} + \bar{W}_{T1}
$$

(2.1.21) for longitudinal waves

$$
\bar{W}_{E1} = \bar{W}_{M1} + \bar{W}_{K1}
$$

(2.1.22) for transverse waves

This section ends with a useful observation following from Eqs (2.1.2) and (1.4.9). For longitudinal waves the direction of $\bar{E}_1$ is fixed, as it is given by the direction of $\vec{k}$. It is also fixed for a transverse wave if the latter is *linearly* polarized (as in (1.7.6)). In both cases the direction of $\bar{V}_{\alpha 1}$ (for all $\alpha$) coincides with the fixed direction of $\bar{E}_1$ (see Eq. (2.1.2)). Thus if we switch on a homogeneous magnetic field $\bar{B}_0$ parallel to the fixed direction of $\bar{E}_1$ and $\bar{V}_{\alpha 1}$, the presence of $\bar{B}_0$ will be ignored by the wave ($\vec{V}_{\alpha 1} \times \omega_{c\alpha} = 0$ in Eq. (1.4.9)).
2.2. Electrostatic waves: Langmuir oscillations and ion-acoustic wave

The dispersion relation for electrostatic waves in an isotropic plasma can be written (assuming \( \omega \) to be real — see Eqs (2.1.6) and (2.1.12)):

\[
\sum_{\alpha} \frac{\omega_{\alpha}^2}{1 - \frac{C_{\alpha}^2}{v_p^2}} = \omega^2
\]  

(2.2.1)

where \( v_p(= \omega/k) \) is the phase velocity. Equation (2.2.1) defines implicitly \( v_p^2 \) in terms of \( \omega^2 \). Notice that the ratio \( C_{\alpha}^2/v_p^2 \) in the denominator of Eq.(2.2.1) comes from the pressure term in Eq.(2.1.1), whereas unity corresponds to the left-hand side of (2.1.1), which is proportional to the inertial term \( (\rho \omega_0 \partial^2 \partial t) \). Thus the pressure terms (thermal corrections), which are zero in a cold plasma, are also negligible in the limit \( v_p^2 \to \infty \) (strictly speaking, for \( C_{\alpha}^2/v_p^2 < 1 \), i.e. if \( v_p^2 \gg v_{T\alpha}^2 \) for all \( \alpha \), which is the essence of the ‘cold plasma approximation’ in application to waves). A ‘cold plasma’ so defined exhibits a constant resonant frequency,

\[
\omega^2 \simeq \omega_{pe}^2 \equiv \sum_{\alpha} \omega_{\alpha}^2 \simeq \omega_{pe}^2 \quad \text{if} \quad v_p^2 \gg v_{T\alpha}^2
\]  

(2.2.2)

which corresponds to electron and ion oscillations in the self-consistent electric field \( E_1 \). The ion response to \( E_1 \) is much weaker than the electron response (as \( q_i/m_i \ll e/m_e \)) and that is why the resonant frequency is mainly determined by the electron properties \( (\omega \equiv \omega_{pe}) \); it becomes exactly equal to \( \omega_{pe} \) in the infinite ion mass limit, \( m_i \to \infty \). Note that \( \nu_{pe} \equiv \omega_{pe}/2\pi \) is uniquely determined by the equilibrium electron concentration \( n_{e0} \),

\[
\nu_{pe} = (1/2\pi) \sqrt{4\pi n_{e0} e^2/m_e} = 8.98 \times 10^3 \sqrt{n_{e0} \text{, Hz , cm}^{-3}}
\]  

(2.2.3)

For laboratory plasmas \( n_{e0} \sim (10^{12} - 10^{19}) \text{ cm}^{-3} \), and so \( \nu_{pe} \sim (10^4 - 10^7) \text{ MHz} \), which is often the largest macroscopic frequency in the plasma system.

The high-frequency plasma oscillations (2.2.2) are usually called Langmuir oscillations, and \( \omega_{pe} \) is known as Langmuir frequency. To examine them in more detail, we take for simplicity the infinite ion mass limit, in which

\[
\mathcal{E} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \gamma_e^2 v_{Te}^2 k^2}
\]  

(2.2.4)

From the dispersion relation \( (e\xi = 0) \) we obtain

\[
\omega^2 = \omega_{pe}^2 \left[ 1 + \gamma_e^2 (k \lambda_{pe})^2 \right]
\]  

(2.2.5)
where

$$\lambda_{De} \equiv \frac{v_{Te}}{\omega_{pe}} = \sqrt{\frac{KTe}{4\pi n_{e0}e^2}} \quad (2.2.6)$$

is the electron Debye length. The Langmuir oscillations ($\omega \approx \omega_{pe}$) are obtained in the long-wavelength limit, $(k\lambda_{De})^2 \ll 1$, i.e. for $\lambda \gg \lambda_{De}$. In this limit the analysis based on the Vlasov equation can also be effectively performed (see Ref.[5], p.383). It leads to complex $\omega(= \omega_t + i\omega_i, \omega_i < 0)$ but $|\omega_i| \ll |\omega_t|$ ('small Landau damping'). For the real part of the frequency, $\omega_t$, an equation of the form (2.2.5) is obtained in which $\gamma_e = 3$. This can be interpreted as the adiabatic compression of the one-dimensional electron gas, which is a reasonable picture if one recalls that the oscillations are longitudinal and notices that

$$\left( k \lambda_{De} \right)^2 \ll 1 \quad \Rightarrow \quad \frac{\omega_{pe}^2}{k^2} = \frac{v_{Te}^2}{\left( k \lambda_{De} \right)^2} \gg v_{Te}^2 \quad (2.2.7)$$

Thus electrons in their thermal motion are very slow compared to the wave, and in one period of oscillation ($T \approx 2\pi/\omega_{pe}$) can travel only at a very small fraction of the wavelength.

The dispersion relation (2.2.5), with $\gamma_e = 3$, was first derived by Bohm and Gross [9]. Nowadays there is also experimental evidence for the validity of this choice of $\gamma_e$ for high-frequency plasma oscillations (see e.g. Ref. [5], p.161).

To examine all possible waves described by Eq.(2.2.1) it is convenient to plot $v_p^2$ versus $\omega^2$. If $v_p^2 \rightarrow C_\alpha^2 + 0$, then $\omega \rightarrow \infty$, which means that each species $\alpha$ produces a separate branch of $v_p^2(\omega^2)$ (see Fig.2). The electron branch is described by Eq.(2.2.5), and we now concentrate on the ion branches.

Note that the velocities $C_\alpha$, defined formally by Eqs (1.4.11) and (1.4.13), can be interpreted as the partial sound speeds in a neutral gas obtained by putting $q_\alpha = 0$. Indeed, for $q_\alpha = 0$, Eq.(2.1.1) leads to $v_{a1} = v_{\alpha1}$, and

$$\left( \omega^2 - k^2C_\alpha^2 \right) V_{\alpha1}^\ell = 0 \quad (2.2.8)$$

which describes a longitudinal wave propagating with a constant phase velocity, $v_p^2 = C_\alpha^2$, i.e. a sound wave. The fact that $v_p^2 \rightarrow C_\alpha^2$, as $\omega \rightarrow \infty$, means that in the high-frequency limit the fluid plasma behaves as a neutral gas, i.e. the pressure term in Eq.(2.1.1) dominates over the electric force term. For $\omega \rightarrow 0$, $v_p^2$ given by Eq.(2.2.1) also tends to a constant, which satisfies

$$\sum_i \frac{\omega_{pi}^2}{v_p^2 - C_i^2} = \frac{\omega_{pe}^2}{C_e^2 - v_p^2} \quad (2.2.9)$$
FIG. 2. High- and low-frequency branches of electrostatic waves in multicomponent plasma. Thick lines denote weakly damped oscillations: (a) Langmuir oscillations; (b) ion-acoustic wave; (c) damped-ion branch.

FIG. 3. Graphical solution of Eq.(2.2.9) (two kinds of ion, $m_{i1} > m_{i2}$).

Plotting the left- and right-hand sides of Eq.(2.2.9) versus $v_p^2$, it can be noticed (see Fig.3) that each kind of ion produces a separate solution: $v_p^2 = C_i^{r2}$ and $C_i^{r2} > C_j^{r2}$. For each ion branch, $v_p^2$ is a decreasing function of $\omega^2$, satisfying (see Fig.2)
We show now that only one of the ion branches has a physical meaning, i.e. represents a weakly damped normal mode ($|\omega_i| \ll |\omega_1|$), which can propagate in a real plasma. All others are strongly damped ($\omega_i < 0$) owing to the energy exchange between the wave and the 'resonant' ions whose velocity is close to the phase velocity of the wave (see e.g. Ref. [5], p.12). This effect, called Landau damping, depends on the details of the particle distribution functions and is therefore beyond the scope of the fluid description (that is why in our analysis $\omega_i = 0$).

For our purposes it is enough to realize that the Landau damping caused by the $\alpha$-type particles will be large only if the phase velocity of the wave is close to the thermal speed $v_{\text{T}\alpha}$. (The damping depends on the slope of the distribution function at $v = v_p$, which is maximum at $v = v_{\text{T}\alpha}$.) Thus only a longitudinal wave with phase velocity well separated from all thermal speeds $v_{\text{T}\alpha}$ will be weakly Landau damped.

Note first that the Langmuir oscillations are weakly damped in view of Eq.(2.2.7) ($v_{\text{T}e}^2 < v_{\text{T}e}^2$ if $T_i/T_e < m_i/m_e \sim 10^3$). The short-wavelength sounds ($\omega \to \infty$, $v_p^2 \approx C_i^2$) are heavily Landau-damped by ions, as $C_i^2 \sim v_{\text{T}\alpha}^2$.

Figure 3 seems to suggest that $C_i^2 = C_{i2}$, which in view of Eq.(2.2.10) would mean that all ion branches are strongly Landau-damped by ions. However, for $T_e \gg T_i$ the electron part of the plot shown in Fig.3 (RHS) shifts to the right, so that $C_i^2$ get away from $C_{i2}$. The ion branch corresponding to the largest $C_i^2$ becomes well separated in that case (for $\omega$ not too large) from all thermal speeds, i.e. it satisfies

$$C_i^2 \ll v_p^2 \ll C_e^2 \quad (2.2.11)$$

For other types of ion, $C_i^2$ remain between the neighbouring $C_i^2$ (see Fig.3) and therefore cannot be well separated from $C_{i2}$. That is because in practice $C_i^2$ do not differ very much.

Thus only the ion branch satisfying condition (2.2.11) (usually called the ion-acoustic wave) will be weakly damped. Using condition (2.2.11) in Eq.(2.1.6), we obtain

$$\mathcal{E}_{\text{ion acoustic}} = 1 - \sum_i \frac{\omega_i^2}{\omega^2} + \frac{\omega_{pe}^2}{C_e^2 k^2} = 1 - \sum_i \frac{\omega_i^2}{\omega^2} + \frac{1}{\gamma_e (k \lambda_{pe})^2} \quad (2.2.12)$$

Thus the dispersion relation ($\mathcal{E} = 0$) for the ion-acoustic wave is

$$\sum_i \frac{\omega_i^2}{\omega^2} = 1 + \frac{1}{\gamma_e (k \lambda_{pe})^2} \quad (2.2.13)$$
which can also be written:

\[ \omega^2 = \frac{k^2 C_s^2}{1 + \frac{\omega_p^2}{\omega_i^2}} \]  

(2.2.13')

where

\[ C_s^2 = \gamma_e \lambda_{De}^2 \sum_i \omega_p^{-2} \equiv \gamma_e k T_e \sum_i \frac{r_i Z_i}{m_i} \]  

(2.2.14)

We have introduced

\[ Z_i = \frac{q_i}{e} \quad \text{ion charge number} \]  

(2.2.15)

\[ r_i = \frac{n_{i0} Z_i}{n_{e0}} \quad \left( \sum_i r_i = 1 \right) \]

Here \( r_i \) gives the fraction of the total number of electrons in the unit volume, \( n_{e0} \), which compensate the given type of ionic charge (in equilibrium); in the two-component plasma (one kind of ion) \( r_i = 1 \). For \((k\lambda_{De})^2 \ll 1\) we obtain

\[ \omega^2 / \sum_i \omega_p^{-2} \ll 1 \quad \text{and} \quad \omega^2 / k^2 \approx C_s^2 \]

Thus \( C_s \) can be interpreted as the (long-wavelength, low-frequency) sound speed in plasma \((C_s^2 = \max C_i^2)\). The ion-acoustic wave in the limit \((k\lambda_{De})^2 \ll 1\) is usually called ion sound and \( C_s \), given by Eq.(2.2.14), is referred to as ion-acoustic speed. Note that \( C_s^2 \approx C_i^2 \equiv \gamma_e k T_e / m_e \), so that one part of condition (2.2.11) is automatically fulfilled in the ion-sound limit. However, we have to satisfy \( C_s^2 \gg C_i^2 \), which leads to (assuming approximately the same temperature \( T_i \) for all types of ion)

\[ T_e \sum_i \frac{r_i Z_i}{m_i} \gg \frac{T_i}{m_i \min} \]  

(2.2.16)

Equation (2.2.16) is a condition (necessary and sufficient) for the existence of ion sound in multicomponent plasma. In two-component plasma (one kind of ion), condition (2.2.16) reduces to

\[ T_e Z_i \gg T_i \]  

(2.2.16')
For $(k\lambda_D)^2 \gg 1$ the dispersion relation (2.2.13) leads to

$$\omega^2 \simeq \sum_i \omega_i^2 \quad \text{for} \quad (k\lambda_D)^2 \gg 1 \tag{2.2.17}$$

so that the plasma system in that limit exhibits approximately a constant resonant frequency (see Fig.2). Notice that this 'short-wavelength' limit, $\lambda \ll \lambda_D$, may nevertheless correspond to $\lambda$ large enough for the validity of the fluid description, i.e. $\lambda \gg r_0$, where $r_0 = n_0^{1/3}$ is the mean distance between electrons. This is because in real plasmas the number of particles in the 'Debye sphere' (of its radius $r = \lambda_D$) is usually large (see e.g. Ref. [5] p.4), so that $r_0 \ll \lambda_D$. Comparison of the dispersion relation (2.2.13), in the limit $(k\lambda_D)^2 \ll 1$, with the result following from the Vlasov theory gives $\gamma_e = 1$ (see e.g. Ref. [5], p.390), which can be interpreted as isothermal compression of the electron gas. This is related to the fact that electrons in their thermal motion are now very fast compared to the wave ($v_T^2 \gg v_p^2$), and travel many wavelengths in one period of oscillation.

The main features of the Langmuir oscillations and ion-acoustic waves, and their main differences, are as follows.

2.2.1. **Frequency, wavelength and electron versus ion velocity amplitudes**

The Langmuir wave is the high-frequency ($\omega^2 \gtrsim \omega_{pe}^2$) long-wavelength $[(k\lambda_D)^2 \ll 1]$ mode, in which the pressure force on both electrons and ions is much less than the electric force. Mainly electrons are therefore oscillating:

$$-\frac{V_e}{V_i} \simeq \frac{m_i}{Z_i m_e} \gg 1 \quad \text{for Langmuir oscillations} \tag{2.2.18}$$

and a good approximation is obtained by completely neglecting all thermal motions and the ion dynamics ($T_e = T_i = 0; \ m_i \to \infty$). In this picture, ions are motionless (a neutralizing background), and electrons oscillate in an ordered way around ions, with frequency $\omega = \omega_{pe}$.

The ion-acoustic wave is the low-frequency mode

$$\left(\omega^2 \lesssim \sum_i \omega_{pi}^2\right)$$

There is more physics to this than to the Langmuir oscillations, because now both electrons and ions participate effectively in the wave motion; furthermore, it requires 'hot electrons and cold ions' (see Eq.(2.2.16)). Ions, as before, move mainly under the influence of the electric field ($C_i^2/v_p^2 \ll 1$), but the electric force on electrons is now almost completely compensated by the pressure force. This is because the inertial term in Eq.(2.1.1) for electrons is negligible compared to
the pressure term \((v_p^2/C_s^2) \gg 1\). (The momentum transfer equation for electrons, in which the inertial term is negligible (for \(\omega \ll \omega_{pe}\), is usually called *Ohm's law*.) With these simplifications, we obtain from Eq. (2.1.2) \((\gamma_e = 1)\):

\[
\frac{V_e}{v_p} \approx i \frac{e \omega}{k T_e k^2} E \tag{2.2.19}
\]

\[
\frac{V_i}{v_i} \approx i \frac{Z_i e}{m_i \omega} E \tag{2.2.20}
\]

Using Eqs (2.2.19), (2.2.20) and (2.2.13'), we find

\[
\frac{V_e}{V_i} \approx \frac{m_i}{Z_i} \left( \sum_i \frac{r_i Z_i}{m_i} \right) \frac{1}{1 + (k \lambda_{de})^2} \sim \frac{1}{1 + (k \lambda_{de})^2} \tag{2.2.21}
\]

Thus in the ion sound limit, \((k \lambda_{De})^2 \ll 1\), the electron oscillations are comparable to the ion oscillations, where, as for \((k \lambda_{De})^2 \gg 1\), mainly ions are oscillating. These low-frequency ion oscillations against a nearly motionless electron background are analogous to high-frequency electron oscillations (Langmuir oscillations). However, while in the latter case the heavy ions have a natural tendency to remain at rest as a neutralizing background and can be cold, for ion oscillations the pressure force on electrons is necessary ('hot electron background') to keep the light electrons from following the ion motion.

**Problem 11**

*Using Eq. (2.2.21), calculate the centre-of-mass velocity associated with the ion-acoustic wave in terms of \(V_e\):*

\[
\overline{V} = A^{-1} \left[ 1 + (k \lambda_{de})^2 \right] V_e
\]

\[
A = \left( \sum_i \frac{r_i m_i}{Z_i} \right) \left( \sum_i \frac{Z_i}{m_i} \right) \tag{2.2.22}
\]
and show in examples that $A \approx 1$, e.g.

(i) $A = 1$ in the two-component plasma;
(ii) $A = 1$ in deuterium + helium ($\text{H}_2^+$) plasma;
(iii) $A \approx 1.04$ in 50% deuterium + 50% tritium plasma, etc.

2.2.2. Electric field and charge density

To express in a simple way the amplitudes $E$ and $\rho_\alpha$ associated with electrostatic waves, it is convenient to introduce the Lagrangian displacement of the fluid element, $\xi_\alpha(\vec{r}, t)$. Thus if the initial position of the $\alpha$-type fluid element is $\vec{r}$ (at $t = 0$, i.e. in equilibrium), its displaced position at time $t$ will be denoted by $\vec{r} + \xi_\alpha(\vec{r}, t)$. With this definition, the time derivative

$$\frac{\partial \xi_\alpha}{\partial t} = \nabla \phi_\alpha(\vec{r}, t) + \xi_\alpha \cdot \nabla \phi_\alpha(\vec{r}, t) \simeq \vec{V}_\alpha(\vec{r}, t)$$  \hspace{1cm} (2.2.23)$$
gives the velocity $\vec{V}_\alpha$ at time $t$, at the displaced position $\vec{r} + \xi_\alpha$ which, however, within the accuracy of the linear theory, is equal to $\vec{V}_\alpha$ at the initial position $\vec{r}$.

For harmonic time dependence,

$$\vec{V}_\alpha = -i \omega \xi_\alpha$$  \hspace{1cm} (2.2.24)$$

The electric field amplitude $E$ can be expressed in terms of $V_x$ from Eq.(2.1.2) (or (2.2.19) in the case of the ion-acoustic wave). Using also the dispersion relation (2.2.5) or (2.2.13'), and Eqs (2.2.22) and (2.2.24), we finally obtain

$$E = 4 \pi \left( e n_{e0} \right) \xi_\alpha$$  \hspace{1cm} for Langmuir oscillations \hspace{1cm} (2.2.25a)$$

$$E = -4 \pi \left( e n_{e0} \right) \xi_\alpha \frac{(k \lambda_{De})^2}{1 + (k \lambda_{De})^2}$$  \hspace{1cm} for ion-acoustic wave \hspace{1cm} (2.2.25b)$$

where $\xi$ denotes the centre-of-mass displacement amplitude.

Formula (2.2.25a) indicates that the amplitude $E$ associated with the Langmuir oscillations is equal to the static electric field which would develop if all electrons from a shell of thickness $\xi_e$ were shifted by $\xi_e$. In that case there would be no electrons left in this shell, where the charge density would be equal to that of ions, i.e.

$$\sum_i q_i n_{i0} = e n_{e0}$$
A similar result follows from Eq. (2.2.25b) in the limit \((k\lambda_{De})^2 \gg 1\) (i.e. for ion oscillations), but now ions should be shifted by \(\xi\). (We recall that \(A \approx 1\); see Problem 11.)

In the ion sound limit \(\left( (k\lambda_{De})^2 \ll 1 \right)\), \(E\), given by Eq. (2.2.25b), is diminished by a factor \((k\lambda_{De})^2\) and tends to zero as \(k \to 0\). This is related to the Debye shielding of ions by electrons which takes place for \(\lambda \gg \lambda_{De}\) in the low-frequency limit. Using Poisson's equation,

\[
\frac{i k E}{\varepsilon_0} = 4\pi \frac{\rho}{\varepsilon_0} \tag{2.2.26}
\]

and Eqs (2.2.25), we can calculate the charge density amplitude:

\[
\frac{\rho}{\varepsilon_0} = \begin{cases} 
-ik\xi & \text{for Langmuir oscillations} \\
-ik\xi^{-1} & \text{for ion-acoustic wave}
\end{cases} \tag{2.2.27a}
\]

\[
\frac{\rho}{\varepsilon_0} = \frac{(k \lambda_{De})^2}{1 + (k \lambda_{De})^2} \tag{2.2.27b}
\]

It can be seen that the ratio \(\rho/\varepsilon_0\) is given by the displacement amplitude to the wavelength ratio (as \(k = 2\pi/\lambda\)) both for the high-frequency Langmuir oscillations and low-frequency ion oscillations \((k\lambda_{De})^2 \gg 1\).

2.2.3. Energy distribution

From formulas for \(\epsilon^l\) and the dispersion relations (2.2.4), (2.2.5), (2.2.12) and (2.2.13), we obtain

\[
\frac{\partial}{\partial \omega} \left( \omega \epsilon^l \right) = \omega \frac{\partial \epsilon^l}{\partial \omega} = \begin{cases} 
2 \left[ 1 + 3 (k \lambda_{De})^2 \right] & \text{for Langmuir oscillations} \\
2 \left[ 1 + \frac{1}{(k \lambda_{De})^2} \right] & \text{for ion-acoustic wave}
\end{cases} \tag{2.2.28}
\]

Using now Eq. (2.1.15') for \(\bar{W}_{K1}\) and the energy equipartition rule (2.1.21), we get

**Langmuir oscillations** \((k \lambda_{De})^2 \ll 1\):

\[
\bar{W}_{T1} = 3 (k \lambda_{De})^2 \bar{W}_{E1} \ll \bar{W}_{E1} \\
\bar{W}_{K1} = \bar{W}_{E1} + \bar{W}_{T1} \approx \bar{W}_{E1} \tag{2.2.29}
\]
Ion-acoustic wave:

\[ \tilde{W}_{T1} = \frac{\tilde{W}_{E1}}{(k\lambda_{pe})^2}, \text{ i.e.} \]

\[ \tilde{W}_{K1} = \tilde{W}_{E1} + \tilde{W}_{T1} \approx \begin{cases} 
\tilde{W}_{T1} & \text{for } (k\lambda_{pe})^2 \ll 1 \\
\tilde{W}_{E1} & \text{for } (k\lambda_{pe})^2 \gg 1
\end{cases} \]  

(2.2.30)

Thus for electron and ion oscillations the thermal energy is very small \((\tilde{W}_{K1} \approx \tilde{W}_{E1})\), whereas in the ion-sound limit the electric energy is negligible \((\tilde{W}_{K1} \approx \tilde{W}_{T1})\).

2.3. Electromagnetic waves

Electromagnetic (i.e. transverse) waves have simpler properties than electrostatic waves because they are not influenced by the thermal motion of the plasma; thus they propagate in the same way in warm and cold plasmas. This becomes clear if one notices that the wave motion is now incompressible \((\tilde{k} \cdot \nabla \alpha = 0; \text{ see Eq.}(2.1.2))\), and it therefore leaves the equilibrium pressure undisturbed, \((p_{\alpha1} = 0; \text{ see Eq.}(1.4.8))\). Obviously there is no thermal energy associated with the EM wave \((\tilde{W}_{T1} = 0)\), and on average the wave energy is equally distributed between the electric energy and magnetic plus kinetic energy \((\tilde{W}_{E1} = \tilde{W}_{M1} + \tilde{W}_{K1}; \text{ see Eq.}(2.1.22))\).

The dispersion relation for EM waves has a simple form (see Eqs (2.1.7) and (2.1.13)):

\[ \omega^2 = k^2 c^2 + \omega_p^2 \]  

(2.3.1)

It defines the propagation band

\[ \omega_p < \omega < \infty \]  

(2.3.2)

where at \(\omega = \omega_p \approx \omega_{pe}\) the EM waves undergo a cutoff \((k = 0)\). The cutoff frequency, \(\omega \approx \omega_{pe}\), below which there is no propagation of EM waves, can be measured by using experimental devices called microwave interferometers \([10]\); with these the plasma density \(n_{eq}\) can be determined (see Eq.(2.2.3)).

Within the propagation band (2.3.2) the phase velocity

\[ v_p = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \omega_p^2/\omega^2}} \geq c \]  

(2.3.3)
is greater than the velocity of light in vacuum, $c$. The vacuum propagation is obtained in the limit $\omega^2 > \omega_p^2$, in which the plasma electron response to the electric field of the wave is negligible (the ion response is in general much weaker than the electron response).

Differentiating Eq. (2.3.1) with respect to $k$, we obtain ($\omega = \omega(k)$)

$$v_g \cdot v_p = c^2$$

where

$$v_g = \frac{d\omega(k)}{dk} = c \sqrt{1 - \omega_p^2/\omega^2} < c$$

is the group velocity.

Group velocity, defined uniquely by the form of the dispersion relation, can often be interpreted as the wave energy transport velocity. A general proof of this statement is given in Ref. [7], and it is instructive to demonstrate it explicitly for the EM waves. (This interpretation can fail only in the presence of beams, $V_{\alpha_0} \neq 0$, which will be shown in Section 4.2.) Using Eqs (1.6.15), (1.6.16) and (2.1.16), we obtain (in view of (2.3.3) and (2.3.4))

$$\overline{W}_{M1} = \left(1 - \frac{\omega_e^2}{\omega^2}\right)\overline{W}_{E1}, \quad \overline{W}_{K1} = \frac{\omega_p^2}{\omega^2} \overline{W}_{E1}$$

$$\overrightarrow{P}_1 = 2\overline{W}_{E1} v_g \hat{k}$$

Thus the Poynting vector (2.3.7), which for $\overline{W}_{T1} = 0$ is equal to the wave energy current (see Eq. (1.5.13)), can be written:

$$\overrightarrow{P}_1 = \left(\overline{W}_{E1} + \overline{W}_{M1} + \overline{W}_{K1}\right) v_g$$

where

$$v_g = \frac{d\omega}{dk} \hat{k} = \frac{\partial \omega}{\partial k}$$

Note that our simple fluid description of the EM waves should be quite accurate, partly because these waves ignore the thermal motion and partly because there is no Landau damping associated with the EM waves (there are no particles in plasma which can move with velocities close to $v_p$, as $v_p > c$).
3. WAVES IN COLD MAGNETIZED PLASMA

This section deals with waves in a homogeneous plasma permeated by a homogeneous magnetic field $\vec{B}_0$ in the absence of beams. For simplicity we adopt the cold plasma approximation, which provides insight into a rather complicated picture arising in the presence of $\vec{B}_0$.

3.1. Dielectric tensor and general dispersion relation

The linearized momentum transfer equation (1.4.9) now becomes

$$-i\omega \vec{V}_{\alpha 1} = \frac{q_\alpha}{m_\alpha} \vec{E}_1 + \vec{V}_{\alpha 1} \times \vec{\omega}_{c\alpha}$$

where $\vec{\omega}_{c\alpha}$, given by Eq. (1.4.10), has the same direction as $\vec{B}_0$. Thus the vector product (magnetic force) in Eq. (3.1.1) has no component along $\vec{B}_0$ and it is convenient to introduce the coordinate system (c. s.) with the z-axis taken along $\vec{B}_0$. Equation (3.1.1) can easily be solved in this c.s., but the solution can also be obtained in vector form by taking first a scalar and then a vector product of Eq. (3.1.1) with $\vec{\omega}_{c\alpha}$, i.e.

$$-i\omega \vec{\omega}_{c\alpha} \cdot \vec{V}_{\alpha 1} = \left( \frac{q_\alpha}{m_\alpha} \right) \vec{\omega}_{c\alpha} \cdot \vec{E}_1$$

$$i\omega \vec{V}_{\alpha 1} \times \vec{\omega}_{c\alpha} = \left( \frac{q_\alpha}{m_\alpha} \right) \vec{\omega}_{c\alpha} \times \vec{E}_1$$

Inserting into Eq. (3.1.3) $\vec{\omega}_{c\alpha} \cdot \vec{V}_{\alpha 1}$ from Eq. (3.1.2) and $\vec{V}_{\alpha 1} \times \vec{\omega}_{c\alpha}$ from (3.1.1), and solving for $\vec{V}_{\alpha 1}$, we obtain

$$\vec{V}_{\alpha 1} = -i\omega \frac{q_\alpha}{m_\alpha} - \frac{1}{\omega^2 - \omega^2} \left[ \vec{E}_1 - \frac{i}{\omega^2} \vec{\omega}_{c\alpha} \cdot \vec{E}_1 \right]$$

Problem 12

Show that the same method can also be used to solve Eq. (1.4.9) in the presence of beams ($\vec{V}_{\alpha 0} \neq 0$, $C_{\alpha}^2 = 0$), leading to [11]
Equation (3.1.5) gives a feeling of how the particle motions in the magnetized plasma become complicated in the presence of beams.

Inserting \( \mathbf{V}_{d1} \), given by Eq. (3.1.4), into Eqs (1.6.6) and (1.4.8) we obtain \( \mathbf{D}_1 \) and can determine the dielectric tensor:

\[
\mathbf{D}_1 = \mathbf{E} \cdot \mathbf{E}_1 = \mathbf{E}_1 + \sum_{d} \frac{\omega_{pd}^2}{\omega_{pd}^2 - \omega^2} \left[ \mathbf{E}_1 - \frac{1}{\omega} \mathbf{V}_{d} \cdot (\mathbf{k} \times \mathbf{E}_1) - \mathbf{i} \frac{1}{\omega} \mathbf{V}_{d} \times \mathbf{E}_1 \right]
\]

which in the chosen c.s. gives

\[
\mathbf{\mathbf{e}} = \begin{bmatrix}
\varepsilon_1 & -i \varepsilon_2 & 0 \\
 i \varepsilon_2 & \varepsilon_1 & 0 \\
 0 & 0 & \varepsilon_0
\end{bmatrix}
\]

where

\[
\varepsilon_1 = 1 + \sum_{d} \frac{\omega_{pd}^2}{\omega_{pd}^2 - \omega^2}
\]

\[
\varepsilon_2 = \frac{\omega_{pe}^2}{\omega} - \sum_{i} \frac{\omega_{ci}}{\omega} - \sum_{i} \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2}
\]

and \( \varepsilon_0 = 1 - \frac{\omega_p^2}{\omega^2} \) is the dielectric constant of the cold unmagnetized plasma (see Eq. (2.1.9)). Note that \( \omega_{pd} \) in Eq. (3.1.8) are defined as positive quantities:

\[
\omega_{pd} = \frac{1}{m_d} \frac{B_0}{c}, \quad i.e. \quad \omega_{pe} = \frac{e}{m_e} \frac{B_0}{c}, \quad \omega_{ci} = \frac{T_{ci} e B_0}{m_i c}
\]
These $\omega_{ce}$ and $\omega_{ci}$, called the electron and ion-cyclotron frequency, are the angular velocities of the cyclotron gyration in the magnetic field, for electrons and ions (see Problem 6).

It can be seen from Eq. (3.1.7) that $D_{iz} = \varepsilon_0 E_{iz}$, as in the absence of $\vec{B}_0$, and that $\varepsilon$ is Hermitian. Thus formulas (2.1.18) and (2.1.20) for the total energy of the wave should also be applicable to $\varepsilon$ given by (3.1.7) if $\omega$ is real.

We recall that the $z$-axis of the coordinate system is chosen along $\vec{B}_0$, and we now also specify the directions of other axes by assuming that $\vec{k}$ lies in the $yz$-plane (Fig. 4). In this c.s. Eq. (1.6.9) for $\vec{E}_1$ can be written:

$$
\begin{bmatrix}
\begin{array}{ccc}
    n^2 - \varepsilon_1 & i \varepsilon_2 & 0 \\
    -i \varepsilon_2 & n^2 \cos^2 \theta - \varepsilon_1 & -n^2 \sin \theta \cos \theta \\
    0 & -n^2 \sin \theta \cos \theta & n^2 \sin^2 \theta - \varepsilon_0
\end{array}
\end{bmatrix}
\begin{bmatrix}
    E_{1x} \\
    E_{1y} \\
    E_{1z}
\end{bmatrix} = 0
$$

(3.1.10)

where we have introduced the index of refraction

$$
n^2 = \frac{c^2 k^2}{\omega^2}
$$

(3.1.11)

(if $\omega$ is real, then $n^2 = c^2/\nu_p^2$); $\Theta$ is the propagation angle, defined in Fig. 4.

Equating to zero the determinant of Eq. (3.1.10), we obtain the dispersion relation, which is usually written in one of the following equivalent forms:

$$
\frac{\omega^2}{\varepsilon_0} \Theta = -\frac{\varepsilon_0 (n^2 - \varepsilon_L)(n^2 - \varepsilon_R)}{(n^2 - \varepsilon_0)(\varepsilon_1 n^2 - \varepsilon_L \varepsilon_R)}
$$

(3.1.12)
where

\begin{align}
\varepsilon_R &= \varepsilon_1 + \varepsilon_2 = 1 - \frac{\omega_p^2}{\omega(\omega - \omega_{ci})} - \frac{4}{\omega} \sum_i \frac{\omega_p^2}{\omega + \omega_{ci}} \\
\varepsilon_L &= \varepsilon_1 - \varepsilon_2 = 1 - \frac{\omega_p^2}{\omega(\omega + \omega_{ci})} - \frac{4}{\omega} \sum_i \frac{\omega_p^2}{\omega - \omega_{ci}}
\end{align}

Equations (3.1.12) and (3.1.12') lead to important general conclusions. First it can be shown that \( B^2 - 4 AC \geq 0 \) (see e.g. Ref. [7]). Therefore Eq. (3.1.12') has always one or two real solutions for \( n^2 \) which can be either positive or negative; the wave can propagate only if \( n^2 > 0 \). Thus for each value of \( \Theta \) and \( \omega \) there are no more than two possible waves in a cold magnetized plasma. (For \( \Theta = 0 \), Eq. (3.1.12) is additionally fulfilled (for arbitrary \( n \)) if \( \varepsilon_o = 0 \), i.e. \( \omega = \omega_p \).)

Keeping the propagation angle fixed, propagation bands for these two waves can be determined by finding the cutoffs (\( n^2 = 0 \)) and resonances (\( n^2 = \infty \)). The cutoff frequencies should satisfy \( C = 0 \), i.e.

\[ \varepsilon_o \varepsilon_L \varepsilon_R = 0 \text{ at the cutoff} \]  

which means that they are independent of the propagation angle. The resonant frequencies satisfy \( A/C = 0 \) and depend on \( \Theta \).

### 3.2. Waves in the low-frequency limit (\( \omega \ll \omega_{ci} \))

For frequencies well below the ion-cyclotron frequencies, the dispersion relation (3.1.12') simplifies considerably for arbitrary \( \Theta \) (and arbitrary number of species). This limit also introduces instructive examples of waves in a magnetized plasma, i.e. the Alfvén wave, and magnetosonic wave.

For \( \omega \ll \omega_{ci} \ll \omega_{ce} \) we obtain from Eq. (3.1.8)
$$\xi_1 = 1 + \sum_\alpha \frac{\omega_{\Phi i}^2}{\omega_{ci}^2} = 1 + \sum_\alpha \frac{\frac{4\pi}{3} n_{\alpha 0} m_{\alpha} c^2}{B_0^2} = 1 + \frac{c^2}{V_A^2} \quad (3.2.1)$$

where

$$V_A = \frac{B_0}{\sqrt{4\pi n_{\alpha 0}}} = \text{Alfvén velocity} \quad (3.2.2)$$

Similarly it can be shown (using in addition the equilibrium charge neutrality) that

$$|\xi_2| \approx \omega \sum_i \frac{\omega_{\Phi i}^2}{\omega_{ci}^3} \ll \sum_i \frac{\omega_{\Phi i}^2}{\omega_{ci}^2} \approx \frac{c^2}{V_A^2}, \quad \text{i.e.} \quad |\xi_2| \ll \xi_1 \quad (3.2.3)$$

At the same time $|\xi_0| \sim \frac{\omega_0^2}{\omega^2} \gg 1$, and only terms proportional to $\xi_0$ can be kept in the dispersion relation $(3.1.12')$, which leads to

$$\left( n^2 \cos^2 \theta - \xi_1 \right) \left( n^2 - \xi_1 \right) = \xi_2^2 \quad (3.2.4)$$

Neglecting the right-hand side of Eq. (3.2.4), in view of (3.2.3), we obtain two solutions:

$$n^2 \cos^2 \theta = \xi_1, \quad \text{i.e.} \quad v_p^2 = V_A^2 \cos^2 \theta, \quad \text{Alfvén wave} \quad (3.2.5)$$

$$n^2 = \xi_1, \quad \text{i.e.} \quad v_p^2 = V_A^2 \quad \text{magnetosonic wave} \quad (3.2.6)$$

where

$$V_A' = \frac{V_A}{\sqrt{1 + v_A^2/c^2}} = \text{generalized Alfvén velocity} \quad (3.2.7)$$

Usually $v_A \ll c$ in typical conditions, and in that case $v_A' = v_A$. In the opposite limit ($v_A > c$) $v_A' \approx c$. The phase velocity of the magnetosonic wave is independent of $\Theta$ (isotropic propagation). For the Alfvén wave it depends on $\Theta$ such that the group velocity is directed along $B_0$:

$$\omega = v_A' k_z, \quad \text{i.e.} \quad \frac{\partial \omega}{\partial k} = v_A' \frac{\Rightarrow}{\Rightarrow} \text{for the Alfvén wave} \quad (3.2.8)$$
This means that the wave packet containing Alfvén waves propagating in various directions will actually move along \( \vec{B}_0 \) with the speed \( v'_A \). Note that the Alfvén wave cannot propagate across \( \vec{B}_0 \) (\( v_p = 0 \) for \( \Theta = \pi/2 \)).

Neglecting in the z-component of Eq. (3.1.10) terms without \( \epsilon_0 \), we obtain \( E_{1z} = 0 \). Thus for both waves \( \vec{E}_1 \) is perpendicular to \( \vec{B}_0 \). In this situation the second term in Eq.(3.1.4) is zero, and the third term dominates. Neglecting \( \omega/\omega_{ca} \) compared to unity in Eq.(3.1.4), we arrive at the following approximate formula:

\[
\vec{V}_{\alpha 1} \approx \frac{c}{\epsilon_1} \left( \vec{E}_1 \times \vec{B}_0 \right)/B_0^2
\]

Thus in the lowest approximation in \( \omega/\omega_{ca} \), all plasma particles move across \( \vec{B}_0 \) and \( \vec{E}_1 \) with the same drift velocity (3.2.9). This picture is completely different from the motion caused by the self-consistent field \( \vec{E}_1 \) in the absence of \( \vec{B}_0 \).

For \( \Theta = 0 \) the Alfvén and magnetosonic waves propagate at the same velocity \( (v_p = v'_A) \) and are both transverse \( (k \cdot \vec{E}_1 = 0) \) and incompressible \( (k \cdot \vec{V}_{\alpha 1} = 0 \) from (3.1.4)). To examine the polarization properties for \( \Theta \neq 0 \) we have to improve the approximate solutions (3.2.5) and (3.2.6). For the magnetosonic wave we find from Eq. (3.2.4)

\[
\epsilon_1^2 = \epsilon_2 = - \frac{\epsilon_2^2}{\epsilon_1 \sin^2 \Theta}
\]

Using this result in the x-component of Eq. (3.1.10), we obtain \( E_y/E_x = -ie_2/(\epsilon_1 \sin^2 \Theta) \). This ratio is small, in view of (3.2.3), if \( \Theta \) is not too close to zero. In that case the dominating component of the amplitudes associated with the magnetosonic wave are \( E_x \) and \( V_{\alpha y} \) (from Eq. (3.2.9)). Similarly, for the Alfvén wave, the dominating components are \( E_y \) and \( V_{\alpha x} \). This means that for \( \Theta \neq 0 \) the Alfvén wave remains (approximately) incompressible, whereas the magnetosonic wave remains transverse. Thus, in practice, the Alfvén wave should not be influenced by plasma thermal motions.

For \( \Theta = \pi/2 \) only a magnetosonic wave can propagate, and it is then approximately a pure compressional wave, i.e. \( \vec{V}_{\alpha 1} \parallel \vec{k} \). That is why the magnetosonic wave propagating across \( \vec{B}_0 \) is usually called magnetic sound.

Note that the magnetic field associated with the magnetic sound is approximately parallel to \( \vec{B}_0 \), i.e. \( \vec{B}_1 \cong \vec{B}_1 \parallel \vec{z} \). This means that the wave perturbs the magnetic pressure \( B^2/8\pi \) but not the direction of \( \vec{B}_0 \). The magnetic field lines become compressed and decompressed under the influence of the magnetic sound, but they remain straight lines parallel to the z-axis.
3.3. Parallel and perpendicular propagation

For waves propagating along \( \vec{B}_0 (\Theta = 0, \text{parallel propagation}) \) and across \( \vec{B}_0 (\Theta = \pi/2, \text{perpendicular propagation}) \) the dispersion relation (3.1.12) factorizes in a simple way, and Eqs (3.1.10) can be effectively solved.

For \( \Theta = 0 \) we obtain three solutions. One, corresponding to \( \epsilon_0 = 0 \), i.e. \( \omega = \omega_p \), is a longitudinal wave (in a cold plasma) propagating along \( \vec{B}_0 \) in the same way as in the absence of \( \vec{B}_0 \). Two other solutions represent the transverse EM waves described by (two-component plasma)

\[
\eta^2 = \epsilon_{R,L} \equiv \epsilon_1 \pm \epsilon_L = 1 - \frac{\omega_p^2}{\omega (\omega \mp \omega_{ce})} - \frac{\omega_i^2}{\omega (\omega \pm \omega_{ci})} \tag{3.3.1}
\]

One can immediately see that these waves have resonances at the cyclotron frequencies, i.e.

\[
\epsilon_R (\omega_{ce}) = \infty \quad \text{and} \quad \epsilon_L (\omega_{ci}) = \infty \tag{3.3.2}
\]

and the propagation bands correspond to \( \omega < \omega_{ce} \). These electron and ion-cyclotron resonances have a clear physical meaning. Thus, using the x-component of Eqs (3.1.10), we obtain

\[
\frac{E_y}{E_x} = i \frac{n^2 - \epsilon_1}{\epsilon_2} = i \frac{\epsilon_1 + \epsilon_2 - \epsilon_4}{\epsilon_2} = \pm i \tag{3.3.3}
\]

which means that the waves (3.3.1) are circularly polarized. The upper sign in Eq. (3.3.3) corresponds to the right circular polarization (with respect to \( \vec{B}_0 \)). Thus at each point the electric field associated with this wave rotates around \( \vec{B}_0 \) with the same sense as electrons in their cyclotron motion (see Problem 6). The resonance is achieved when the electric field and electrons rotate with the same angular velocity, i.e. at \( \omega = \omega_{ce} \). A similar picture is obtained at \( \omega = \omega_{ci} \), where the left circularly polarized wave \( (n^2 = \epsilon_L) \) resonates with ions.

The cutoff frequencies \( (n^2 = 0) \) for the waves (3.3.1) are (neglecting the ion contributions to \( \epsilon_{R,L} \))

\[
\omega_{2,4} = \sqrt{\omega_p^2 + (\omega_{ce}/2)^2} \pm \omega_{ce}/2 , \quad [\epsilon_R (\omega_2) = 0 , \epsilon_L (\omega_4) = 0]. \tag{3.3.4}
\]

They are good for waves propagating at an arbitrary angle (see Eq. (3.1.14)). Note that the higher cutoff frequency corresponds to the right circularly polarized wave, which also has higher resonance frequency \( (\omega_{ce}) \). For \( \omega \gg \omega_{ce} , \omega_p , \) \( n^2 \to 1 \) for
both waves (3.3.1) (vacuum propagation), and for \( \omega \ll \omega_{ci} \) both tend to Alfvén waves: \( n^2 \rightarrow c^2/v_A^2 \). With this information one can easily sketch the plot of \( n^2 \) versus \( \omega \), shown in Fig. 5, and find the propagation bands.

The dispersion relation (3.3.1) takes a single form for intermediate frequencies:

\[
\frac{n^2}{v_A^2} = 1 \pm \frac{\omega_{pe}^2}{\omega \omega_{ce}} \quad \text{for} \quad \omega_{ci} \ll \omega \ll \omega_{ce} \tag{3.3.5}
\]

If the plasma is sufficiently dense and the magnetic field not too strong, so that \( \omega_{pe}/\omega_{ce} > 1 \), then only the right circularly polarized wave can propagate. Such conditions occur, for example, in the ionosphere, where typical frequencies are
$\omega_p \sim 10^7$, $\omega_{ce} \sim 10^6$, $\omega_{ci} \sim 10^2$ ; $\omega \sim 10^4$ \hfill (3.3.6)

In that case the right circularly polarized wave (3.3.5) is called a **whistler mode**. Note that its phase (and group) velocity increases with $\omega$ and is smaller than $c$,

$$V_p = \frac{\omega}{c} \ll c \quad \text{for a whistler mode} \hfill (3.3.7)$$

For $\Theta = \pi/2$ (perpendicular propagation) the wave propagates along the $y$-axis. The dispersion relation (3.1.12) has two solutions:

\[ n^2 = n^2 \sim \frac{\omega_p^2}{\omega^2} , \quad \text{i.e.} \quad \omega^2 = n^2 c^2 + \omega_p^2 \hfill (3.3.8) \]

\[ n^2 = \frac{\varepsilon_R \varepsilon_L}{\varepsilon_L} \hfill (3.3.9) \]

For the two-component plasma, $\varepsilon_{R,L} (= \varepsilon_1 \pm \varepsilon_2)$ are defined in Eq. (3.3.1), and

\[ \varepsilon_1 = \frac{1}{\omega} + \frac{\omega_p^2}{\omega_{ce}^2 - \omega^2} \quad + \quad \frac{\omega_p^2}{\omega_{ci}^2 - \omega^2} \hfill (3.3.10) \]

\[ \varepsilon_2 = \frac{\omega_{ce} \omega_p^2}{\omega} - \frac{\omega_{ce}^2}{\omega} \quad - \quad \frac{\omega_{ci}^2}{\omega} \hfill (3.3.11) \]

The first solution (3.3.8) represents a pure transverse EM wave whose electric field ($E_{1z}$) is polarized along $B_0$. Therefore this wave, usually called an **ordinary** wave is not influenced by the magnetic field. The second solution (3.3.9), usually called an **extraordinary** wave, depends on $B_0$. Its electric field is perpendicular to $B_0$ and in general contains both longitudinal ($E_{1y}$) and transverse ($E_{1x}$) components. Thus for $\Theta = \pi/2$, waves do not separate into pure longitudinal and transverse.

The cutoffs, as for arbitrary $\Theta$, are now at $\omega = \omega_p$ (for an ordinary wave) and at zeros of $\varepsilon_{R,L}$ (for an extraordinary wave), i.e. at $\omega = \omega_{1,2}$ (see Eq. (3.3.4)).

An ordinary wave has no resonance, and for an extraordinary wave the resonances occur at zeros of $\varepsilon_1$. Assuming in Eq. (3.3.10) $\omega > \omega_{ce}$, we obtain the high-frequency resonance called the **upper hybrid** resonance (neglecting the ion contribution),

$$\omega_H = \sqrt{\omega_p^2 + \omega_{ce}^2} \hfill (3.3.12)$$
Similarly, for \( \omega < \omega_{ce} \) we obtain the lower hybrid resonance \( (Z \equiv Z_i) \):\[
\omega_{LH} = \sqrt{\frac{\omega_{ci}}{\omega_{ce}}} \sqrt{\frac{Z m_e}{m_i}} + \frac{1}{1 + \omega_{ce}^2 / \omega_{pe}^2} \quad (3.3.13)
\]
Usually \( \omega_{LH} \equiv \sqrt{\omega_{ci} \omega_{ce}} \), and \( Z m_e / m_i \) in Eq. (3.3.13) is insignificant (only for \( \omega_{ce}^2 / \omega_{pe}^2 \gg m_i / Z m_e \) do we obtain \( \omega_{LH} \gg \omega_{ci} \)). Note that there is no additional resonance at \( \omega_{ce} \) or \( \omega_{ci} \) because at these frequencies the numerator and denominator in Eq. (3.3.9) have the same singularity and \( n^2 \) is finite.

From the \( y \)-component of Eq. (3.1.10) we obtain
\[
\frac{E_y}{E_x} = -i \frac{\epsilon_2}{\epsilon_1} \quad (3.3.14)
\]
which defines the polarization of the extraordinary wave. For \( \omega \to \omega_{ce} \), \( \epsilon_2 / \epsilon_1 \to 1 \), and the wave is left circularly polarized (\( E_1 \) rotates with the opposite sense to the electrons). At the hybrid resonances, \( \epsilon_2 / \epsilon_1 \to \infty \) so that the wave
becomes longitudinal. This feature is characteristic of all 'non-cyclotron' resonances (see Eq. (1.6.9), where $\frac{\omega}{c}$ given by (3.1.7) is finite, and $k \to \infty$). For $\omega \ll \omega_{ci}$ the wave is transverse, in view of Eq. (3.2.3), and it becomes the magnetosonic wave discussed in Section 3.2. A schematic plot of $n^2$ versus $\omega$ for waves propagating across $\mathbf{B}_0$ is shown in Fig. 6.

Note that, in general (two-component plasma),

$$\omega_1 < \omega_{pe} < \frac{\omega_1 + \omega_2}{2} < \omega_{le} < \omega_2$$

\hspace{1cm} (3.3.15)

$$\omega_{pi}^2/\omega_{pe}^2 = \omega_{ce}/\omega_{ce} = (\omega_{pe}^2/\omega_{ce}^2)v_A^2/c^2 = Zm_e/m_i$$

\hspace{1cm} (3.3.16)

4. WAVES AND STABILITY OF A MAGNETIZED PLASMA WITH BEAMS

4.1. Dielectric tensor and dispersion relation

The dielectric tensor of a cold magnetized plasma with beams ($\mathbf{V}_{a0} \neq 0$) is much more complicated than (3.1.7), and depends not only on $\omega$ but also on $k$ (see Eq. (3.1.5)). An effective discussion is possible only with an appropriate choice of $k$.

Here we consider only an important special case, i.e. parallel propagation ($\hat{k} \parallel \mathbf{B}_0$). We shall demonstrate that the dielectric tensor in that case has the same form as in the absence of beams, i.e. (3.1.7), but we have to modify $\epsilon_0$, $\epsilon_1$ and $\epsilon_2$ appropriately. Hence the general results of Section 3.3, pertinent to the parallel propagation, will be applicable. As in Section 3.1, we introduce the coordinate system with the z-axis chosen along $\mathbf{B}_0$ (and $\hat{k}$). All macroscopic velocities $\mathbf{V}_{a0}$ (which must be parallel to $\mathbf{B}_0$ (see Section 1.3)) will have only the z-components ($\mathbf{V}_{a0} = V_{a0z}$).

We first calculate the electric displacement $\mathbf{D}_1$ caused by a longitudinal electric field. Thus $\mathbf{E}_1 = \mathbf{E}_{10}$, $\mathbf{B}_1 = c\hat{k} \times \mathbf{E}_1/\omega = 0$, and the linearized equation for $\mathbf{V}_{a1}$, (1.4.9), differs from that in the absence of beams only in that $\omega$ is replaced by $\omega - kV_{a0}$. The solution of this equation is longitudinal and therefore independent of $\mathbf{B}_0$. Thus $\mathbf{V}_{a1}$ will be given by formula (2.1.2), derived for $\mathbf{B}_0 = 0$, if we replace $\omega \to \omega - kV_{a0}$. At the same time, for longitudinal velocity perturbations, (1.6.6) can be written (also using Eq. (1.4.8)):

$$\mathbf{D}_1 = \mathbf{E}_1 + 4\pi i \sum_{a} \frac{q_a n_{a0} \mathbf{V}_{a0}}{\omega - kV_{a0}}$$
which finally leads to \( \vec{D}_1 = \vec{D}^\parallel_1 = \varepsilon^t \vec{E}_1 \), where

\[
\varepsilon^t = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{(\omega - kV_{\alpha 0})^2 - C_{\alpha}^2 k^2}
\]  

This longitudinal dielectric constant (independent of \( \vec{B}_0 \)) should be used in Eq. (3.1.7) instead of \( \varepsilon_0 \).

For pure transverse perturbations, \( \vec{E}_1 = \vec{E}^t_1 \), assuming the plasma is cold, we can reduce Eq. (1.4.9) to (\( \vec{B}_1 = c\vec{k} \times \vec{E}^t_1 / \omega \)):

\[
-i \omega \vec{V}_{d1}^t = \frac{q_d}{m_d} \vec{E}^t_1 + \frac{\omega}{\omega - kV_{d0}} \vec{V}_{\alpha 1}^t \times \vec{\omega}_{c\alpha}
\]

This indicates that the fluid motion is incompressible (\( \vec{k} \cdot \vec{V}_{\alpha 1} = 0 \)), and Eq. (4.1.2) will also be valid in a warm plasma. Furthermore there will be no additional term in Eq. (1.6.6) for \( \vec{V}_{\alpha 0} \neq 0 \), as \( n_{\alpha 1} = 0 \). Thus the only effect of \( \vec{V}_{\alpha 0} \neq 0 \) in the case of transverse perturbations is the presence of the factor \( \omega / (\omega - kV_{\alpha 0}) \) in Eq. (4.1.2). Incorporating this factor into \( \vec{\omega}_{c\alpha} \), we can formally reduce the present case to that without beams. This leads to (warm plasma, \( \vec{V}_{\alpha 0} \neq 0, \vec{k} \parallel \vec{B}_0 \))

\[
\varepsilon^t = \begin{bmatrix}
\varepsilon_1 & -i \varepsilon_2 & 0 \\
-i \varepsilon_2 & \varepsilon_1 & 0 \\
0 & 0 & \varepsilon^t
\end{bmatrix}
\]

where \( \varepsilon_1, 2 \) are given by formulas (3.1.8), if we replace

\[
\omega_{c\alpha} \rightarrow \frac{\omega}{\omega - kV_{\alpha 0}}
\]

The dispersion relation can now be written by using Eq. (3.1.12) for \( \Theta = 0 \), which yields

\[
1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{(\omega - kV_{\alpha 0})^2 - C_{\alpha}^2 k^2} = 0 \quad \text{for longitudinal waves}
\]

\[
h^2 = \overline{\varepsilon}_{R,L} = \overline{\varepsilon}_1 \pm \overline{\varepsilon}_2 \quad \text{for EM waves}
\]
The EM waves are circularly polarized, in view of Eq. (3.3.3). For the two-component plasma (with arbitrary mean velocities of electrons and ions) we can make use of formula (3.3.1) to write the dispersion relation (4.1.6) as

$$\omega^2 = c^2 k^2 + \omega_{pe}^2 \frac{\omega - kV_{te}}{\omega - kV_{te} \pm \omega_{ce}} + \omega_i^2 \frac{\omega - kV_{io}}{\omega - kV_{io} \pm \omega_i}$$

(4.1.7)

This dispersion is more complicated than (3.3.1). Note, however, that it reduces to the dispersion relation for the EM waves in isotropic plasma (Eq. (2.3.1)) if \(B_0 = 0\). Thus in the absence of \(B_0\) the EM waves propagate along the drift velocities in the same way as for \(V_{a0} = 0\).

Equation (4.1.7) is a quartic in \(\omega\). It can be shown graphically [11] that if \(V_{e0} > 0\), and \(V_{i0} = 0\) (electrons drifting against ions), then Eq. (4.1.7) has four real roots \(\omega(k)\), which means stability against the pure EM perturbations.

4.2. Electrostatic waves in a cold drifting plasma

In application, the plasma as a whole often drifts in the laboratory frame (along \(B_0\) if \(B_0 \neq 0\), i.e. \(V_{e0} = V_{i0} = V_D\)). Assuming the plasma is cold, and neglecting the ion dynamics, we obtain from Eq. (4.1.5) the following dispersion relation for the longitudinal (electrostatic) waves in the drifting plasma:

$$\varepsilon^2 = 1 - \frac{\omega_{pe}^2}{(\omega - kV_D)^2} = 0$$

, i.e.

$$\omega = \pm \frac{2\omega}{k}$$

(4.2.1)

The phase and group velocities for these waves are

$$V_p = V_D \pm \frac{\omega_{pe}}{k}$$

, \(V_g = \frac{2\omega}{\partial k} = V_D$$

(4.2.2)

The wave with \(V_p > V_D\) (upper sign) is called fast space-charge wave, and that with \(V_p < V_D\) a slow wave. For both, the group velocity is equal to \(V_D\).

The space-charge waves in drifting plasma have interesting energy transport properties. It was shown in Section 1.5 that the kinetic energy and energy current derived from the linearized fluid equations differ from those obtained by averaging the exact energy-conservation equation. We now make a detailed comparison of these results. As \(\bar{W}_{M1} = 0\) and \(\bar{P}_1 = 0\) (electrostatic waves), the total energy and energy current are (see Eqs (1.5.13) and (1.5.24))

$$\bar{W}_1 = \bar{W}_{E1} + \bar{W}_{K1}$$

, \(\bar{\eta}_4 = \bar{\eta}_{K1}$$

(4.2.3)
From Eqs (1.4.8) and (1.4.9) we obtain

\[ \frac{\mathbf{v}_e}{e} = -i \frac{e \mathbf{E}}{m_e (\omega - k \mathbf{V}_D)} \]  
(4.2.4)

and we can calculate \( \bar{W}_{K1} \) given by Eq. (1.5.29) (see also (4.2.1)):

\[ \bar{W}_{K1} = \frac{\omega_{pe}^2}{(\omega - k \mathbf{V}_D)^2} \bar{W}_{\mathbf{E}_1} = \bar{W}_{\mathbf{E}_1} \]  
(4.2.5)

Thus from (1.5.29) and (4.2.3) we get

\[ \bar{W}_1 = 2 \bar{W}_{\mathbf{E}_1}, \quad \bar{F}_1 = \frac{3}{2} \bar{W}_{\mathbf{E}_1} \mathbf{V}_D, \quad U = \frac{\bar{F}_1}{\bar{W}_1} = \frac{3}{2} \mathbf{V}_D \]  
(4.2.6)

It can be seen that for both fast and slow waves the total energy is positive and independent of \( \mathbf{V}_D \). An interesting feature is that the energy transport velocity \( U \) is different from the group velocity.

Starting now from Eqs (1.5.27) and (1.5.28), derived from the 'linear' conservation equation, we get (in view of (4.2.4), (4.2.5) and (4.2.1))

\[ \bar{W}_{1}^{lin} = \bar{W}_{\mathbf{E}_1} + \bar{W}_{K1}^{lin} = \pm 2 \frac{\omega}{\omega_{pe}} \bar{W}_{\mathbf{E}_1}, \quad U_{lin} = \frac{\bar{F}_1^{lin}}{\bar{W}_1^{lin}} = \mathbf{V}_D \]  
(4.2.7)

Thus the situation is now quite different. The quantity \( \bar{W}_{1}^{lin} \) is not positive definite; it can be negative for the slow wave (if \( k \mathbf{V}_D > \omega_{pe} \)). That is why the slow space-charge wave is called the \textit{negative energy wave} (see e.g. Ref. [5], p. 140). This is interpreted as a decrease of the total energy of the system when the slow wave is excited in a drifting plasma. The 'linear' result (4.2.7) can more easily be obtained from formula (2.1.18), specified to \( \mathbf{e}^\ell \) given by (4.2.1), as

\[ \frac{\partial}{\partial \omega} (\omega \mathbf{e}_1^\ell) \bigg|_{\mathbf{e}_1^\ell=0} = \omega \frac{\partial \mathbf{e}_1^\ell}{\partial \omega} = \pm 2 \frac{\omega}{\omega_{pe}} \]  
(4.2.8)
4.3. Two-stream instability

In a thermodynamic equilibrium there is no relative motion of the various plasma components, i.e. all mean velocities \( V_{\alpha 0} \) are the same. Relative motions in equilibrium can be the energy source leading to an instability. Similar instabilities can develop if there are two or more groups of particles of the same sort which move with different mean velocities. In both situations the plasma equilibrium deviates from a thermodynamic equilibrium in the velocity space, and the instabilities caused by this are called velocity-space instabilities. Their detailed description requires a kinetic approach. An approximate analysis is also possible within the framework of the fluid model if each species and each stream is treated as a separate fluid. This is meaningful if the thermal velocities \( v_{\alpha} \) are small compared to the shifts between the mean velocities \( V_{\alpha 0} \).

Simplest and most important are the electrostatic instabilities, described by Eq. (4.1.5). This equation for \( \omega(k) \) is algebraic, with real coefficients. Thus the solutions \( \omega \) will either be real (stable modes) or will appear in pairs as complex conjugate roots,

\[
\omega = \omega_r \pm \omega_i, \quad \omega_i > 0.
\]

In the latter case the system is unstable because the upper sign root has a positive imaginary part.

We now show that in a cold plasma the relative motion always leads to instability; thermal corrections in Eq. (4.1.5) are negligible if

\[
\left( v_p - V_{\alpha 0} \right)^2 + \left( \frac{\omega_i}{k} \right)^2 \gg v_{\alpha}^2 \quad \Rightarrow \quad v_{\alpha}^2 = \frac{k T_{\alpha}}{m_{\alpha}}
\]

If for two or more species there are only two distinct macroscopic velocities, two streams, i.e. \( V_{\alpha'0} = V_{01} \) and \( V_{\alpha''0} = V_{02} \), then the dispersion relation (4.1.5) for cold plasma can be written:

\[
\frac{\omega_{\alpha 1}^2}{(\omega - k V_{01})^2} + \frac{\omega_{\alpha 2}^2}{(\omega - k V_{02})^2} = 1
\]

where \( \omega_{\alpha 1}' = \sum \omega_{\alpha'2}^2 \) and \( \omega_{\alpha 2}'' = \sum \omega_{\alpha''2}^2 \) are the effective plasma frequencies of the two streams. Equation (4.3.3) is a quartic in \( \omega \), and a number of its real solutions can be determined graphically (see Fig. 7). As the left-hand side of Eq. (4.3.3) has two poles, at \( \omega = k V_{01} \) and \( \omega = k V_{02} \), and tends to zero for \( \omega \rightarrow \pm \infty \), two real roots can always be found, denoted by \( \omega_1 \) and \( \omega_4 \) in Fig. 7.
The remaining two roots will be complex conjugate if the pole separation $|kV_D|$ is sufficiently small, where $V_D = V_{02} - V_{01}$ is the relative drift velocity. That will be the case when the minimum of the left-hand side of Eq. (4.3.3) becomes larger than one. Thus Eq. (4.3.3) leads to the two-stream instability.

**Problem 13**

*Show that the onset of the two-stream instability (the minimum of the left-hand side of Eq. (4.3.3) equal to one, as in Fig. 7) corresponds to*

$$|kV_D| = |kV_D|_0 = \left( \frac{\omega_{p1}^{2/3} + \omega_{p2}^{2/3}}{2} \right)^{3/2}$$  \hspace{1cm} (4.3.4)

The instability develops for

$$0 < |kV_D| < |kV_D|_0$$  \hspace{1cm} (4.3.5)

i.e. for $0 < k < k_0$, if $V_D$ is fixed (long-wavelength instability). Note that Eq. (4.3.3) defines $(\omega - kV_{01})$ in terms of three parameters: $\omega_{p1}$, $\omega_{p2}$ and $kV_D [\omega - kV_{02} = (\omega - kV_{01}) - kV_D]$. Thus, for real $k$,

$$\omega_i = \omega_i (\omega_{p1}, \omega_{p2}, kV_D)$$  \hspace{1cm} (4.3.6)

The growth rate of the two-stream instability vanishes at the boundary points of the interval (4.3.5). (For $kV_D \rightarrow 0(\omega - kV_{01})^2 \rightarrow \omega_{p1}^2 + \omega_{p2}^2$.) Thus the growth rate must have a maximum in the unstable interval. This behaviour can be
illustrated by solving Eq. (4.3.3) for $\omega_{p1}^2 = \omega_{p2}^2$. In practice we obtain $\omega_{p1}^2 = \omega_{p2}^2$ for two electron streams of equal intensity ($n_e0/2$) drifting in opposite directions against the neutralizing ionic background ($n_e0 = Zn_i0$) which, in general, can also be moving. For such a configuration the dispersion relation (4.1.5) in the cold plasma approximation gives

$$\frac{\omega_{pe}^2/2}{(\omega - k V_{01})^2} + \frac{\omega_{pe}^2/2}{(\omega - k V_{02})^2} + \frac{\omega_{pi}^2}{(\omega - k V_{i0})^2} = 1 \quad (4.3.7)$$

where

$$\omega_{pe}^2 = 4\pi n_e0 e^2/m_e, \quad \omega_{pi}^2 = 4\pi n_i0 (Ze)^2/m_i$$

and $V_{01,2} = V_{i0} \mp V_D/2$ (we assume equal velocities of both streams in the ion rest frame to avoid $j_0 \neq 0$). Assuming that $|\omega_{pi}^2/(\omega - k V_{i0})^2| \ll 1$ and neglecting the ion contribution in Eq. (4.3.7), we arrive at the two-stream dispersion relation (4.3.3), which can be written

$$\frac{1}{(x + y)^2} + \frac{1}{(x - y)^2} = 1 \quad (4.3.8)$$

where

$$x = \frac{k V_D}{\sqrt{2} \omega_{pe}}, \quad y = (\omega - k V_{i0}) \frac{\sqrt{2}}{\omega_{pe}} \quad (4.3.9)$$

Solving Eq. (4.3.8) (a quadratic in $y^2$), we obtain

$$y_{1,2}^2 = (x^2 + 1) \pm (4x^2 + 1)^{1/2}$$

$$\omega = k V_{i0} \pm \frac{\omega_{pe}}{\sqrt{2}} y_{1,2} \quad (4.3.10)$$

(General results pertinent to Eq. (4.3.3) with $\omega_{p1}^2 = \omega_{p2}^2$ are obtained on replacing $\omega_{pe}/\sqrt{2} \to \omega_{p1}$, and $V_{i0} \to (V_{01} + V_{02})/2$ in Eqs (4.3.9) and (4.3.10).)

The instability ($y_{1,2}^2 < 0$) arises for $0 < x^2 < 2$, in agreement with Eq. (4.3.5), where $|k V_D|_0 = 2 \omega_{pe}$. Maximum growth rate is
Equations (4.3.10) and (4.3.11) indicate that the ion contribution in Eq. (4.3.7) can safely be neglected:

\[
\left( |\omega - k V_i| \right)^2 \sim \omega_{pe}^2 \gg \omega_{pi}^2
\]

except for the immediate vicinity of the marginal stability, i.e. for \( y_2 \) when \( k V_D \to 0 \) or \( |k V_D| \to 2 \omega_{pe} \).

The dispersion relation (4.3.3) can also be effectively solved when one of the plasma frequencies is much larger than the other (see e.g. Ref. [12]), in which case it is convenient to write Eq. (4.3.3) in the rest frame for the stream of greater plasma frequency. Denoting the drift velocity of the other stream in this frame by \( u \), we obtain

\[
1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_p'^2}{(\omega - ku)^2} = 0 , \quad \omega_p^2 \gg \omega_p'^2
\]

Specifically, we assume \( u > 0 \). (We account for \( u < 0 \) by replacing \( u \to -u \) and \( \omega \to -\omega \), which leaves the growth rate unchanged, as the non-real roots of Eq. (4.3.12) are complex conjugate.) A dispersion relation of the form (4.3.12) is obtained, for example, when ions in a two-component neutral plasma are drifting against electrons (\( \omega_p^2 = \omega_{pe}^2, \omega_p'^2 = \omega_{pi}^2 \)), or when a dilute electron beam (of large cross-section) penetrates a neutral plasma

\[
\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2 = \omega_{pe}^2 = 4 \pi n_{eo} e^2 / m_e
\]

In both examples \( j_0 \neq 0 \) (see Section 1.3).

Two roots of Eq. (4.3.12) are close to \( \omega = \pm \omega_p \), and the remaining two roots satisfy \( \omega \approx ku \). Putting \( \omega = \pm \omega_p \) in the last term in Eq. (4.3.12), and solving for \( \omega \), we obtain two (stable) solutions:

\[
\omega_\pm = \pm \omega_p \left[ 1 + \frac{\omega_p'^2}{2 \left( ku \mp \omega_p \right)^2} \right]
\]
For the plasma-beam system they represent Langmuir oscillations in a cold plasma, slightly modified by the electron beam. The root \( \omega_+ \) becomes a poor approximation for \( ku \approx \omega_p \). The remaining two roots can be looked for by introducing a small quantity \( \eta \):

\[ \omega = ku + \eta, \quad |\eta| \ll ku \]  

(4.3.14)

Neglecting \( \eta \) in \( \omega_p^2/\omega^2 \), we obtain

\[ \eta = \pm \frac{\omega_p'}{\sqrt{1 - (\omega_p/ku)^2}} \]  

(4.3.15)

Equation (4.3.15) predicts an instability for \( ku < \omega_p \), where the growth rate increases with \( ku \). For \( (ku)^2 < \omega_p^2 \) we obtain \( |\omega| = (\omega_p'/\omega_p) ku \). Again the results (4.3.15) become inapplicable for \( ku \approx \omega_p \), and to examine this parameter range we write Eq. (4.3.12) in terms of \( \eta \):

\[ \eta^3 (\eta + 2ku) + \eta^2 [(ku)^2 - \omega_p^2 - \omega_p'^2] - \omega_p^2 ku (2\eta + ku) = 0 \]  

(4.3.16)

and define a small parameter:

\[ \varepsilon = ku - \omega_p \]  

(4.3.17)

Assuming the ordering,

\[ |\varepsilon| \ll |\eta| \ll \omega_p \approx ku \]  

(4.3.18)

we neglect in Eq. (4.3.16) \( \eta/ku \) and \( \varepsilon/\omega_p \) compared to unity, but keep \( \varepsilon/\eta \). This leads to a cubic

\[ \eta^3 + \varepsilon \eta^2 = \frac{1}{2} \omega_p \omega_p'^2 \]  

(4.3.19)

For \( \varepsilon \) satisfying condition (4.3.18), Eq. (4.3.19) defines three values for \( \omega \), the fourth being \( \omega_+ \) given by Eq. (4.3.13).

**Problem 14**

Find graphically the number of real roots of Eq. (4.3.19) and show that the marginal stability corresponds to
\[\varepsilon = \varepsilon_0 = \frac{3}{2} \omega_p \left( \frac{\omega_p^2}{\omega_p^2} \right)^{1/3}\]  

(4.3.20)

Compare Eq. (4.3.20) with (4.3.4) and check condition (4.3.18) at the marginal stability.

Equation (4.3.19) can be used to determine the maximum growth rate of the two-stream instability in the limit \(\omega_p^2 \gg \omega_p^2\). Differentiating Eq. (4.3.19) with respect to \(\varepsilon\), we get

\[\eta \left( 3 \frac{d\eta}{d\varepsilon} + 1 \right) + 2 \varepsilon \frac{d\eta}{d\varepsilon} = 0\]  

(4.3.21)

At maximum growth rate \(\eta\) should be complex but \(d\eta/d\varepsilon\) real. That, in view of Eq. (4.3.21), is possible if and only if \(\varepsilon = 0\), i.e. in the resonant conditions, characterized by \(kU = \omega_p\). From Eq. (4.3.19) we obtain

\[\eta \mid_{kU = \omega_p} = \frac{3}{\sqrt{1}} \omega_p \left( \frac{\omega_p^2}{2 \omega_p^2} \right)^{1/3} \frac{3}{\sqrt{1}} \omega_p' \left( \frac{\omega_p^2}{2 \omega_p'} \right)^{1/3}\]  

(4.3.22)

where

\[\frac{3}{\sqrt{1}} = [1, - (1/2) \pm i \sqrt{3}/2]\]

Thus

\[\omega_{i_{\text{max}}} = \frac{\sqrt{3}}{2} \omega_p \left( \frac{\omega_p^2}{2 \omega_p^2} \right)^{1/3} = \frac{\sqrt{3}}{2} \omega_p \times \left( \frac{Z m_e}{2 m_i} \right)^{1/3}\]  

(4.3.23)

where the upper result refers to drifting ions and the lower to the plasma-beam system. It can be seen from Eq. (4.3.22) that

\[\mid \eta(kU = \omega_p) \mid \equiv \mid \omega - kU \mid \sim \omega_p\]

if \(\omega_p'/\omega_p\) is not too small. For example, for hydrogen plasma with streaming ions \((Z = 1, m_i/m_e = 1836), \omega_{pi} = \omega_{pe}/43\), and \(\mid \eta \mid \approx \omega_{pe}/15\). Physically it means
that in resonant conditions the Doppler-shifted frequency seen by drifting ions, \( \omega - ku \), is small and comparable to the resonant frequency of ions \( \omega_{pi} \). At the same time, the frequency seen by electrons is large and close to the resonant frequency of electrons, \( \omega \cong \omega_{pe} \). Consequently, both electrons and ions participate effectively in the wave motion. Maximum growth rate in hydrogen plasma with streaming ions is \( \omega_{i\text{max}} \cong \omega_{pe}/18 \).

Note that for all unstable modes described by Eq. (4.3.12) the phase velocity \( \omega/k \cong u \) (see Eqs (4.3.14) and (4.3.18)). Thus in our examples the unstable modes propagate with phase velocity approximately equal to the velocity of drifting ions, or the velocity of the electron beam.

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NONLINEAR EFFECTS IN WAVE PROPAGATION

A. SEN
Physical Research Laboratory,
Ahmedabad,
India

Abstract

NONLINEAR EFFECTS IN WAVE PROPAGATION.

Some simple nonlinear effects that can arise for large-amplitude wave propagation in a plasma are examined. Choice of simple models allowed these effects to be isolated and studied individually, and near analytic solutions are obtained. The paper contains sections on wave-steepening effects, solitons and relativistic effects.

INTRODUCTION

Wave propagation studies constitute an important area of plasma theory as they provide useful information on the collective properties of the media. It is customary to begin with a 'linear' analysis of the wave motion whereby, under the assumption of the small-amplitude approximation, all nonlinear terms (i.e. terms in which the dependent variable occurs in higher than the first power) are neglected. In such a case it is convenient to Fourier-analyse any physical perturbation, since each harmonic can be treated independently. The propagation characteristics of the wave are usually expressed in the form of a dispersion relation, \( \omega = \omega(k) \), which relates the frequency and wavenumber of the wave in terms of plasma parameters. The waveform is purely sinusoidal.

The 'linear' approximation can break down very easily. For example, in an experiment, if the wave is externally launched and is of large amplitude, the nonlinear terms can no longer be neglected. Similarly, if the wave is unstable, even in the linear sense, it quickly gets out of the linear regime since the predicted growth is exponential. In the nonlinear regime a variety of effects can come into play and drastically change the linear picture of wave propagation. For instance, the waveform could suffer distortion through the coupling of various harmonics, or the original wave could get coupled to other normal modes of the system and thereby excite them to large amplitudes. This process could be repeated several times in the system. It can also interact strongly with the particles by trapping them in its potential trough as well as modifying the plasma properties by distorting the density, raising the temperature, etc. These interactions are all
incorporated in the Vlasov-Maxwell formalism but a complete nonlinear solution of these equations is not yet available. The mathematical complexities are formidable and therefore to make any progress in nonlinear theory it is necessary to isolate these interactions and study them under various approximations. A logical first step is to study weakly nonlinear situations where perturbative methods can be used to estimate the changes occurring as a result of slight departures from the linear situation. This has been by far the most successful approach. It is also convenient to classify the interactions into coherent (where phase information of the waves can be kept track of) and turbulent interactions (where the interaction of many waves makes it impossible to keep track of the phase information).

We confine ourselves here to coherent situations and further restrict ourselves to macroscopic nonlinear phenomena associated with single large-amplitude waves. Our aim is to highlight certain basic nonlinear effects in a simple physical way and with the aid of not too complicated mathematics. Discussions will therefore be centred on the three basic modes in an unmagnetized plasma: the electron plasma waves, the ion acoustic waves and the electromagnetic waves. In Section 1, wave-steepening effects in finite-amplitude electron plasma waves are discussed. The nonlinear equations are solved for special steady solutions as well as for arbitrary time-dependent solutions through the more formal method of Lagrangian variables. In Section 2, finite-amplitude ion acoustic waves are used to introduce a special nonlinear wave solution, the soliton, which has very interesting properties and a special place in nonlinear wave propagation studies. Two nonlinear partial differential equations are discussed in this context as model evolution equations for certain classes of wave propagation problems. In the final section we look at nonlinearities arising from relativistic effects and show how they can lead to propagation of electromagnetic waves in overdense media. A useful ‘mechanical analogy’ is also pointed out in most cases, which helps us towards a qualitative understanding of the nonlinear solutions discussed.

1. WAVE-STEEPENING EFFECTS

Let us consider the simple longitudinal oscillations of the electrons in a plasma: plasma oscillations, which can be modelled by a one-dimensional cold plasma (since $\omega/k \gg v_e$) in the fluid approximation. For high-frequency plasma waves, the ions can be assumed to be immobile and to provide a uniform background of positive charge. The electrons satisfy the following equations:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (nv) = 0 \quad (1)$$
The nonlinear terms arise in the convective derivative terms of both the continuity equation (for density \( n \)) and the equation of motion (for velocity \( v \)). These are usually neglected in the linear analysis so that \( n \) and \( v \) can be solved in terms of the electrostatic potential \( \phi \). Use of Poisson's equation (3) then yields the linear dispersion relation. We proceed to solve Eqs (1)–(3) by retaining the nonlinear terms. A simple way of doing this is to seek special steady-state solutions which are functions of \( \xi = x - ut \); i.e. we look for travelling wave solutions with velocity \( u \). Transforming to variable \( \xi \), by using

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial t} = -u \frac{\partial}{\partial \xi}
\]

Eqs (1) and (2) can be converted to

\[
\frac{\partial}{\partial \xi} \left[ n(v-u) \right] = 0
\] (4)

\[
\frac{\partial}{\partial \xi} \left[ \frac{m}{2} (v-u)^2 - e\phi \right] = 0
\] (5)

Solving (4) and (5) we get

\[
n(v-u) = -n_0 u
\] (6)

\[
(v-u) = \pm \left( \frac{2e\phi}{m} + u^2 \right)^{1/2}
\] (7)

In (7) we choose the negative sign, since when \( \phi = 0 \), \( v = 0 \), the left-hand side is negative. Substituting now in Poisson’s equation, one obtains

\[
\frac{d^2 \phi}{d\xi^2} = -4\pi e n_0 \left[ 1 - \frac{u}{\left( \frac{2e\phi}{m} + u^2 \right)^{1/2}} \right]
\] (8)
Integrating Eq. (8) once gives

\[
\frac{1}{2} \left( \frac{d\phi}{d\xi} \right)^2 + 4\pi n_0 m \left[ \frac{e\phi}{m} - u \left( \frac{2e\phi}{m} + u^2 \right)^{1/2} \right] = C
\]  

(9)

The constant C can be evaluated by noting that the electric field \(|d\phi/d\xi| = E_m\) (maximum or minimum) when \(d^2\phi/d\xi^2 = 0\). This occurs when \(\phi = 0\) (from Eq. (8))

\[
\therefore C = \frac{E_m^2}{2} - 4\pi n_0 m u^2
\]  

(10)

Notice that Eq. (9) is in the form of the expression for the total energy of a 'particle' in a potential well if we identify \(\frac{1}{2}(d\phi/d\xi)^2\) with a 'kinetic energy' and the second term as a potential energy (\(\phi\) and \(\xi\) take the roles of \(x\) and \(t\), respectively). Thus the wave solutions are related to the orbits of a 'particle' in the potential well,

\[
V(\phi) = 4\pi n_0 m u^2 \left[ \frac{e\phi}{m u^2} - 1 + \frac{E_m^2}{8\pi n_0 m u^2} - \left( \frac{2e\phi}{m u^2} + 1 \right)^{1/2} \right]
\]  

(11)

Such an analogy is quite useful for qualitative understanding of the wave solutions. For example, when \(e\phi/(mu^2) \ll 1\) the potential well becomes parabolic, yielding simple harmonic motion for the particle orbit and corresponding sinusoidal waves for \(\phi\), which is the linear limit of plasma waves. In the nonlinear regime we can see what happens to the sine wave by following the peaks and troughs of \(\phi\) (i.e. points given by \(d\phi/d\xi = 0\)). At such points Eq. (9) can be rewritten as \([1]\)

\[
\psi + 2 - \theta = 2 (1 + \psi)^{1/2}
\]  

(12)

where

\[
\psi \equiv \frac{2e\phi}{mu^2} \quad ; \quad \theta \equiv \frac{E_m^2}{4\pi n_0 m u^2}
\]

A graphical solution of (12) is illustrated in Fig.1. The intersections, which represent the maximum and minimum of \(\psi\), lie close to each other and almost symmetrically about \(\psi = 0\), near \(F_1\), where \(\theta\) is small. This is the linear regime, where the oscillations are simple harmonic. As \(\theta\) increases, the distance between the intersections increases asymmetrically around \(\psi = 0\) and the oscillations
FIG. 1. Graphical solution of Eq. (12). The straight lines are the left-hand side for $\theta = 0$ and $\theta = 1$ respectively. The intersections with the solid curve (RHS) within the area defined by the two straight lines represent physical solutions.

FIG. 2. Plot of electric field for varying amplitudes depicting nonlinear wave-steepening effects.

become increasingly anharmonic. This is shown in Fig. 2, where the electric field is plotted for increasing amplitudes, and the waveform is found to steepen. Beyond $\theta = 1$ there is a discontinuity in the slope of $\psi$ and no solutions of the periodic type are possible. Note that $\theta = e^2 E_m^2 / (m^2 \omega_p^3)$ (since $u = \omega/k \equiv \omega_p / k$), so that physically the critical point $\theta = 1$ corresponds to the situation where the electron excursion length $eE_m / (m\omega_p^3)$ becomes comparable to the wavelength $k^{-1}$. In such a situation, wavebreaking occurs.

Thus one characteristic of nonlinearity is wave steepening. As seen from Fig. 2, in the nonlinear regime the wavelength becomes a function of the wave
amplitude, which implies that the phase velocity is also a function of the wave amplitude. Thus, for a given finite amplitude wave, the top travels faster than the lower portion, giving rise to wave steepening. Beyond a certain amplitude the wave breaks over. Such a phenomenon is not restricted merely to the special solutions we have analysed but can be seen for solutions of Eqs (1)—(3) with any arbitrary time dependence. Such general solutions can be obtained by going over from the Eulerian variables \((x, t)\) (i.e. laboratory frame) to Lagrangian variables \((x_0, \tau)\) (fluid frame). We define

\[
\tau \equiv t , \quad x_0 \equiv x - \int_0^\tau dv(x_0, \tau') \quad (13)
\]

Then the space and time derivatives transform according to

\[
\frac{\partial}{\partial x} \equiv \left[ 1 + \int_0^\tau dv(x_0, \tau') \frac{\partial}{\partial x_0} \right]^{-1} \frac{\partial}{\partial x_0} \quad (14)
\]

and

\[
\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial \tau} - v(x_0, \tau) \left[ 1 + \int_0^\tau dv(x_0, \tau') \frac{\partial}{\partial x_0} \right]^{-1} \frac{\partial}{\partial x_0} \quad (15)
\]

Equations (1) and (2) now become

\[
\frac{\partial}{\partial \tau} \left[ n(x_0, \tau) \left[ 1 + \int_0^\tau dv(x_0, \tau') \frac{\partial}{\partial x_0} \right] \right] = 0 \quad (16)
\]

\[
\frac{\partial v}{\partial \tau}(x_0, \tau) = -\frac{e}{m} E(x_0, \tau) \quad (17)
\]

Taking the \(\tau\) derivative of (17) and using (16) along with the time-derivative form of (3), to eliminate \(\partial E/\partial \tau\) in terms of \(v(x_0, \tau)\), allows us to obtain a simple equation:

\[
\frac{\partial^2 v}{\partial \tau^2}(x_0, \tau) + \omega_p^2 v(x_0, \tau) = 0 \quad (18)
\]
We have thus reduced the nonlinear set of equations (1) – (3) to a linear one, effectively through a nonlinear transformation given by (13). Of course, this is not always possible and the Lagrangian method fails in certain circumstances. The general solutions to Eq.(18) can be written simply as

\[ v(x_0, \tau) = V(x_0) \cos(\omega_p \tau) + \omega_p X(x_0) \sin(\omega_p \tau) \]  

(19)

Correspondingly, using the equations for E and n, we can get

\[ E(x_0, \tau) = \left( m \middle/ e \right) \omega_p V(x_0) \sin(\omega_p \tau) - m \middle/ e \omega_p^2 X(x_0) \cos(\omega_p \tau) \]  

(20)

and

\[ n(x_0, \tau) = \frac{n(x_0,0)}{1 + \frac{1}{\omega_p} \frac{\partial V(x_0)}{\partial x_0} \sin(\omega_p \tau) + \frac{\partial X(x_0)}{\partial x_0} (1 - \cos(\omega_p \tau))} \]  

(21)

where \( V(x_0) \) and \( X(x_0) \) are initial excitation oscillates coherently and indefinitely at the electron plasma frequency in a sinusoidal fashion. To see what happens in the laboratory frame we have to transform back to Eulerian variables using relation (13). For this it is necessary to specify the initial conditions. As a simple example we can choose

\[ n(x_0,0) = n_0 (1 + k \cos(k x_0)) \]  

(23)

\[ v(x_0,0) = V(x_0) = 0 \]  

(24)

Then from (22), \( X(x_0) = (\Delta/k) \sin(k x_0) \), and Eq. (20) becomes

\[ E(x_0, \tau) = -\frac{m}{e} \omega_p^2 \frac{\Delta}{k} \sin(k x_0) \cos(\omega_p \tau) \]  

(25)

The coordinate transformation in Eq.(13) may be expressed as
\[ \tau = t \quad ; \quad kx = kx_0 + \alpha(\tau) \sin(kx_0) \]  

(26)

where

\[ \alpha(\tau) = 2 \Delta \sin^2(\omega_p \tau/2) \sin(kx_0) \]

For \( \Delta \ll 1 \), which is the linear limit, \( x \approx x_0 \), and we have the same phenomena in the laboratory frame as in the Lagrangian frame. For larger amplitudes let us write down the explicit analytic expression for \( E(x,t) \) in the laboratory frame. This can be done by expressing

\[ \sin[kx_0(x,t)] = \sum_{n=1}^{\infty} a_n(t) \sin(nkx) \]  

(27)

where

\[ a_n(t) = \frac{k}{2 \pi} \int_{0}^{2\pi/k} dx \sin(nkx) \sin[kx_0(x,t)] \]

\[ = (-1)^{n+1} \frac{2}{n\alpha(t)} J_n(n\alpha(t)) \]  

(28)

(using Eq.(26)). \( J_n \) is the Bessel function of the first kind of order \( n \). Therefore,

\[ E(x,t) = -\frac{m}{e} \frac{\omega_p^2}{k} \Delta \sum_{n=1}^{\infty} (-1)^n + 1 \frac{2}{n\alpha(t)} J_n(n\alpha(t)) \sin(nkx) \cos(\omega_p t) \]  

(29)

Similar expressions can be obtained for \( V \) and \( n \). We notice immediately from Eq.(29) that the waveform in the laboratory frame is no longer sinusoidal but is distorted through the generation of higher harmonics of \( (kx) \). Another interesting feature is the additional time dependence (apart from \( \omega_p t \)) which enters through the argument of the Bessel functions and corresponds to a nonlinear shift in the frequency of the oscillation.

Note that from Eq.(26) we require that \( \alpha < 1 \) in order that \( x \) be single valued. This implies that \( |\Delta| < 1/2 \) and hence \( kX(x_0) < 1 \), which is the restriction obtained earlier for the steady-state solutions. Thus when wavebreaking occurs we cannot use the Lagrangian method either. General restrictions on the class
of initial value problems that can be treated by the Lagrangian method may be derived from Eq.(21) by requiring that the density remain non-negative and finite for all time (for more details see Ref. [2]).

Before leaving nonlinear electron plasma oscillations, let us examine the effect of including finite temperature effects. This can be done simply by including an additional term

\[
\left(-\frac{1}{n_e m} \frac{\partial P}{\partial x}\right)
\]

in the momentum equation. To close the set we need one more equation, which can be conveniently taken to be the equation of state:

\[
\frac{P}{n^3} = \gamma \quad \text{(good for } \frac{\omega_P}{k} \gg v_e) \quad \text{(30)}
\]

Transforming again to Lagrangian variables, one now obtains

\[
\left(\frac{\partial^2}{\partial \tau^2} + \omega_p^2\right) v(x_0, \tau) = \frac{3}{mn(x_0,0)} \frac{\partial}{\partial x_0} \left\{ \frac{P(x_0,0)}{\tau} \frac{\partial}{\partial x_0} v(x_0, \tau) \right\}
\]

\[
\left[ 1 + \int_0^{\tau} d\tau' \frac{\partial v}{\partial x_0}(x_0, \tau') \right]^{4}
\]

which, unlike the cold plasma case, is now a nonlinear equation and hence not easily tractable. For \(k^2 \lambda_D^2 \ll 1\) we could treat the thermal contribution as a small correction and linearize Eq. (31) (for \(P(x_0,0) = P_0, \ n(x_0,0) = n_0\) and \(E(x_0,0) = 0\)) to get

\[
\left(\frac{\partial^2}{\partial \tau^2} + \omega_p^2\right) v(x_0, \tau) = 3 \omega_p^2 \lambda_D^2 \frac{\partial^2 v}{\partial x_0^2}
\]

\[
(32)
\]

The solution of Eq. (32) for initial periodic perturbations is again similar to those obtained earlier for the cold plasma case, i.e. temporal oscillations that persist indefinitely and have period \(2\pi/\omega(k)\). The only modification in this case is that there is a slight frequency shift from the cold plasma result, since

\[
\omega(k) = \omega_p (1 + 3k^2 \lambda_D^2)^{1/2}
\]

\[
(33)
\]

However, if the initial perturbation is localized, the dispersive effects of finite electron temperature cause the disturbance to spread in space indefinitely with
\( v(x_0, t \to \infty) = 0 \). This is in contrast to the cold plasma case, where coherent oscillations are maintained indefinitely over the region of initial excitation. This can be understood physically as follows. From (33), which is the linear dispersion relation, we see that the phase velocity is now a function of the wavenumber. Therefore for a localized initial perturbation, its various Fourier components will travel at different velocities, causing it to spread. In the laboratory frame this tendency will appear together with the steepening effects associated with harmonic generation, and the two effects oppose each other. Interesting situations arise when the two are comparable, and instances of this will be seen in the next section.

2. SOLITONS

We now look at another characteristic wave in a plasma: the low-frequency ion acoustic mode. For this, consider an unmagnetized plasma, with cold ions but warm electrons \((T_i \ll T_e)\). Ion acoustic waves can be simply described in such a plasma by the following equations (in one dimension):

\[
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} (n_i v_i) = 0 \tag{34}
\]

\[
\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = -\frac{e}{m_i} \frac{\partial \phi}{\partial x} \tag{35}
\]

\[
0 = \frac{e \partial \phi}{\partial x} - \frac{T_e}{n_e} \frac{\partial n_e}{\partial x} \tag{36}
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = 4\pi e (n_e - n_i) \tag{37}
\]

We have neglected electron inertia terms so that Eq.(36) gives a Boltzmann distribution for electrons, \(n_e = n_0 \exp(e \phi/T_e)\). This can be substituted in the Poisson equation (37). It is convenient at this stage to make the equations dimensionless. Accordingly we introduce

\[
x \equiv x/\lambda_D, \quad t \equiv t \omega_{pi}, \quad \phi \equiv \omega \phi/T_e, \quad n \equiv \frac{n_i}{n_0}, \quad v \equiv \frac{v_i}{c_s}
\]
where $\lambda_D$ is the Debye length and $c_s$ is the ion sound velocity. The equations now reduce to

$$\frac{\partial^2 \phi}{\partial x^2} = e^\phi - n \quad (38)$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (nv) = 0 \quad (39)$$

$$\frac{\partial v}{\partial t} + \frac{v \partial v}{\partial x} = - \frac{\partial \phi}{\partial x} \quad (40)$$

Note that linearization at this stage about the uniform equilibrium $n = 1$, $v = 0$, $\phi = 0$ would reproduce the usual ion sound dispersion relation, $\omega^2 = [1 + 1/k^2]^{-1}$, for perturbations of the form $\exp[i(kx - \omega t)]$. It is also important to realize the importance of the Poisson equation, which depicts the effects of charge separation through $d^2 \phi/dx^2$. In fact, if charge neutrality is assumed ($n = e^\phi$) then Eq. (40) would reduce to

$$\frac{\partial v}{\partial t} + \frac{v \partial v}{\partial x} = - \frac{1}{n} \frac{\partial n}{\partial x} \quad (41)$$

Equations (41) and (39) lead to wave solutions with unlimited steepening of an initial perturbation until breaking occurs. However, when spatial gradients become large it is no longer permissible to neglect $\partial^2 \phi/\partial x^2$ in (38) and it is this dispersive effect of deviations from charge neutrality which eventually limits the build-up of short-wavelength components to the disturbance. We shall come back to this point later, but first, as in Section 1, let us look for special solutions with dependence on $\xi = x - Mt$ (M is the Mach number since velocities are normalized to the speed of sound). Then Eqs (38)–(40) can again be cast in an analogous form to Eq. (9):

$$\frac{1}{2} \left( \frac{d\phi}{d\xi} \right)^2 - [e^\phi + M(M^2 - 2\phi)^{1/2} + C] = 0 \quad (42)$$

The constant $C = -(1 + M^2)$ if we require that $V(\phi) = 0$ at $\phi = 0$. Therefore the quasipotential $V(\phi)$ takes the form

$$V(\phi) = 1 - e^\phi + M^2 \left[ 1 - \left( 1 - \frac{2\phi}{M^2} \right)^{1/2} \right] \quad (43)$$
For $M$ lying in a certain range this function has the shape sketched in Fig.3. Note that a ‘particle’ entering from the left will go to the right-hand side of the well, reflect and return to its original position, making a single transit. It does not go back again since $V(0) = (dV(\phi)/d\phi|_0 = 0$. This orbit corresponds to a very special solution in $\phi$: a single-pulse solution called a soliton (see Fig.4). To digress slightly, if the well had any friction in it (e.g. if we had included any dissipation in our system) the particle would not quite have come back to $\phi = 0$ and then it would oscillate back and forth with decreasing amplitude. The corresponding solution is a shock wave, which is another nonlinear solution for systems with nonlinearity and dissipation. Coming back to the soliton, we can get an analytic expression for it by solving (42) in a slightly approximate way. For $M$ close to unity (i.e. $\delta M = M - 1 \ll 1$) and finite but small amplitudes (so that $e^\phi$ can be expanded), Eq.(42) can be simplified to

$$\left(\frac{d\phi}{d\xi}\right)^2 = \frac{2}{3} \phi^2 (3\delta M - \phi)$$

which can be integrated to give

$$\phi = 3\delta M \text{sech}^2 \left[\sqrt{\frac{1}{3}}\delta M \right] (x - Mt)$$
This is an expression for a small-amplitude soliton, travelling slightly faster than the Mach number of unity. In general, going back to (42), such solitary waves can exist in the range of $1 \leq M \leq 1.6$. This can be surmized easily by examining $V(\phi)$ and requiring it to be negative, i.e. a well, and to cross the $\phi$ axis for some $\phi > 0$, so that the 'particle' can become reflected. (As an exercise, the reader can check the potential given by Eq.(9) where the second condition is not satisfied and hence there are no soliton solutions for these waves.)

Solution (45), obtained in the weakly nonlinear regime ($\phi_{\text{max}} \ll 1$) belongs to an important class of exact solutions of certain nonlinear partial differential equations. It is interesting to derive such a partial differential equation, since it can serve as a useful model evolution equation for weakly nonlinear ion acoustic waves. To do this we note that the argument of the sech function in (45) can be rewritten as

$$\frac{1}{\sqrt{2}} \left[ \delta M^{1/2} (x-t) - \delta M^{3/2} t \right]$$

indicating two different scalings in space and time variables (in a frame moving with $M = 1$) that should be incorporated in the equation. Accordingly we construct the following stretched variables:

$$\xi = e^{1/2} (x-t) ; \quad \tau = e^{3/2} t \quad (46)$$

where $\epsilon$ is a small expansion parameter. Next we perturb the physical parameters about their equilibrium values

$$n = 1 + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \ldots$$

$$\phi = \epsilon \phi^{(1)} + \epsilon^2 \phi^{(2)} + \ldots$$

$$v = \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \ldots \quad (47)$$

From (46),

$$\frac{\partial}{\partial x} = \frac{\epsilon^{1/2} \partial}{\partial \xi} ; \quad \frac{\partial}{\partial t} = \frac{\epsilon^{3/2} \partial}{\partial \tau} - \frac{\epsilon^{1/2} \partial}{\partial \xi} \quad (48)$$

Following standard expansion procedures, one can now obtain in the lowest order

$$n^{(1)} = \phi^{(1)} = v^{(1)} \quad (49)$$
In the next order one gets

$$\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = \phi^{(2)} + (\phi^{(1)})^2 - n^{(2)}$$  \hspace{1cm} (50)

$$- \frac{\partial n^{(2)}}{\partial \xi} + \frac{\partial n^{(1)}}{\partial \tau} + \frac{\partial}{\partial \xi} (n^{(1)} \psi^{(1)}) + \frac{\partial v^{(2)}}{\partial \xi} = 0$$  \hspace{1cm} (51)

$$- \frac{\partial v^{(2)}}{\partial \xi} + \frac{\partial v^{(1)}}{\partial \tau} + \psi^{(1)} \frac{\partial v^{(1)}}{\partial \xi} = - \frac{\partial \phi^{(2)}}{\partial \xi}$$  \hspace{1cm} (52)

Adding (51) and (52) eliminates $v^{(2)}$. Differentiating (50) with respect to $\xi$ once, substituting $(\partial \phi^{(2)}/\partial \xi) - (\partial n^{(2)}/\partial \xi)$ in terms of first-order quantities, and, further, using (49) one can get

$$\frac{\partial n^{(1)}}{\partial \tau} + \frac{n^{(1)} \partial n^{(1)}}{\partial \xi} + \frac{1}{2} \frac{\partial^3 n^{(1)}}{\partial \xi^3} = 0$$  \hspace{1cm} (53)

Equation (53) is known as the Korteweg-DeVries equation and was first derived by them in 1895 to model shallow water waves [3]. In the present case, it serves to model weakly dispersive and weakly nonlinear ion acoustic waves in a plasma. Its nonlinear contribution comes from convective terms of the $[(v \cdot \nabla)v]$ type and the dispersive term arises in this case from the deviation from exact charge neutrality in Poisson's equation. These two terms have opposite tendencies — non-linearity tends to steepen the wave whereas dispersion tends to spread it out. In principle therefore, it is possible to obtain a stationary solution of some stable form if these two exactly balance each other. Such a solution is in fact the 'solitary' wave solution we have obtained earlier and can be easily checked by substituting in the equation. One of the remarkable properties of these solitary waves is that they are quite stable and retain their identity even through a collision process (Fig.5). In other words, if two solitons are set on a collision course they will emerge from it retaining their initial sizes and speeds. Zabusky and Kruskal [4] invented the term 'soliton' to distinguish the class of solitary waves that behave in this remarkable fashion. They were also among the first to check out the soliton nature of the KdV solutions by doing a numerical calculation. Their early work is interesting from several points of view and, in historical perspective, stimulated and motivated much of the recent work on solitons.

They chose an initial sinusoidal perturbation and solved the KdV equation numerically. Their results are depicted in Fig.6. At first they observed a steepening of the wave (due to predominance of the nonlinear term). After the wave has steepened sufficiently the third term becomes important and serves to prevent
the formation of a discontinuity. Oscillations of small wavelength develop on the left of the front. Amplitudes of the oscillations grow, and finally each oscillation achieves an almost steady-state amplitude (which increases linearly from left to right) — the shapes are identical to simple solitons. Finally, each soliton begins to move uniformly at a rate linearly proportional to its amplitude. The larger solitons soon overtake the smaller ones; there is a period of overlap, and eventually they come out retaining their original shape and size. In the author's words this is a 'nonlinear physical process in which interacting localized pulses do not scatter irreversibly'. There is a further important point about reversibility in their results in that after a long enough time they found the solution reverting to the initial condition, i.e. there is a phenomenon of recurrence.

Several important points were established by these calculations. One was, of course, the existence of solitary wave-train solutions for the KdV. Secondly, it established the property of preservation of identity of solitons in a collision. Thirdly, the phenomenon of recurrence is also very interesting and seems to throw some light on the Fermi-Pasta-Ulam (FPU) problem, which is related to one of the
important tacit assumptions of classical statistical mechanics, that 'small non-linearities' lead to equipartition of energy. In a strictly linear theory, as we know, all the energy in a vibrating system must remain in those normal modes into which it is placed by the initial conditions. In practice, however, such systems are very often observed to be 'thermalized' into a state for which approximately equal energy is shared by each normal mode. Thus it seems reasonable to suppose that a small but unavoidable nonlinearity leads from arbitrary initial conditions to the thermalized (ergodic) state.

This assumption was first critically examined in a series of computer experiments by Fermi, Pasta and Ulam [5] in the 1950s. They studied the vibration of 64 mass particles connected by nonlinear springs arranged as an approximation to a nonlinear vibrating string. The results were surprising — no tendency toward thermalization was observed. Instead, if the energy was put into one particular mode it kept coming back to it after interaction with a few other modes. Zabusky and Kruskal's results can be viewed as a manifestation of a similar recurrence phenomenon. It seems to indicate that a class of nonlinear interactions exists (in our case soliton-type interactions) which leads to a non-ergodic state. Over the last twenty years a fairly extensive literature has developed to record computer experiments and analytic attempts to understand the FPU problem. Solitons have figured prominently in them.

Returning to the KdV equation and the soliton solution, apart from the extensive numerical investigations, there have been various analytical efforts to evolve general methods to solve such equations. What we have done so far is to obtain a very special solution — the soliton — by transforming to the travelling wave frame; this is certainly not the most general solution. What one ideally desires is the solution of the initial value problem, i.e. how does an initial perturbation evolve in time? Does any general method exist for the nonlinear KdV? The answer is the inverse scattering method, the brainwave of Gardner, Greene, Kruskal and Miura [6]. This general technique is seen by many as a very major mathematical triumph of recent times. In essence it is a nonlinear generalization of the Fourier transform method where solitons play the role of basis vectors and allow a nonlinear equation to be solved through a host of linear operations. It is not universally applicable to all nonlinear partial differential equations but to a large class of them which have certain common properties (e.g. admitting infinite conservation laws). The interested reader is referred to Refs [7] and [8] for some excellent descriptions of the method.

The KdV equation, which we derived here to describe nonlinear ion-acoustic waves, can model other nonlinear waves in the plasma too. In fact, any wave with a linear dispersion relation of the form

\[ \omega = ak + bk^3 \]  \hspace{1cm} (54)
can be shown to obey the KdV equation in the weakly nonlinear regime. Examples of such waves are the Trivelpiece-Gould mode, surface waves on a cylindrical plasma, magnetosonic waves, etc. Besides the KdV equation there are a host of other nonlinear partial differential equations which are exactly solvable by the inverse scattering method and which can serve as model equations for nonlinear wave propagation problems. One such important equation, which is quite useful for many plasma wave propagation problems, is the nonlinear Schrödinger equation:

$$\frac{i\partial \tilde{E}}{\partial \tau} + \frac{\partial^2 \tilde{E}}{\partial x^2} + |\tilde{E}|^2 \tilde{E} = 0$$ (55)

It is pertinent to waves with a linear dispersion relation of the form

$$\omega = a + bk^2$$ (56)

and describes the nonlinear evolution of an envelope of carrier waves satisfying the above relation. As an example, we consider Langmuir waves in a warm plasma and represent the wave amplitude as

$$E(x, t) = \tilde{E}(x, t) e^{i(kx - \omega t)}$$ (57)

where \(\tilde{E}\) is a slowly varying amplitude. Rather than go through the formal derivation from the set of fluid equations, we give a more heuristic derivation which can also serve as a quick recipe for arriving at nonlinear evolution equations from the linear dispersion relation. The linear propagation equation can be written as

$$D(\omega, k, \alpha_j) E(k, \omega) = 0$$ (58)

where \(D(\omega, k, \alpha_j)\), the dispersion function, is a function of \(\omega, k\) and various profile parameters \(\alpha_j\) (e.g. \(n, T, \) etc.). In the weakly nonlinear regime, assuming a form for \(E\) as given by (57), we can express (58) as

$$e^{i(kx - \omega t)} D\left(k + \frac{i\partial}{\partial x}, \frac{\omega - i\partial}{\partial t}, \alpha_j + \Delta \alpha_j\right) \tilde{E}(x, t) = 0$$ (59)

where the derivatives are now in terms of slow variables and operate on the envelope. \(\Delta \alpha_j\) is the nonlinear modification of the profile and is assumed small and slowly varying. Taylor-expanding around the linear state, we get
\[ D \approx D(k, \omega, \alpha_j) + i \frac{\partial D}{\partial k} \frac{\partial}{\partial x} - \frac{1}{2} \frac{\partial^2 D}{\partial k^2} \frac{\partial^2}{\partial x^2} + \ldots \]

\[ - \frac{i \partial D}{\partial \omega} \frac{\partial}{\partial t} + \ldots + \sum_j \Delta \alpha_j \frac{\partial D}{\partial \alpha_j} + \ldots \]  

(60)

Transforming to a moving frame with velocity

\[ v_g = - \frac{\partial D/\partial k}{\partial D/\partial \omega} \]

we can get rid of the \( \partial D/\partial k \) term. Further, since \( D(\omega, k, \alpha_j) = 0 \) (linear dispersion relation), Eq.(59) can now be rewritten as

\[ \int (kx - \omega t) \left( i \frac{\partial D}{\partial \omega} \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2 D}{\partial k^2} \frac{\partial^2}{\partial x^2} + \sum_j \Delta \alpha_j \frac{\partial D}{\partial \alpha_j} \right) \bar{E} = 0 \]  

(61)

which to lowest order reduces to

\[ \left[ i \frac{\partial}{\partial t} + \frac{p \partial^2 D}{\partial x^2} + F(\Delta \alpha_j) \right] \bar{E} = 0 \]

(62)

Here

\[ p = - \frac{1}{2} \frac{\partial^2 D}{\partial k^2} \frac{\partial D}{\partial \omega} = \frac{1}{2} \frac{\partial v_g}{\partial k} \]

(63)

and

\[ F = \sum_j \left( \frac{\partial D}{\partial \alpha_j} / \frac{\partial D}{\partial \omega} \right) \Delta \alpha_j \]

(64)

If \( \Delta \alpha_j \approx k |\phi|^2 \) we recover the nonlinear Schrödinger equation. Such a nonlinear modification of a profile parameter occurs in several situations. A simple example is the effect on density due to the ponderomotive force or radiation pressure force.
Such a force can be explained by a simple model calculation. The equation of motion for an electron interacting with an oscillating field is

\[
\frac{dv}{dt} = -\frac{e}{m} E(x)\sin(kx - \omega t)
\] (65)

For a slow electron the first-order perturbed orbit can be obtained as

\[
x^{(1)} = x_0 - \frac{eE(x_0)}{m\omega^2} \sin(kx_0 - \omega t)
\] (66)

This describes the rapid oscillations driven by the high-frequency field. Substituting this perturbed orbit into the equation of motion then yields terms with rapid oscillations as well as terms with a slow time dependence. Averaging out the fast variation, we get

\[
\frac{d}{dt} \langle v \rangle = -\frac{\partial}{\partial x} \left[ \left( \frac{e}{m\omega} \right)^2 \frac{E^2(x)}{2} \right]
\] (67)

which describes a slow drift superimposed on the rapid oscillations. The term on the right-hand side is like a low-frequency pressure arising out of the self-interaction of the high-frequency wave. The effect of this pressure is to push the electrons out of these regions in the plasma where the electric field has a local maximum. This effect is called the ponderomotive force and is analogous to the radiation pressure effects that can be produced by intense light beams.

When electrons are expelled they also drag the ions out because of ambipolar fields. This leads to a density depression or cavity. Such a cavity can further act as a potential well within which the Langmuir waves can get trapped, leading to further enhancement of the local field and further digging. Now if the rate at which energy is being trapped within the cavity is exactly equal to the energy which is leaking out due to convection, a stationary state can result. In the Schrödinger equation, the term proportional to \( \nabla^2 \vec{E} \) is a measure of the convection and the \( |\vec{E}|^2 \) term leads to wave trapping. For a closed system we can treat the energy in the electric field as a constant, i.e.

\[
\int |\vec{E}(x,t)|^2 d^n x = \text{constant}
\] (68)

where \( n \) indicates the dimensionality of the system. Therefore if the localized electric field has a scale length \( L \), then (68) implies that

\[
|\vec{E}|^2 \sim \frac{1}{L^n}
\] (69)
The convection term, on the other hand, always scales as $1/L^2$. Therefore, for short scale lengths $L$, the $|\mathbf{E}|^2$ term due to the density cavity can always overpower the convection term for $n = 2, 3$ but not for $n = 1$. Thus in one dimension a balance can be struck and the state obtained is again a soliton solution, in this case an envelope soliton (Fig. 7). We can write the expression for such an envelope soliton for Langmuir waves as

$$E(x, t) = E_m \text{sech}(k_0 x) e^{i(kx - \omega t)}$$  \hspace{1cm} (70)

with

$$k_0 \lambda_D = E_m/(48\pi n T)^{1/2}$$  \hspace{1cm} (71)

Such a nonlinear state can also occur for intense electromagnetic waves and is analogous to self-focussing effects. Physically it represents a highly localized concentration of the electric field and is often identified with the nonlinear state of filamentation instability. Some of the other common names in the literature are spikons (because the electric field appears as a spike) or cavitons (because the local density is depressed and resembles a cavity). They play an important role in laser-plasma interactions, as well as in RF heating schemes.

3. RELATIVISTIC EFFECTS

So far we have restricted ourselves to rather weak amplitudes of the propagating wave and have looked for some nonlinear solutions of electrostatic waves. In this section we consider very large amplitude electromagnetic waves propagating in a cold plasma. When the wave amplitude is large enough for the induced electron velocity $v_{\text{ind}} \equiv eE_0/m\omega_0$ to become comparable to the velocity of light, relativistic effects become important and lead to a new nonlinearity in the equations, which essentially arises from the variation in the electron mass. For simplicity, we still assume the ions to be stationary (essentially this restricts the electric field amplitude such that the induced ion velocity is still appreciably
smaller than c, but since this is down by the electron-to-ion mass ratio it is not a very restrictive condition). The relevant equations are now simply the Maxwell equations and the relativistic form of the momentum equation for electrons:

\[ \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \]  

\[ \nabla \times \vec{B} = \frac{4\pi}{c} n e \vec{v} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \]  

\[ \nabla \cdot \vec{E} = 4\pi e (n - n_0) \]  

\[ \nabla \cdot \vec{B} = 0 \]  

\[ \frac{\partial \vec{p}}{\partial t} + (\vec{v} \cdot \nabla) \vec{p} = e \vec{E} + \frac{e}{c} (\vec{v} \times \vec{B}) \]  

where \( \vec{p} = m \vec{v}/((1-v^2/c^2)^{1/2}) \). We again look for special (z-ut)-dependent solutions and define the following dimensionless quantities:

\[ \xi = \frac{\omega}{c} (z-ut) \]  

\[ \beta = \frac{u}{c} ; \quad \rho = \frac{\vec{p}}{mc} = \frac{\vec{v}/c}{\sqrt{1-v^2/c^2}} \]  

Equations (72)—(76) can then be reduced to the following set of coupled equations:

\[ \frac{d^2 \rho_{x,y}}{d\xi^2} + \frac{\beta \rho_{x,y}}{(\beta^2-1) [\beta \sqrt{1+\rho^2} - \rho_z]} = 0 \]  

\[ \frac{d^2}{d\xi^2} [\beta \rho_z - \sqrt{1+\rho^2}] + \frac{\rho_z}{[\beta \sqrt{1+\rho^2} - \rho_z]} = 0 \]  

These equations were analysed by several authors [9—12] some time ago and yield exact or semianalytical results in various limits. One such simple limit is the
purely transverse case ($p_z = 0$). For bounded solutions (i.e. $p_x^2 + p_y^2 = \text{const}$) one finds circularly polarized waves:

$$
\rho_x = \rho \cos \left[ \omega (t-z/u) \right] ; \quad \rho_y = \rho \sin \left[ \omega (t-z/u) \right]
$$

(79)

where

$$
\omega = \beta \omega_p (\beta^2 - 1)^{-1/2} (1 + \rho^2)^{-1/4}
$$

(80)

Using (80) one can express the wave phase velocity $u$ in terms of $\beta$ as

$$
u = \beta c = c \varepsilon^{-1/2}
$$

(81)

where $\varepsilon$, the dielectric constant of the plasma, is given by

$$
\varepsilon = 1 - \frac{\omega_p^2}{\omega^2} (1 + \rho^2)^{-1/2}
$$

(82)

Using Eqs (72) and (76), it can be shown that

$$
E_x = \left( \frac{mc \omega \rho}{\varepsilon} \right) \sin \left( \frac{\omega t}{u} \right) , \quad E_y = \left( \frac{mc \omega \rho}{\varepsilon} \right) \cos \left( \frac{\omega t}{u} \right)
$$

(83)

so that

$$
\rho^2 = \frac{e^2 E_0^2}{m^2 c^2 \omega^2} \quad \text{where} \quad E_0^2 = E_x^2 + E_y^2
$$

Thus (82) can now be written:

$$
\varepsilon = 1 - \frac{\omega_p^2}{\omega^2} \left( 1 + \frac{e^2 E_0^2}{m^2 c^2 \omega^2} \right)^{-1/2}
$$

(84)

Equation (84) is an interesting result from the point of view of wave propagation, since it leads to the conclusion that waves in the frequency range

$$
\omega_p \left( 1 + \frac{e^2 E_0^2}{m^2 c^2 \omega^2} \right)^{-1/2} < \omega < \omega_p
$$

(85)

can also propagate in the plasma. This is in contrast to what would be expected from linear theory where waves with $\omega < \omega_p$ cannot propagate. The power
requirement for the relativistic effect to be important is roughly determined by
\( \frac{eE}{(m \omega c)} \sim 1 \) (for a ruby laser, for example, with \( \omega = 2.6 \times 10^{15} \), this is roughly
\( 10^{18} \text{ W} \cdot \text{cm}^{-2} \) and easily attainable with modern technology). A simple physical
way of interpreting this propagation into overdense plasmas is by arguing that since
the electrons become more massive they lower the effective plasma frequency of
the medium and hence allow the wave to propagate as long as its frequency is
above this effective plasma frequency.

Similar propagation conditions have also been obtained for other interesting
limits, namely the pure longitudinal case \( (\rho_x = \rho_y = 0, \rho_z = \rho) \) and the more
general coupled longitudinal-transverse waves case \( (e.g. \rho_x, \rho_z \neq 0, \rho_y = 0) \).
Max and Perkins [12] have obtained an analytic solution for a linearly polarized
wave in the latter case for arbitrary amplitudes and \( \beta \gg 1 \). The propagation
condition they obtained is very similar to (85):

\[
\frac{1}{\beta^2} = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\pi}{2} \cdot \frac{\omega_p^2}{\omega} \cdot \frac{mc}{E} > 0
\]  (86)

Condition (86) was obtained for a weakly inhomogeneous plasma with \( n(x), E(x) \)
slowly varying with \( x \), and the authors pointed out the interesting possibility that
if \( E \) increases faster than \( n(x) \) there could be continuous propagation into an
overdense plasma. They showed that this was possible for \( kL \gg 1 \) (where \( L \) is
the density scale length and \( k \) the wavenumber) as the wave amplitude increase due
to WKB enhancement was faster than the first power of \( \omega_p^2 \). For \( kL \sim 1 \), the
WKB concepts break down and in general there would be wave reflection at some
point in the plasma. The physical reason why strong electromagnetic waves can
penetrate continuously (in the \( kL \gg 1 \) limit) is that the plasma current is limited
to \( (n_e c) \) instead of increasing with \( E \) as \( n_e v_{ind} = (n_e^2 E/m\omega) \), so that it does not
become large enough to cancel the displacement current and hence there is no
wave reflection. Relativistic effects thus diminish the ability of the plasma to act
as a dielectric.

The set of coupled equations (77)—(78) can also be derived from a
Hamiltonian:

\[
H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \beta \sqrt{\beta^2 - 1} + x^2 + z^2 + z
\]  (87)

so that we can once again relate the wave solutions to the orbits of a particle in
a potential well:

\[
V(x, z) = \beta \sqrt{\beta^2 - 1} + x^2 + z^2 + z
\]  (88)
FIG. 8. A typical surface of section plot \((x = 0, \ x > 0)\) for the system
\[ E = \frac{1}{2}(\dot{x}^2 + \dot{z}^2) + \beta \sqrt{\beta^2 - 1 + x^2 + z^2 + z}. \]

Since this is a two-dimensional well, the orbits are more complicated and not so easy to visualize. Their projections or crossings can be examined on a particular plane (e.g. \(x = 0, \ x > 0\)) and such surface-of-section plots can be useful in analysing the particle motion. Figure 8 is a typical surface-of-section plot (at \(x = 0, \ x > 0\)) for the system (87). It is essentially a phase-space plot \((\dot{z}, z)\) for fixed \(\beta\) and total energy \(E\) of the particle. Rewriting (87) as

\[ E = \frac{\dot{x}^2}{2} + \frac{\dot{z}^2}{2} + \beta \sqrt{\beta^2 - 1 + x^2 + z^2 + z} \]  \( (89) \)

we see that for given \(E\) and \(\beta\) the condition \(\dot{x}^2 \geq 0\) (otherwise \(\dot{x}\) would become imaginary) restricts \(\dot{z}\) and \(z\) to a certain region in the plane \(x = 0\). The bounding curve in Fig.8 is essentially defined by \(x = 0\). The two intersections on the \(z\)-axis can be easily found from (89) as

\[ z_{1,2} = -\frac{E}{\beta^2 - 1} \pm \sqrt{\frac{E^2}{(\beta^2 - 1)^2} - \frac{\beta^2(\beta^2 - 1) - E^2}{(\beta^2 - 1)^2}} \]  \( (90) \)

Point \(z_0\) is known as a fixed point (of the lowest order) and corresponds to the minimum of the potential well in the limit \(E \to \beta^2 - 1\). Thus the closed curves around \(z_0\) correspond to small-amplitude solutions, since \(\dot{z}, z\) are small in that region. Similarly, the outer curve is near the large-amplitude limit and the solutions of Max and Perkins could be correlated to one of the curves in that region. All the other orbits in Fig.8 are periodic solutions which have not appeared
in previous analytic studies. One can see higher-order fixed points around which islands have formed. These fixed points indicate periodic nonlinear waves with the periods of x and z a ratio of integers. There are also quasiperiodic solutions which almost (but never quite) repeat themselves because the periods of x and z have a ratio which is an irrational fraction. The 'islands' (with five-fold symmetry in this case) constitute another interesting class of orbits. The z-width of an island surrounding the fixed point corresponds to an amplitude excursion of a nonlinear periodic wave with the 5/2 symmetry. The islands thus correspond to amplitude-modulated waves, and the modulation envelope itself may be periodic or quasiperiodic. A separatrix orbit (between the islands and other curves) corresponds to a modulation period of infinity; it is thus a nonlinear solitary envelope wave (in this case with less than 100% amplitude modulation). The present technique has not been widely used for coupled stationary wave problems in plasmas but, as is clear, it can be very useful for finding a rich variety of nonlinear solutions.

REFERENCES

LOW-FREQUENCY MICROSCOPIC STABILITY ANALYSIS IN FUSION DEVICES

A.M. EL NADI
Electronics and Communications Department,
Faculty of Engineering,
Cairo University,
Giza, Egypt

Abstract

LOW-FREQUENCY MICROSCOPIC STABILITY ANALYSIS IN FUSION DEVICES.
A plasma confined in a fusion device is characterized by the presence of gradients in the equilibrium parameters, such as density, temperature, magnetic field, etc. Such gradients may then result in loss of equilibrium and escape of the plasma from the confinement system. Using a microscopic analysis, a review is presented of the low-frequency, gradient-driven instabilities in fusion devices, including both the shearless case (local approximation) and the case when magnetic shear is present. A detailed analysis of instabilities related to the tokamak configuration, such as trapped-particle modes, is also given.

1. INTRODUCTION

By definition, a confined plasma must possess an exterior boundary and is thus to be considered basically as an inhomogeneous medium. The free energy associated with the gradients can give rise to instabilities with a resulting plasma escape. To investigate the linear plasma behaviour in various confinement devices, one may use the linearized one-fluid, two-fluid or microscopic prescription. In the latter approach, the equilibrium condition for each species is prescribed by a local Maxwellian velocity distribution which allows for the spatial dependence of the density, temperature, magnetic field, electrostatic potential and average velocity. One then proceeds by solving the linearized Boltzmann equation for the perturbed distribution function in terms of the electric and magnetic field perturbations. The linear behaviour is then fully determined through Maxwell’s equations.

Obviously the well-known closure problem is not met with in this approach, and velocity-dependent effects such as particle trapping and wave-particle resonance are accounted for. Also, the commonly treated collisionless regime is particularly simple to handle since the collision term in the Boltzmann equation may then be ignored and the resulting Vlasov equation integrated directly using the method of characteristics.

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The purpose of this review is to present a linearized study of the basic instabilities associated with the spatial gradients which may be expected to be present in fusion devices. For simplicity, we shall consider only low-frequency ($\omega < \Omega_i$) long-wavelength ($k_x a_i < 1$) perturbations in a two-component plasma, where $\Omega_i$ is the ion cyclotron frequency, $a_i$ the ion mean Larmor radius, and $k_x$ the component of the wave vector perpendicular to the confining magnetic field. It will be assumed that the quasineutrality condition ($\lambda_D < a_i$), where $\lambda_D$ is the Debye radius, holds throughout and that no equilibrium electric fields are present. Electron and ion temperatures will be taken as equal except when specifically stated. Collisional effects will be treated through a number-conserving Krook operator. The basic set of equations to be solved is

$$
\frac{\partial f_{ij}}{\partial t} + \mathbf{V} \cdot \frac{\partial f_{ij}}{\partial \mathbf{V}} + \frac{1}{M_i} \left( e_i \mathbf{V} \times \mathbf{B}_0 + M_i \mathbf{g}_i \right) \cdot \frac{\partial f_{ij}}{\partial \mathbf{V}} = - \frac{e_i}{M_i} \left( \mathbf{E} + \mathbf{V} \times \mathbf{B} \right) \cdot \frac{\partial f_{ij}}{\partial \mathbf{V}} + \frac{\partial f_{ij}}{\partial t} \bigg|_{\text{collisions}}
$$

$$
\nabla \times \mathbf{B} = - \mu_0 \sum_i e_i \int f_{ij} \mathbf{V} \, d^3 \mathbf{V} \quad \Rightarrow \quad \sum_i e_i \int f_{ij} \, d^3 \mathbf{V} = 0
$$

and

$$
\nabla \times \mathbf{E} = - \mathbf{B} \quad \Rightarrow \quad \nabla \cdot \mathbf{B} = 0
$$

where $\mathbf{B}_0$ and $\mathbf{B}$ are the equilibrium and perturbation magnetic fields, $\mathbf{E}$ the perturbation electric field, $\mathbf{g}$ any equivalent gravitational force, $f_{ij}$ the perturbation distribution function and $f_{0ij} = f_{0ij}(C_{1ij}, C_{2ij}, C_{3ij})$ is the equilibrium distribution function appropriately formed from the constants of motion, $C_{ij}$, to describe the plasma equilibrium in the particular fusion device under study. The normal procedure for solving this set of equations is to Laplace-transform all linearized variables in time and Fourier-transform in the spatial coordinates along which no (or negligibly small) inhomogeneity is present. One may then use the method of characteristics to solve the Vlasov first-order partial differential equation and substitute for $f_{ij}$ into Maxwell’s equations. The inverse Laplace integral of any appropriate perturbed field component is then expressed as a sum of normal modes each $\sim \exp(-i\omega_n t)$ together with any necessary branch-cut contribution. Absolute stability is ensured if all $\omega_n$ have negative imaginary parts and no branch cut is present.
We may note here that a slab model can be used when the curvature effects are ignorable, and a suitably chosen gravitational force may then be introduced to simulate the magnetic curvature drift. Radial derivatives may also be ignored if $k_i \gg |\partial / \partial r|$, $K_i = K_0(r)$ (no magnetic shear) and if a radially localized solution can be found (e.g. near an inflection point in the density profile). In some instances, the mode variation along the field lines cannot be ignored and the "ballooning" effects must then be considered.

Standard notation is used, and only unfamiliar terms will be defined. In most cases, only the necessary assumptions, basic equations and results are presented, while the mathematical steps are left as an exercise for the interested reader. The basic literature for each section is cited in the Bibliography at the end of the review.

2. LOCAL APPROXIMATION

Assuming parallel equilibrium magnetic field lines (no magnetic shear) pointing in the $z$-direction, and choosing the $x$-axis to point in the direction of decreasing density, we may write down the expression for the equilibrium distribution function in the slab limit as

$$\rho_{e,0} = \frac{N}{(2\pi v_i^2)^{3/2}} \exp \left[ -\frac{(v^2 - 2 g_{i0} x)/2 v_i^2}{L_n} \right] \left\{ 1 - \left[ \frac{1}{L_n} + \frac{g_{i0}}{v_i^2} + \frac{\eta_i}{L_n} \left( \frac{v^2}{2 v_i^2} - \frac{3}{2} \right) \right] \right\}$$

where

$$\frac{1}{L_n} = - \frac{\partial \eta_0(x)/\partial x}{\eta_0(x)} = \left| \frac{\partial \eta_0(x)/\partial x}{\eta_0(x)} \right|, \quad T_i = M_i \cdot v_i^2$$

$$\eta_i = \frac{\partial T_i/\partial x}{T_i} \cdot \frac{\partial \eta_0(x)/\partial x}{\eta_0(x)}, \quad u_{i0} = \langle v_i \rangle \quad \text{and} \quad g_i = \frac{2 v_i^2}{R_m}$$
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simulates the magnetic curvature and is > 0 if the field lines curve towards the plasma. Using Ampère's law, it is easy to see that the plasma diamagnetic current introduces a gradient in the vacuum magnetic field such that

$$\vec{B}_o = \vec{B}_{vac} \left( 1 + \frac{\beta}{2} \frac{x}{L_T'} \right) \hat{e}_x$$

where

$$\beta = \frac{2 NT}{B_o^2/2 \mu_o}$$

and

$$\frac{1}{L_T'} = \frac{1}{L_n} + \frac{1}{L_T} + \frac{2}{R_m}$$

The particle drift velocity associated with this gradient is given by

$$v_{m,j} = \frac{v_j^2}{4 \Omega_j} \frac{\beta}{L_T'}$$

In the collisionless limit, assuming that all perturbed quantities \( \sim \exp(-i\omega t + iky + iKz) \) and ignoring the weak \( x \)-dependence of \( f_{0j} \) (local approximation), we integrate the linearized Vlasov equation to obtain

$$f_{\omega} = \frac{e_i p f_{M,j}}{RT} \left\{ \left( E_y + u_{o,j} B_x - \left[ \omega + \frac{k T_j}{\Omega_j} - K u_{o,j} + \omega_{*j} \left( 1 + \frac{\nu_j}{2 \nu_j} - \frac{3}{2} \right) \right] \right) \right\}

\times \left[ \left( J_0 E_y + J_0 v_z B_x - i J_1 v_z B_z \right) \left( J_0 - 2 i J_1 \frac{v_x}{v_{\perp}} \right) \right]

\omega_{*j} = \frac{k T_j}{c B_o L_n}, \quad j J_\nu = J_\nu (kv_\perp/\Omega_j)$$

where \( f_{M,j} \) is the Maxwellian part of \( f_{0j} \).
is the Bessel function of order $v$, and terms multiplied by $v_y$ were ignored since
the $y$-component of the current density will not be used owing to its dependence
on the weak $x$-behaviour, ignored in the local approximation.

The following homogeneous set of equations may now be solved for $\omega$:

$$n_i = n_e \quad i R B_x = -\mu_0 J_z$$

and

$$i ( k + \frac{K^2}{\kappa^2} ) B_2 = \mu_0 J_x$$

It may be easily verified that the magnetic perturbation $B$ is related to $\beta$ and can
hence be ignored in the zero-$\beta$ (or zero $\omega/K \nu_A$ where $\nu_A$ is the Alfvén speed
$= (B_0^2/\mu_0 NM)^{1/2}$) electrostatic limit. In the next section a variety of gradient-
driven instabilities are presented using the above procedure.

2.1. Density gradient drift instability

Assuming: $u_{oi} = \beta = \gamma = 0 , \left| K v_i - \frac{R g}{\Omega} \right| \ll \left| \omega \right| \ll \left| K v_y + \frac{R g}{\Omega} \right|$

and using the quasineutrality condition $n_i = n_e$, one obtains

$$2 = \frac{\omega + \omega_\star + \frac{R g}{\Omega}}{\omega + \frac{R g}{\Omega}} \left( 1 - \frac{K^2 v_i^2}{\omega^2} + \frac{K^2 v_i^2}{(\omega + R g/\Omega)^2} \right)$$

$$- i \sqrt{\frac{\pi}{2}} \left( \omega - \omega_\star - \frac{R g}{\Omega} - K u_e \right) / |K| V_e$$

where $\omega_\star = \omega_\star i$ and $a_i$ is the mean ion Larmor radius $= (v_i^2/\Omega_i^2)^{1/2}$. Solving
for $\omega$, we get

$$\omega = \omega_\star \left( 1 - \frac{K^2 a_i^2}{\omega_\star^2} + \frac{2 K v_i^2}{\omega_\star^2} \right) \frac{R g}{\Omega} + i \sqrt{\frac{\pi}{2}} \frac{\omega_\star}{|K| V_e} \left( \omega_\star K^2 a_i^2 + \frac{2 K g}{\Omega} + K u_e \right) \frac{2 K v_i^2}{\omega_\star}$$
Note the stabilizing effect of the ion sound term, and the destabilizing effects of $k^2 a_i^2$, $L_n/R > 0$ (bad curvature) and $u_e/(\omega/K) > 0$ (wave propagating along the electron parallel average velocity). If $L_n/R \leq -1/4$ (strong magnetic well), the wave is stable for all $k$. The phase and group velocities are in the direction of the electron diamagnetic current, and the wave feeds on the electron parallel energy through Landau inverse damping. The wave front is slightly inclined from $B_0(a_e/L_n < K/k < a_i/L_n)$.

In the limit $|\omega/\omega_*| \ll 1$ and $|\omega/KV_e| \gg 1$, $n_i = n_e \rightarrow$

$$0 = \frac{k^2 a_i^2}{2} + \frac{k^2 V_i^2}{\omega^2} \rightarrow \omega = \pm i \frac{K}{R} \sqrt{\frac{M_i}{M_e}}$$

and a purely growing instability is obtained.

2.2. Temperature gradient drift instability

Assuming $u = g = \beta = 0$, $KV_i \ll \omega \ll KV_e$, then $n_i = n_e \rightarrow$

$$2 = \frac{\omega + \omega_* - \frac{1}{\omega} \left( \frac{k^2 a_i^2}{2} - \frac{k^2 V_i^2}{\omega^2} \right) (\omega + \omega_*(1+\gamma_i)) - i \frac{\sqrt{\pi}}{2} \frac{\omega - \omega_* + \omega_* \gamma_i/2}{KV_e}$$

$$\omega = \omega_* \left[ 1 - \left( \frac{k^2 a_i^2}{2} \frac{k^2 V_i^2}{\omega^2} \right) (1+\gamma_i/2) \right] + i \frac{\sqrt{\pi}}{2} \frac{\omega_*}{KV_e} \left[ \left( \frac{k^2 a_i^2}{2} \frac{k^2 V_i^2}{\omega^2} \right) (1+\gamma_i/2) - \frac{\gamma_i}{2} \right]$$

Thus $\eta_i < 0$ is destabilizing.

In the limit $|\omega| \ll \omega_*$, $|\omega| \ll KV_i$, $n_i = n_e \rightarrow$

$$2 = \left[ \omega + \omega_*(1 - \gamma_i/2) \right] \frac{1}{KV_i} \left( \frac{\omega}{KV_i} - i \frac{\sqrt{\pi}}{2} \right) \frac{\eta_i \omega_* \omega}{2 k^2 V_i^2}$$

or

$$\omega = \frac{2 k^2 V_i^2}{\omega_* (1 - \gamma_i/2)} + \frac{i \sqrt{\pi}}{2} K V_i \frac{1 - \gamma_i/2}{1 - \gamma_i/2}$$

unstable for $\eta_i < 1$, $\eta_i > 2$. 
2.3. Parallel velocity gradient instability (Kelvin-Helmholtz)

Assuming
\[
\frac{T_i}{T_e} \ll 1, \quad \frac{1}{L_n} = \frac{1}{L_T} \frac{\nu = 1}{R_m} = \beta = 0 \quad \text{or} \quad n_i = n_e \rightarrow
\]
\[
\frac{1}{T_i} \left[ 1 - \int \frac{dV_i}{(2\pi V_i^2)^{3/2}} \frac{e^{-\frac{V_i^2}{2 \nu_i^2}}}{\omega - K \frac{V_i}{K \Omega_i}} \right] = -\frac{i}{T_e}
\]
or
\[
\omega^2 = -K^2 \frac{\omega}{\Omega_i} \left[ \frac{\nu_i}{K \Omega_i} - 1 \right]
\]
unstable if \( \nu_i / K \Omega_i > 1 \).

2.4. Drift dissipative instability

Assuming
\[
\nu_e / \omega \ll 1 \quad \nu_e = \gamma = g = u = \beta = 0 \quad |\omega| \gg |K \nu_i| \gg |K \nu_e|
\]
expressing the electron collision term as
\[
\frac{\partial f_e}{\partial t} \bigg|_{\text{coll}} = -\nu_e \left( f_{e1} - \frac{n e1}{(2\pi \nu_e^2)^{3/2}} \right) \left( \frac{\nu^2}{2 \nu_e^2} \right) \equiv \text{Number}
\]
conserving the Krook collision operator, and using the quasineutrality condition, we obtain
\[
1 - \left( 1 + \frac{\omega^*}{\omega} \right) \left( 1 - \frac{K^2 a_i^2 \omega^*}{2} \right) = -\frac{\omega^*}{\omega} + \frac{K^2 v_e^2}{\omega^2} \left( 1 - \frac{\omega^*}{\omega} \right) \left( 1 - \frac{i \nu_e}{\omega} \right)
\]
which is a cubic in \( \omega \).

If \( \frac{K^2 a_i^2 \omega^*}{K^2 v_e^2} \ll 1 \) one obtains
\[
\omega = \omega^* + i \frac{\nu_e}{\omega} \frac{K^2 a_i^2 \omega^*}{K^2 v_e^2}
\]
2.5. Interchange instability (Rayleigh-Taylor)

Assuming $K = u = \beta = \nu = 0$, and using $n_i = n_e$, we obtain

$$2 = (\omega + \omega_*) + \frac{kg}{\omega n} \left( \frac{1}{\omega} - \frac{kg}{\omega^2 n} \right) - \frac{k^2 a_i^2}{2} \left[ \omega + \omega_*(1 + \gamma_i) + \frac{kg}{\omega n} \right] \left( \frac{1}{\omega} - \frac{kg}{\omega^2 n} \right)$$

$$+ \left( \omega - \omega_* - \frac{kg}{\omega n} \right) \left( \frac{1}{\omega} + \frac{kg}{\omega n} \right)$$

or

$$\frac{\omega}{\omega_*} = -\frac{1}{2} \left[ (1 + \gamma_i) \pm \left\{ (1 + \gamma_i)^2 - \frac{32 \ln R}{R k^2 a_i^2} \right\}^{1/2} \right]$$

stable if $k^2 a_i^2 > (32 \ln R)/(1 + \eta_i)$ \equiv finite ion Larmor radius stabilization. Allowing for a velocity spread in $g$ leaves a small residual resonant growth rate when $k^2 a_i^2 > (32 \ln R)/(1 + \eta_i)$. Note that the wave travels along the ion diamagnetic drift direction.

2.6. Density gradient drift instability in a finite-$\beta$ plasma

Assuming

$$u = g = \nu = \gamma = 0 , \quad \beta \ll 1 \quad (B_{i\theta} \approx 0) , \quad KV_i \ll \omega \ll KV_e$$

using $n_i = n_e$ and $ikB_x = -\mu_0 J_z$, we obtain

$$\left[ \frac{\omega - \omega_*}{\omega + \omega_*} \left( 1 + i \frac{\pi}{2} \frac{\omega}{KV_e} \right) - \frac{k^2 V_i^2}{\omega^2} \right] \left[ 1 - \frac{\omega (\omega + \omega_*)}{k^2 V_A^2} \right] + \frac{k^2 a_i^2}{2} = 0$$

Thus the drift wave ($\omega = \omega_*$) is coupled through the finite ion Larmor radius to the shear Alfvén wave:

$$\frac{\omega}{\omega_*} = \frac{1}{2} \left\{ 1 \pm \left( 1 + 4 \frac{k^2 V_A^2}{\omega_*^2} \right)^{1/2} \right\}$$
The interaction occurs only for $\beta > m/M$, and stabilizes the low-K part of the drift branch. The fast Alfvén wave remains stable while coupling with the drift branch stabilizes the large-K part of the slow Alfvén branch (Fig. 1).

![Dispersion relation of the drift wave in a finite-\(\beta\) plasma.](image)

2.7. Infinite-\(\beta\) zero-pressure gradient drift instability

Assuming

$$u = g = v = 0, \quad \eta = -1, \quad \beta = \infty, \quad KV_i \ll \omega \ll KV_e$$

and noting that the presence of the six components of $\vec{E}$ and $\vec{B}$ must now be considered, we have:

$$n_i = n_e \rightarrow E_2 + \frac{i K V_i^2}{\omega_i (1 - \omega^*)} B_2 = 0$$

$$J_x = 0 \rightarrow E_2 - i 2 K V_i^2 (1 - \omega^*) B_2 = 0$$

Thus

$$\omega = \frac{2 \omega^*}{3} \left( 1 \pm \frac{i}{\sqrt{2}} \right)$$

and the drift wave is unstable even though resonance effects were ignored.
2.8. Interchange instability in a finite-$\beta$ plasma

Assuming $u = K = \nu = 0$, $\beta < 1$, it follows that

$$i \frac{dB_\parallel}{dt} = \mathcal{N}_0 J_x \Rightarrow B_\parallel = -\frac{\beta (\nu p/p)}{2i\omega} E_y$$

$$-iJ_z v_\parallel B_\parallel / E_y = (-k v_m / \omega).$$

Considering the expression for $f_{ij}$, we notice that the self-dug well is unimportant to the interchange instability and is carried along by the drifting plasma.

3. THE ONE-DIMENSIONAL PROBLEM

3.1. Effect of magnetic shear

Parallel plasma current produces a shear component in the magnetic field

$$\left( \frac{\partial B_{y_0}}{\partial x} = \frac{B_{z_0}}{L_s} = \mu_0 n_e (u_1 - u_e) \right) \Rightarrow \vec{B} = B_0 \left( \vec{e}_z + \frac{x}{L_s} \vec{e}_y \right)$$

This results in $\Delta v_y = v_z x / L_s$ and $v_\parallel = v_z + v_y x / L_s$ being nearly a constant of motion. The local expression for $f_{ij}$ obtained in the previous section is thus modified as:

$$K \rightarrow K_\parallel = K + k x / L_s (= k x / L_s \text{ if } x \text{ is measured from } K_\parallel = 0)$$

and

$$\int_{-\infty}^{t} \phi(t', x') \, dt' = \left[ \exp \left( -i \omega t + i ky \right) \right]$$

$$\times \int_{-\infty}^{\infty} dk_x e^{ik_x x} \int_{0}^{\infty} \frac{(k_x v_\perp)}{\Omega} \int_{-\infty}^{\infty} \phi(x'') e^{-ik_x x''} \, dx''$$

$$= \left\{ \phi(x) + \frac{v_\perp^2}{2 \Omega^2} \left( \frac{\partial^2 \phi}{\partial x^2} - k^2 \right) + \ldots \right\}$$
Truncation of the expansion beyond the second term is justified only if $\phi(x)$ is slowly varying (otherwise one has to deal with an integral equation). In the zero-$\beta$ limit, Laplace-transforming Poisson's equation in time, we obtain:

$$\phi''(\omega, x) + \nabla(\omega, x) \phi(\omega, x) = g(\omega, x)$$

together with the causality condition:

$$\lim_{\omega \to +\infty} \phi(\omega, x) = \text{finite or } \lim_{\omega \to 0} \phi(\omega, x) \to 0 \text{ or outgoing wave energy}$$

If $\phi_1$ and $\phi_2$ are acceptable solutions for $x \to \pm \infty$, the general solution may be written as

$$\phi = \frac{\phi_1 \int_{-\infty}^{x} \phi_2 g \, dx - \phi_2 \int_{-\infty}^{x} \phi_1 g \, dx}{\phi_2 \phi_1' - \phi_1 \phi_2'}$$

Analytical continuation to the lower $\omega$ half-plane then produces a complete set of normal modes if $\phi$ is analytic except for poles at $\omega_n$ with $\text{Im } \omega_n > 0$ (from the zeros of the Wronskian). For $\text{Im } \omega_n < 0$, this holds only along certain paths in the complex $x$-plane, the real $x$ solution being the analytic continuation of the complete orthogonal solution.

Thus, the plasma normal modes are obtained by solving the homogeneous equation (or a system of equations in the finite-$\beta$ case) subject to the causality restriction. It should be noted that while the normal-mode approach may rule out the presence of an absolute instability, convective amplification of waves emitted by a noise source may still be present:

$$A \sim \exp \left[ \int \frac{\gamma(x)}{\partial \omega / \partial k_x} \, dx \right]$$

3.1.1. Density gradient drift instability

Assuming $u = g = \beta = \eta = \nu = 0, (1/L_s(x)) = (1/L_n)(1- x^2 / b^2)$, ignoring $|x| < x_e \sim \omega L_s / k v_e$, and using $n_i = n_e$, we obtain

$$E_y'' = \frac{2}{a_i^2} \left[ \frac{\omega - \omega_*}{\omega + \omega_*} + \frac{k^2 a_i^2}{2} + x^2 \left( \frac{\omega_*}{\omega + \omega_*} - \frac{1}{b^2} - \frac{k^2 v_i^2}{L_s^2 \omega_*^2} \right) + i \sqrt{\frac{\pi}{2}} \frac{\omega (\omega - \omega_*)}{|K_i| v_e (\omega + \omega_*)} \right] E_y$$
The modes are not localized for $L_n/L_s > a_i/2b$. To estimate the shear damping, let $b \to \infty$ and treat the electron residue term as a perturbation. The solution may then be expressed in terms of Weber functions, and the lowest-order mode, least stabilized by shear, is

$$
\phi = \exp \left[ -ix^2 \left( 1 - i\gamma/\omega_i \right)/a^2(L_s/L_n) \right]
$$

$$
\omega = \omega_i + i\gamma = \omega_* \left( 1 - k^2 a_i^2 \right) - 2i\omega_* \frac{L_n}{L_s} + i\omega_* \delta
$$

$$
\delta = \text{Im} \left( \frac{n_1}{2} \right)^{1/2} k^2 a_i^2 \frac{L_s}{kv_e} \omega_*
$$

$$
\times \int_{x_e}^{\infty} \frac{1}{t} \exp \left[ -2t^2/a_i^2(L_s/L_n) \right] dt \int_{x_e}^{\infty} \exp \left[ -2t^2/a_i^2(L_s/L_n) \right] dt
$$

$$
= k^2 a_i^2 \left( \frac{mL_s}{ML_n} \right)^{1/2} \ln \left( \frac{ML_n}{mL_s} \right)^{1/2}
$$

where the electron contribution is obtained through standard perturbation theory. As $k^2 a_i^2$ increases, the damping rate decreases and the complex turning points of the exact equation approach each other until it is no longer possible to ignore the details of the electron response for $|x| \lessgtr x_e$ and one must then solve the more exact equation:

$$
E^\gamma = \frac{2}{\rho_s^2} \left[ 1 - \frac{\omega_{se}}{\omega} + \frac{R_{se}^2}{2} - \frac{R_{cs}^2 x^2}{L_s^2 \omega^2} + \frac{\omega - \omega_{se}}{\sqrt{2} |K_n| Ve} \right] E^\gamma \quad (1)
$$

valid for $T_i = 0$ ($a_i \ll |x_e|$), where

$$
\omega_{se} = \frac{k_{ei} T_e}{eBLn}, \quad \rho_{se}^2 = q_i \frac{T_e}{T_i}, \quad \frac{R_{cs}^2}{L_s^2} = \frac{V_i^2}{T_i},
$$

and

$$
\mathcal{Z} (\xi) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - \xi} \, dt
$$

is the plasma dispersion function (Fig. 2(a)).
Numerical solution of Eq.(1) has shown that no absolute density gradient drift instability can occur however small \( L_\eta / L_s \) may be, and that the damping rate approaches zero asymptotically for large \( k^2 a_i^2 \) (Fig.2(b)).

\[ \psi = \frac{x_0}{x} Z \left( \frac{x_0}{x} \right) \]

versus \( x \), where \( Z \) is the plasma dispersion function.

(b) Plot of the imaginary part of the drift wave frequency versus \( k^2 a_i^2 \) for a zero-\( \beta \) plasma confined by a sheared magnetic field.

To study convective amplification, we may still use Eq.(1) with \( \omega \), the frequency of the noise source, being real. In the resonant case, \( \omega = \omega_0 (1 - k^2 \rho_e^2 / 2) \), and assuming very weak shear \( (x < x_T) \) we get

\[ E_y'' = \left[ -i \sqrt{\frac{\pi}{2}} \frac{x_\infty}{x} k^2 \right] E_y \]

The solution may be directly obtained in terms of Bessel functions and shows the large amplification caused by electron resonance interaction in the weak shear case.

3.1.2. Temperature gradient drift instability

Assuming \( u_0 = g = \beta = \nu = 0 \) and treating the electron response as a perturbation to the shear damping, one obtains directly

\[ E_y'' = \frac{2}{a_i^2} \left[ \frac{k^2 a_i^2 \rho_e^2}{2} + \frac{\omega - \omega_0}{\omega + \omega_0 (1 + \zeta_i)} - \frac{k^2 v_i^2 x}{\omega^2 L_s^2} \right] + i \sqrt{\frac{\pi}{2}} \frac{\omega (\omega - \omega_0 + \omega_0 \eta_e / 2)}{k v_i v_e \left[ \omega + \omega_0 (1 + \zeta_i) \right]} \]

\[ \omega = \omega_0 \left[ 1 - k^2 a_i^2 (1 + \zeta_i / 2) \right] - 2 i \frac{\eta_e}{L_s} \left( 1 + \zeta_i \right) + i \frac{m_e \omega_0}{m_i v_e} \left( \frac{1}{2} \right) \frac{\beta}{\eta_e} \]
which shows the favourable effect of $\eta_i$, $\eta_e > 0$. Again, numerical solution of the exact equation has shown stability to be maintained for a wide range of $k^2a_i^2$, $\eta_e$ and $\eta_i$ even for $L_n/L_S \ll 1$.

### 3.1.3. Parallel velocity drift instability

Assuming $\eta = \beta = g = \nu = k^2a_i^2 = 0$, $u_i = 0$ and using $n_i = n_e$, we obtain

$$E'' \gamma = \frac{2}{\rho_i^2} \left[ \frac{\omega - \omega_*}{\omega + \omega_*} - \frac{k_i^2 \nu_i \dot{x}^2}{\omega^2 L_s^2} - i \sqrt{\frac{\pi}{2}} \frac{ue}{V_e} \frac{x}{|x|} \right] E' \gamma$$

To obtain the amount of shear required for stability, let $\omega = \omega_*$, $\xi = \sqrt{\xi} x$ to obtain a well for $\xi > 0$ with a barrier at $\xi = 0$. The WKB approximation is then valid on both sides of $\xi = 0$ where the logarithmic derivatives can be matched to obtain the eigenvalue condition. Again the lowest-order mode is least stabilized by shear, and the stability condition is

$$\frac{L_n}{L_s} > \frac{1}{4} \sqrt{\frac{\pi}{2}} \frac{u_e}{V_e}$$

This result is obviously insensitive to the fine structure for $|x/x_e| \leq 1$ as long as $k^2a_i^2$ is negligible, otherwise numerical integration must be used. The effect of magnetic shear on the parallel velocity gradient was also investigated and it was found that a large amount of shear is required for stabilization.

### 3.1.4. Drift dissipative instability

Assuming $\nu_i = \eta = \beta = g = u = 0$, $v_e \gg (\omega, k\nu_e/L_s)$, $T_i = 0$, we get

$$\eta_i = \frac{eE_N}{iR T_e} \left( - \frac{\omega \nu_e (x)}{\omega} + \frac{k_i^2 \rho_i^2}{2} \right) \frac{K_n^2 C_s^2}{\frac{\omega^2}{\omega}} = \frac{eE_N}{iR T_e} \left( - \frac{\omega \nu_e (x)}{\omega} + \frac{1 - i \nu \omega}{k_n^2 \nu_e^2} \right)$$

or

$$E'' \gamma = \frac{2}{\rho_i^2} \left[ \frac{k_i^2 \rho_i^2}{2} + \frac{1 - \frac{\omega \nu_e (x)}{\omega}}{1 - i \nu \omega} - \frac{K_n^2 C_s^2}{\frac{\omega^2}{\omega}} \right] E \gamma$$
The frequency is determined by matching the inner solution

\[ \left| \frac{K_{l}v_{e}^{2}}{\omega \nu} \right| \sim 1 \gg \frac{K_{l}c_{s}^{2}}{\omega^{2}} \]

to the outer solution

\[ \left| \frac{K_{l}v_{e}^{2}}{\omega \nu} \right| \gg 1 \]

where the former may be expressed in terms of associated Legendre functions and the latter as a Kummer's confluent hypergeometric function. If \( \omega_{*}(x) \) localizes the solution, \( \gamma \propto \nu_{e}^{1/2} \); otherwise resistivity enhances the shear damping by a term \( \propto \nu_{e}^{1/2} \).

### 3.1.5. Density gradient drift instability in a finite-\( \beta \) plasma

Assuming \( \nu = \eta = g = u = 0, \ K_{l}V_{i} < \omega < K_{l}V_{e}, \ \beta \ll 1 \), and using \( n_{i} = n_{e}, \)

\( \frac{1}{ik}B_{x}'' - ikB_{x} = \mu_{0}J_{x}, \) one obtains

\[ E_{y}'' - k^{2}E_{y} = \frac{2}{\alpha_{s}^{2}} \left[ \frac{\omega - \omega_{*}}{\omega + \omega_{*}} - \frac{K_{l}^{2}c_{s}^{2}}{\omega^{2}L_{s}^{2}} + i \frac{\pi}{2} \frac{\omega(\omega - \omega_{*})}{|K_{l}V_{e}(\omega + \omega_{*})|} \right] \left[ E_{y} + \frac{\omega}{K_{l}}B_{x} \right] \]

\[ B_{x}'' - k^{2}B_{x} = - \frac{\omega + \omega_{*}}{K_{l}^{2}A_{s}^{2}} \left( E_{y}'' - k^{2}E_{y} \right) \]

Treating terms multiplied by \( k^{2} \) together with the electron residue term as a perturbation, one obtains

\[ \left[ \frac{E_{y}''}{(\omega - \omega_{*})} - \frac{K_{l}^{2}c_{s}^{2}}{\omega^{2}} \right] = \frac{2}{\alpha_{s}^{2}} \left[ 1 - \frac{\omega(\omega + \omega_{*})}{K_{l}^{2}A_{s}^{2}} \right] E_{y}' \]

where the MHD branch

\[ \left[ E_{II} = \frac{K_{l}}{k} \left( E_{y} + \frac{\omega}{K_{l}}B_{x} \right) \right] \]

is ignored by rejecting the solutions

\[ \lim_{x \to \pm \infty} E_{y} = C_{1} \text{ or } C_{2}/x \]
The solution to Eq.(2) may be obtained exactly by writing:

\[
E_y = e^{-i\frac{\omega t}{\alpha' L_s/L_n}} \sum_{n=0}^{\infty} a_n x^{z_n e^{\delta}} \quad E_y''(x) = e^{-i\frac{\omega t}{\alpha' L_s/L_n}} \sum_{n=0}^{\infty} b_n x^{z_n e^{\delta}}
\]

solving for \(\delta\), relating \(a_n\) to \(b_n\) and then determining the condition for the termination of the series. Standard perturbation theory is then used to determine the effect of \(k^2\) and the electron residue term. Finite-\(\beta\) basically reduces the electron destabilization \((\sim \beta (M/m))\) and also changes the rate of energy outflow by modifying the shape of the envelope through \(x^5 (\sim (\partial/\partial x)(V_{B1}/I))\) which may be easily seen to result in opposite effects for even and odd modes. Thus, as expected, it has been found that while increasing \(\beta\) in the presence of magnetic shear has a stabilizing effect on the lowest even mode, it reduces the damping rate of the lowest odd mode. It was also found numerically that no absolute instability can occur for any \(L_n/L_s\) for both the drift and the MHD (shear Alfvén) branch.

### 3.1.6. Interchange instability in a finite-\(\beta\) plasma

We assume \(v = \eta = u = 0\), \(\beta \ll 1\), \(K_1 V_i \ll \omega \ll K_1 V_e\). While magnetic curvature slightly modifies the finite-\(\beta\) drift waves, it has a strong influence on the MHD branch \((E_\parallel \approx 0)\). From \((\nabla \times B_1) \cdot e_z = \mu_0 J_z\) and \(E_\parallel = 0\) we obtain

\[
B_x'' - k^2 B_x = -\frac{\omega_0 Ne^2}{T K_{u1}} \left[ (\omega + \omega_*) \frac{a_x^2}{2} (E_y'' - k^2 E_y) - \frac{2 k_{g_0} \omega x}{\omega} E_y \right]
\]

\[
E_y + \frac{\omega}{K_{u1}} B_x = 0
\]

Defining

\[
\omega (\omega + \omega_*) = \frac{k^2 V_{A1} x_A^2}{L_s^2}
\]

we combine the two equations:

\[
\left[ E_y' (x^2 - x_A^2) \right]' = E_y \left[ k^2 (x^2 - x_A^2) - \frac{\beta L_s^2}{R L_n} \right]
\]

to be solved subject to the boundary condition of even or odd \(E_y\) and

\[
\lim_{x \to -\infty} E_y = 0
\]
By matching the inner solution \( x^2 \ll \beta L_s^2 / k^2 R L_n \) — Legendre equation — to the outer solution \( x^2 \gg \beta A^2 / k^2 R L_n \) — Bessel equation — over the common range \( x^2 \ll x^2 \ll \beta L_s^2 / k^2 R L_n \), one obtains the eigenvalue condition for \( \omega \). It turns out that magnetic shear and the ion Larmor radius effect are stabilizing, while \( \beta \) is destabilizing. In the zero Larmor radius limit, the stability condition takes the simple form \( \beta L_s^2 / R L_n < \frac{1}{4} \) for stability (the Suydam criterion).

3.1.7. The tearing mode

Assuming \( k^2 / k_0^2 = \eta = u = g = \nu_l = 0, \beta \ll 1 \), ignoring the ion sound term and using a Krook number-conserving electron collision operator, one obtains

\[
\rho_{oe} = -e \frac{f_{om}}{i \pi T} \left[ E_y - \frac{(\omega + i \nu_e - \omega^*) E_x + (\omega - \omega^*) V_z B_x}{\omega + i \nu_e - \gamma} \right]
\]

\( n_i = n_e, (\nabla \times \vec{B}) \cdot \vec{e}_2 = \mu_0 J_z \) lead to

\[
B_x^\perp = \frac{(\omega - \omega_e)}{\gamma} \frac{2 (\omega - \omega_e)}{\gamma^2} \frac{1 + \mu \gamma}{1 + \frac{\nu_e}{\omega + i \nu}} Z(z) \gamma^2 E_y + \frac{\omega}{\gamma} B_x
\]

where \( z = \omega + i \mu / (|K_0| \sqrt{2} v_e) \). Note that that the parallel conductivity is constant in the inner region and drops as \( 1/x^2 \) in the outer region \(|z| \ll 1\) with an effective current channel width, \( w \), given by \( k^2 w^2 v_e^2 / L_s^2 = |\omega (\omega + i \nu)| \). For \(|x/w| > 1\), one may assume \( E_y \approx 0 \) and solve the equation:

\[
B_x^\prime = \frac{\omega (\omega + i \nu)}{k_i \gamma \gamma^2} \left( \frac{B_x}{k_i} \right)^\prime
\]

allowing for the proper boundary conditions and the effect of \( J_{01}(x) \) on \( K_0(x) \) to obtain

\[
\frac{B_x}{B_x} \bigg|_{x = w} = \Delta^\prime
\]

Anticipating a dominant role for the electromagnetic perturbations, we have for \(|x/w| < 1\)

\[
B_x^\prime = \frac{2 (\omega - \omega_e)}{\gamma} \frac{1}{\gamma^2} B_x, \quad E_y^\prime = \frac{2 (\omega - \omega_e)}{\gamma} \frac{1}{\gamma^2} \frac{B_x^2}{\gamma^2} \left( x^2 \frac{E_y + \omega}{\gamma} B_x \right)
\]
Assuming that $B_x$ is even and slowly varying, we obtain

$$
\Delta' = \frac{4(\omega - \omega_*) V_e}{\alpha_i^2 V_A^2 (\omega + i \nu)} x_T \quad \text{where} \quad x_T^4 = \frac{\alpha_i^2 \omega (\omega + i \nu) (\omega + \omega_*)}{2 (\omega - \omega_*) k_n^2 V_e^2}
$$
or

$$
\omega (\omega + \omega_*) (\omega - \omega_*)^3 (\omega + i \nu)^3 = -k_n^2 V_A^2 \left( \frac{e}{\omega_p} \right)^6 \left( \frac{\Delta'}{2} \right)^4
$$

In the limit $|\omega_*/\omega|, |\omega/\nu| \ll 1$, the stability condition to the tearing mode may be written simply as $\Delta' < 0$. Note that the driving energy of the tearing instability is the transformation of the stored magnetic energy into particle kinetic energy through the effect of resistivity, and the magnetic field changes its topology to form magnetic islands whose number depends on $k$ (Fig. 3).

If collisions are absent, the tearing mode is driven by the electron resonant interaction and it follows directly that

$$
\Delta' = \frac{2 \omega (\omega - \omega_*)}{k_n^2 \alpha_i^2 V_A^2} \int_{-\infty}^{\infty} \frac{1 + z^2 Z(z)}{x^2} d\chi
$$

By exchanging the order of integration one obtains

$$
\omega = \omega_* + i \frac{k_n^2 \alpha_i^2 V_A^2}{2 \sqrt{\pi} V_e} \Delta'
$$

stable if $\Delta' < 0$.

3.2. Ballooning instabilities

If $\vec{B} = B(x, z) \vec{e}_z$ (Fig. 4), one should consider that:
(a) \[ \phi = \phi(x,z) \exp[-i(\omega t - ky)] \]
\[ = \sum_n \psi_n(x,z) \exp\left[-i\left(\omega t - ky - \frac{2\pi nz}{L}\right)\right] \]
where \( \psi_n(x,z) \) is slowly varying in the z-direction;

(b) The curvature drift = \(-g(z)/\Omega\);

(c) Trapped particles \([n_i < \mu(B_{\text{max}} - B)]\) are present and it may be expected that the expansion-free energy may be released by modes localized longitudinally in regions of bad curvature, even if the average curvature is favourable to stability.

Let \( \nu = \beta = \eta = u = 1/L_s = 0, \) \( \vec{B} = B_0[1 + (x/R)(1 - \delta \cos Kz)]\vec{e}_z. \) \( \phi \) may be expanded as \( \phi = \exp[-i(\omega t - ky)] \sum \phi_\ell \exp(i\ell Kz), \) \( \phi_\ell = \text{const.} \) Ignoring trapped particles, \( n_i = n_e \) leads to

\[ 2 \phi_n = \sum_{v_j} \sum_{r,m} \phi_r \left[ (\omega + \omega_{v_j})(1 - \frac{\delta^2}{2}) \int \frac{dV}{2\pi V^2} \frac{J_m \left( \frac{R V_i}{K V_j} \right) J_{n+m-r} \left( \frac{R V_i}{K V_j} \right)}{\omega + R V_{o,i} - (m+n) K V_j} \right] \]
where

\[ V_{o,i} = \frac{2T}{\epsilon_{j} R B}, \quad V_{i,i} = \delta V_{o,i} \]
and the dispersion relation is a Hill’s determinant. Keeping only \( \ell = 0, \pm 1, \) one finds an instability whose condition coincides with the condition that the mode balloons in regions of bad curvature (max. B). Although the above case was collisionless, one can also find ballooning instabilities in the presence of collisions or finite-\( \beta. \)

4. LOW-FREQUENCY PHENOMENA IN TOKAMAKS

In a tokamak machine (Fig.5), the toroidal magnetic field = \( B_0 R_0 / R. \) The poloidal magnetic field is related to the toroidal current and the externally induced electric field by:

\[ \text{FIG.5. Coordinate system in a tokamak geometry.} \]
where $\rho$ is the parallel resistivity. $B_\theta(t)$ will be considered as constant. The total magnetic field may be written as

$$ \vec{B} = \frac{B_0}{1 + \epsilon \cos \theta} \left( \vec{\xi} + \frac{\epsilon}{\varphi} \vec{\theta} \right) $$

where

- $\epsilon = \text{toroidalicity factor (inverse aspect ratio)} = r/R \ll 1$
- $q = \text{tokamak safety factor} = r B_t / R B_\theta = 2\pi / \iota$
- $\iota = \text{rotational transform} = 2\pi R B_\theta / r B_t$

The constants of motion are:

$$ E = \frac{m}{2}(v_\parallel^2 + v_\perp^2) + e \phi = C_1 $$

$$ \mu = v_\perp^2 / B = C_2 $$

$$ P_\xi = R(m v_\perp + e A_\perp) \rightarrow v_\xi(1 + \epsilon \cos \theta) - r \Omega_p = C_3 $$

Trapping occurs if

$$ \left| \frac{v_\parallel}{v_\perp} \right| < \sqrt{2\epsilon} \cos \frac{\theta}{2} $$

and the trapped particle fraction $= \sqrt{2\epsilon} (\cos \theta / 2) = \sqrt{\epsilon}$ at $\theta = \pi/2$. From $P_\xi = \text{const.}$, it follows directly that the trapped particles drift inwards with $v_\xi = - E_\xi^{\text{ext}} / B_\theta$.

The longitudinal particle motion is described by

$$ \ddot{\theta} + \frac{\epsilon v_\parallel^2}{2 q^2 R^2} \sin \theta = 0 $$

which may be solved in terms of elliptic integrals. For deeply trapped particles

$$ \theta = \frac{V_{\parallel}(a)}{\sqrt{\epsilon/2} v_\perp} \sin (\omega_b t) $$

where

$$ \omega_b = \sqrt{\epsilon/2} \frac{v_\perp}{q R} $$

is the bounce frequency. All trapped particles will also perform this sinusoidal motion if $\vec{B} = B_0 [1 - \epsilon(1 - \theta^2 / 2)] (\vec{\xi} \epsilon + (e/\varphi) \vec{\theta})$ but the trapped particle fraction then increases to $\sqrt{\epsilon/2} (\pi^2 - \theta^2)^{1/2}$. We may also note that all particles are continuously drifting in the vertical direction with...
even though their orbits in the \( r, \theta \) plane are closed because of the rotational transform. The components of the drift motion are

\[
\vec{v}_{Mj} = - \frac{2V_\theta^2 + V_{\psi}^2}{2R \Omega_{ij}} \vec{e}_z = - V_{gij} \vec{e}_z
\]

and

\[
\vec{v}_M^\theta = V_{gij} \frac{\Omega_t}{\Omega_\theta} \left( \cos \theta + \frac{r q'}{q} \sin \theta \right)
\]

By manipulating the expressions for the constants of motion one gets:

\[
r + \frac{v_\theta}{\Omega_t} - \frac{e \cos \theta (2v_\parallel^2 + v_\psi^2)}{2 \Omega_\theta} \left( v_\parallel + \frac{E_r}{\Omega_\theta} \right) = \text{const. for circulating particles}
\]

\[
r + \frac{v_\theta}{\Omega_t} - \frac{v_\parallel}{\Omega_\theta} = \text{const. for trapped particles}
\]

and it follows that:

\[
J_\theta (\text{diamagnetic}) = \frac{2T}{B_t} \frac{\partial N}{\partial r}
\]

\[
J_\xi (\text{rotational transform}) = -4e \cos \theta \frac{T}{B_\theta} \frac{\partial N}{\partial r}
\]

These two constants of motion do not describe the plasma equilibrium properly because of the discontinuity in \( v_\parallel \) at the interface between trapped and circulating particles. Expecting that this discontinuity will be smoothed down by the friction between particles (giving rise to the bootstrap current), we describe the plasma equilibrium by

\[
\mathbf{f}_{0j} = N \mathbf{f}_{Mj} \left[ 1 - \frac{1}{\mathbf{I}_n} \left( 1 + \eta \left( \frac{v_\parallel^2}{2v_j^2} - \frac{3}{2} \right) \right) \right]
\]

where we have ignored the radial electric field, the Ware pinch effect and the interface distortion. It is interesting to note that this expression for \( \mathbf{f}_{0j} \) will produce the same dispersion relation as that resulting from allowing explicitly for the poloidal diamagnetic current, for modes localized in the neighbourhood of rational flux surfaces (the surfaces on which the magnetic field lines close after a few traverses around the machine).
Since both the normal and the geodesic components of the magnetic drift velocity, together with the trapped particles spatial distribution, are functions of the poloidal angle $\theta$, one has to deal with a two-dimensional problem. Using a Krook number-conserving trapped-particle collision operator to describe the trapping and detrapping collisions, noting that $n^c \sim (-\epsilon_j \phi/T)N$ in the small phase velocity limit, ignoring the effect of magnetic shear on the toroidal drift velocity, the modulation in the latter and the Larmor gyration of trapped particles, employing the sinusoidal representation for the trapped particle motion, expressing the perturbations as

$$\phi_0(r, \theta, \xi) = \exp (-i \omega t + i m \theta - i \xi) \sum_{p=-\infty}^{\infty} E_{m+p} \exp (ip \theta)$$

and considering only electrostatic perturbations, we may write

$$f_{i,j,m} = -\frac{e_j f_{mj}}{T} \left[ \phi_m - (\omega + \omega*_{i,j}) g(\nu, x) \right]$$

where

$$g_{x,j} = \sum_s \frac{J_0^2 \left( \frac{x}{\delta} \hat{\theta} \right)}{\omega + i \nu + \omega_{M,j} - s \omega_b} \left( \phi_m^* + \frac{v^2}{2 \omega^2} \phi_m'' \right) + \sum_{p \neq 0} \frac{J_0 \left( \frac{x}{\delta} \hat{\theta} \right) J_0 \left[ \left( \frac{x}{\delta} - p \right) \hat{\theta} \right]}{\omega + i \nu + \omega_{M,j}} \left( \phi_{m+p}^* + \frac{v^2}{2 \omega^2} \phi_{m+p}'' \right)$$

for $\omega \ll (\omega_b)$, where $m = q_0(r_0)k$ at the rational surface from which $x$ is measured, $\delta = 1/q'$ is the spacing between the rational flux surfaces, $\hat{\theta}$ is the maximum angular excursion in the $\theta$-direction,

$$\omega_{M,j} = \frac{\ell \nu^2}{2 R^2 \omega^2_{0,j}} = \omega_{M0j} \left( \frac{\nu^2}{2 \nu^2} \right)$$

and

$$\omega*_{i,j} = \frac{\ell \nu^2}{R \omega_{0,j} L_n} = \frac{R}{L_n} \omega_{M0j}$$

For a cold species we get

$$g_{ij} = J_0^2 \left( \frac{m \nu}{\ell \omega} \right) \sum_p \left\{ \phi_{m+p} + \frac{\alpha^2}{2} \phi_{m+p}'' + \frac{(x/\delta - p)^2}{q^2 \omega^2} \phi_{m+p} \right. \left. - \frac{m \nu}{2 \ell \omega} \left[ \phi_{m+p+1} + \frac{\tau}{m} \phi_{m+p+1} + \phi_{m+p-1} - \frac{\tau}{m} \phi_{m+p-1} \right] \right\}$$
We may note that $K_t = (x/\delta - p)/qR$ so that the effective shear length $= |q^2/\varepsilon q'|$. For trapped particles, both bounce and magnetic drift resonant contributions were accounted for. For circulating particles, resonance occurs for $V_i \sim \omega qR$, $v/L \leq \omega qR/\nu_j \sqrt{\varepsilon}$ if we ignore the effect of the magnetic drift and notice that the main contribution occurs near the trapping interface. Thus the resonant contribution

$$\sim \left( \frac{\omega}{\omega_b} \right)^3 \left( 1 - \frac{3}{2} \eta \right)$$

and is much smaller than the corresponding case in the slab model. Notice also that we have suppressed the pitch-angle dependence of the magnetic drift velocity of trapped particles, since only for those particles near the trapping interface is a significant change found (the drift becomes favourable for stability).

In the following sections, we discuss the instabilities related to the toroidal geometry, i.e. those related to the presence of trapped particles and the $\theta$-modulation of the magnetic drift velocity of the circulating particles. Instabilities discussed in the slab limit are obviously still allowed in a tokamak geometry, with minor modifications, alongside those about to be discussed.

4.1. Collisionless trapped-particle effects

By noting that the rotational transform current is in the same direction as that due to trapped particles on the outer side of the torus, an instability similar to the interchange mode may be expected.

(1) Assumptions: local in $r, \theta$; $K_t V_i \gg \omega$; magnetic shear and resonance ignored:

$$2 \approx \frac{1}{\varepsilon} \left[ \left( \frac{\omega + \omega_*}{\omega} \right) \left( 1 - \frac{3}{2} \frac{\omega^0}{\omega} + \frac{1}{2} k_r^2 \Delta^2 \right) + \left( \frac{\omega - \omega_*}{\omega} \right) \left( 1 + \frac{3}{2} \frac{\omega^0}{\omega} \right) \right]$$

or

$$\omega^2 + \frac{1}{4} \omega \omega^* k_r^2 \Delta^2 + \frac{3}{2} \varepsilon \omega^* \omega^0 = 0 \quad \text{(3)}$$

where $\Delta$ is the average half banana width for trapped particles $= qa_i/\sqrt{\varepsilon}$ and its stabilizing role is clear from Eq.(3) (similar to the finite ion Larmor radius effect in a slab geometry). An oscillating poloidal magnetic field also provides a similar stabilizing mechanism with

$$k_r^2 \Delta^2 \rightarrow \frac{m^2}{\gamma^2} \frac{\varepsilon V_{ei}^2 B_i^2}{\rho^2 \beta_t^2}$$
(2) Assumptions: as above but $\beta$ is finite. Since

$$\frac{\partial B_t}{\partial r} = \beta \frac{1}{L_n} B_t,$$

and

$$\omega_{Mj} = -\omega_{ej} \left( \frac{\beta}{2} - \frac{L_n}{R} \right)$$

it follows that stability is ensured if $\beta > 2 L_n/R$. Note that the magnetic field dug by the plasma cannot be carried along by the small number of trapped particles. The circulating particles flow essentially along the field lines and do not undergo a significant $\vec{E} \times \vec{B}$ drift.

(3) Assumptions: local in $\theta$, weak shear, $K_l V_e \ll \omega$:

$$2 = \sqrt{\epsilon} \left[ \left( \frac{\omega_+ \omega_*}{\omega} \right) \left( 1 - \frac{3}{2} \frac{\omega^2}{\omega} + \frac{1}{2} \Delta^2 \frac{\phi''}{\phi} \right) + \left( \frac{\omega- \omega_*}{\omega} \left( 1 + \frac{3}{2} \frac{\omega^2}{\omega} \right) \right) \right]$$

$$+ \frac{\omega_+ \omega_*}{\omega} + \frac{\omega_- \omega_*}{\omega} \left( 1 + \frac{K_{11}^2 V_e x^2}{\omega^2} \right)$$

stable if $d(\ln q(r))/d(\ln r) \geq 0.1 \epsilon^{3/4}$. Note that the instability still persists for $x > x_i$, and sufficient magnetic shear is required to reverse the direction of the average trapped-particle drift.

(4) Assumptions: local in $r$ (no magnetic shear):

$$\omega^2 E_m = -\sqrt{\epsilon} \omega^* \omega^1 \omega^1 \left( \sum_{\rho} E_{m\rho} \frac{\rho^2}{2 \rho^2} J_0 (\rho \theta) \right)_t$$

When the exact elliptic integral formulation is used, the resulting mode structure in the poloidal direction is given by

$$\phi \sim e^{im\theta} (1 + 1.4 \cos \theta - 0.4 \cos 2\theta)$$

Note the ballooning to the outside, and the zero amplitude at $\theta = \pi$ (no trapped particles present).

(5) Effect on the drift wave:

The banana width plays the same role as the ion Larmor radius in reducing the frequency below $\omega_*$ and hence contributes to the instability. On the other hand, as discussed above, the resonance occurs only for the barely circulating particles if $\omega \ll (\omega_b)$ with the resulting reduction in the growth rate.
4.2. Trapped-ion mode

For lower temperatures, the effect of collisions on both trapped ions and electrons must be considered. The effective trapping and detrapping collision frequency is given by

\[ \nu_j = \frac{\nu_j}{e} = \frac{\nu_{oj}}{e} \left( \frac{2^3 v_j}{v} \right)^3 \]

where we have accounted for the localization of the trapped particles in pitch angle, and the diffusional nature of the collision process. Trapping must be considered to be present if \( \nu_{oj} \ll \langle \omega_{bj} \rangle \) and to contribute to the perturbation if \( \omega \ll \langle \omega_{bj} \rangle \).

(1) Assumptions: local in \( r, \theta \); neglect energy spread of \( \nu_j \); \( \omega \ll \langle \omega_{bi} \rangle \ll \langle \omega_{be} \rangle \), \( \nu_i/e \ll \omega \ll \nu_i/e \), \( K_b V_i \gg \omega \), magnetic drift ignored. \( n_i = n_e \rightarrow \)

\[ 2 = \sqrt{\varepsilon} \left[ \frac{\omega + \omega_*}{\omega + i \nu_i/e} - \frac{\omega - \omega_*}{\omega + i \nu_i/e} \right] \]

or

\[ \omega = \frac{\sqrt{\varepsilon}}{2} \omega_* + i \left( \frac{\varepsilon^2}{4} \frac{\omega^2}{\nu_e} - \frac{\nu_i}{\varepsilon} \right) \]

stable if \( \nu_e \nu_i > e^2 \omega_*^2 / 4 \) (high density).

This shows that ion collisions tend to stabilize the instability driven by electron collisions. The mode travels in the direction of the electron rotational transform current. The energy spread of the collision frequencies should obviously be considered if a temperature gradient is present, and was found to be destabilizing for \( \eta > 0 \). Note also that, according to previous discussions, the circulating ion Landau damping term is destabilizing for \( \eta > 2/3 \).

Local calculations are still valid near an inflection point in the density profile \( \omega_* (x) = \omega_{*0} (1 - x^2 / b^2) \), \( b \ll r \).

(2) Assumptions: local in \( \theta \); \( 1/L_s = 0 \); realistic density profile: \( n \sim \exp[-(r/a)^{7/2}] \), \( \nu_i = \eta = 0 \). By accounting for the ion banana width, we get from \( n_i = n_e \)

\[ \frac{\partial^2 \phi}{\partial r^2} + \frac{2 \rho / R}{q^2 a_i} \left[ 1 + \frac{i \omega \rho / R}{\nu_e} - \frac{2 \omega / \omega_*}{r^2 / R y_0 a^{3/2}} \right] \phi \]

where

\[ \omega_*^0 = \omega_* |_{r=a} = \frac{7 m v^2}{2 q a^2} \], \( q = \text{const.} \) (no shear)
Clearly this equation possesses one turning point for \( r < a \), and reflection from the boundary \( r = a \) provides the second turning point, where \( \phi = 0 \). Noting the phase shifts of \( \pi/4 \) and \( \pi/2 \), the WKB eigenvalue condition may be written as

\[
\int_{r_1}^{r_2} [v(r)]^{1/2} dr = (n + \frac{3}{4}) \pi
\]

and the instability still persists. For small \( n (\text{small } k_r) \), the local results still hold, as expected.

4.3. Trapped-electron mode

Trapped particles are also expected to influence drift waves in tokamaks \( (K_i \nu_l < \omega < K_i \nu_e) \) if \( \omega, \nu_e/e < \langle \omega_{pe} \rangle \). The instability is insensitive to the degree of collisionality of the trapped ions because of the large phase velocity assumption.

(1) Assumptions: local in \( r, \theta; \nu_e^i = (\nu_e/e)(2 \nu_e^2/\nu_T^2)^{3/2} \). Using \( n_e = n_e \), we obtain

\[
2 = \left( 1 + \frac{\omega^*}{\omega} \right) \left( 1 - \frac{\omega^*}{2} \right) + \sqrt{\varepsilon} \left( \frac{\omega - \omega^* \left[ 1 + \frac{\nu_e}{e \nu_T^2} \left( \frac{3}{2} \right) \right]}{\omega + i \nu_e^i} \right)
\]

or

\[
\omega = \omega^* \left( 1 - \frac{\omega^*}{2} \right) - \sqrt{\varepsilon} \omega^* \int_0^\infty \frac{\nu_e^i e^{\nu_e^i} x^2}{1 + i \nu_e/e} \frac{d x}{\omega^*}
\]

The integral may be approximately evaluated for \( (\nu_e/e)/\omega^* \ll 1 \) and \( \gg 1 \) and the result shows that \( k^2 a_i^2 \) is destabilizing for all \( \nu_e/e\omega^* \) while \( \eta_e > 0 \) is destabilizing for \( \nu_e/e\omega^* \geq 1 \). Assuming no shear, one may solve for the poloidal structure of the mode, which turns out to be approximately given by \( \phi(\theta) = (1 + \cos \theta) e^{im\theta} \).

(2) The trapped-electron quasimode:

On the basis of symmetry, one expects to find global modes such that \( \phi_{m+p} = \phi(p - x/\delta) \). The trapped-electron contribution may then be written as:

\[
\sqrt{\varepsilon} \left< \sum_{\rho} J_0 \left( \frac{x}{\delta} \langle \hat{\theta} \rangle \right) J_0 \left[ \left( \rho - \frac{x}{\delta} \right) \langle \hat{\theta} \rangle \right] \phi \left( \rho - \frac{x}{\delta} \right) \right> \approx \sqrt{\varepsilon} \left< \sum_{\rho} J_0 \left( \frac{x}{\delta} \langle \hat{\theta} \rangle \right) \int J_0 \left( \rho' \langle \hat{\theta} \rangle \right) \phi \left( \rho' \right) d\rho' \right.
\]
The resulting integral equation may be solved by perturbation methods. The contribution of more rational flux surfaces to the $E \times B$ drift increases the growth rate as expected, the increase being not very marked since the contributions are incoherent:

$$\left( \frac{v_e}{E \times B} \right) \sim \frac{E_0}{B_t} J_0 (K_q R)$$

### 4.4. Drift-wave ballooning

Assumptions: $\eta = T_i = 0$; no trapped electrons $\to \langle \omega_{be} \rangle e^{-3/2} > \nu_e / e \to \langle \omega_{be} \rangle$.

Using the quasimode formulation, substituting in the previously obtained expression for the response of a cold species, and employing the approximations:

$$\phi_{m+1} + \phi_{m-1} = 2 \phi_m + \frac{\partial \phi_m}{\partial m^2}, \quad \phi_{m+1} - \phi_{m-1} = 2 \frac{\partial \phi_m}{\partial m}, \chi = x - \rho \delta$$

where

$$\frac{\partial \phi_m}{\partial m} = -\delta \frac{\partial \phi_m}{\partial x} \quad \text{and} \quad \frac{\partial \phi_m}{\partial m^2} = \delta^2 \frac{\partial^2 \phi_m}{\partial x^2}$$

one obtains

$$\frac{\partial^2 \phi}{\partial x^2} \left[ -\frac{\partial^2}{2} - \frac{L_n}{R} \left( \frac{2r}{m} \delta - \delta^2 \right) \right] + \phi \left[ 1 + \frac{m^2 \rho_s^2}{2 r^2} - \frac{\omega x \epsilon}{\omega} \left( 1 - \frac{z L_n}{R} - \delta \delta - \frac{\epsilon \gamma}{\epsilon^2} \frac{x^2}{\delta^2 \nu^2 \omega^2} \right) \right] = 0$$

where $\delta$ is the electron destabilizing residue term. The above Weber equation shows that shear stabilization is lost, and localized unstable solutions are possible if

$$\frac{\rho_s^2}{2} < \frac{L_n}{R} \left( \frac{2r}{m} \delta - \delta^2 \right)$$

Thus the toroidal coupling between neighbouring rational flux surfaces prevents the shear outward energy convection. Noting that the strong coupling approximation (quasimode formulation) is justified if $\delta < \rho_s \sqrt{L_s / L_n}$, the above two conditions may be combined to obtain $rq'/q < 1/(2 + 1/q)$, which was confirmed numerically.

### 4.5. Finite-$\beta$ ballooning instability

Assumptions: local in $r$ ($1/L_s = 0$),

$$\frac{g_i(\theta)}{\mathcal{A}_i} = \frac{2 \mathcal{V}_i}{R \mathcal{A}_i} \cos \theta, \quad \omega \gg \frac{\mathcal{V}_i}{q R}, \quad \beta \ll 1$$
The previously obtained equation for radial dependence of the interchange mode in a finite-β plasma may be slightly modified to show the existence of a localized interchange mode near $\theta = 0$. Assuming a weakly favourable average magnetic curvature seen by the electrons, $ge/\Omega e \approx 0$, $\cos \theta \rightarrow 1 - \theta^2/2$ and $K_n \rightarrow (1/qR)(\partial/\partial \theta)$, one obtains
\[
\frac{\partial^2 E_\theta}{\partial \theta^2} + \left[ \frac{q^2e^2}{V_A^2} \omega (\omega + \omega_n) + \frac{q^2\beta R}{2L_n} \left(1 - \frac{\theta^2}{2}\right) \right] E_\theta = 0
\]

This simple Weber equation predicts a $\theta$-localized instability for $\beta > L_n/q^2R$ in the MHD limit ($\omega_n \rightarrow 0$), thus setting an upper limit to the maximum particle pressure attainable in tokamaks.

**BIBLIOGRAPHY**

**SECTION 2: LOCAL APPROXIMATION**


**SECTION 3: ONE-DIMENSIONAL PROBLEM**

3.1. Effect of magnetic shear

3.1.1. Density gradient drift instability


3.1.2. Temperature gradient drift instability


3.1.3. Parallel velocity drift instability


3.1.4. Drift dissipative instability


3.1.5. Density gradient drift instability in a finite-\(\beta\) plasma


3.1.6. Interchange instability in a finite-\(\beta\) plasma


3.1.7. The tearing mode


3.2. Ballooning instabilities

SECTION 4: LOW-FREQUENCY PHENOMENA IN TOKAMAKS


4.1. Collisionless trapped-particle effects


4.2. Trapped-ion mode


4.3. Trapped-electron mode

4.4. Drift-wave ballooning

Part 2

SELECTED LECTURES
ON ADVANCED TOPICS
IN FUSION RESEARCH
FIELD-REVERSED MIRRORS* 

H. L. BERK 
Lawrence Livermore National Laboratory, 
University of California, 
Livermore, California, 
United States of America 

Abstract 

FIELD-REVERSED MIRRORS. 
Progress in field-reversed mirrors is reviewed. The status and research trends of field-reversed configurations are summarized. Problems and future requirements are discussed. 

1. INTRODUCTION 

The success of standard mirror-machine concepts rests on whether configurations can be achieved where microinstabilities due to the presence of an intrinsic loss cone can be eliminated or quenched to a low enough level. Open field-line configurations demand that part of phase space should correspond to uncontained particles which drive loss-cone modes [1]. To fill the loss cone requires a large power drain as particles dwelling in the loss cone transit out of the machine in one bounce time [2]. The obvious remedy is to prevent the energy drain by closing the field lines. This makes a field-reversed configuration desirable. It is possible if the diamagnetism of the plasma currents closes the external field. 

The field-reversal concept has been investigated since the inception of the controlled fusion programme. The two best known paths have been the Astron [3], where high-energy particles (either electrons or ions whose Larmor radius is larger than or comparable to the plasma size) are used to create currents to reverse the field, and the tokamak [4], where the dominant field is an external toroidal one, but whose equilibrium consists of a mirror field that focuses a plasma current which is reversing the applied field. These projects have reached different levels of success and both are facing some practical engineering difficulties. The experiments of Fleischmann [5] with relativistic electron rings clearly demonstrate that field-reversed configurations are achievable and are stable to magnetic perturbations over a long time period, ~1 ms. However, the energy investment required to build the 'coils' from high-energy beams is high, and a lower energy current carrier is therefore desirable. Progress in the tokamak has reached the 

point where the plasma community is convinced that controlled thermonuclear conditions are achievable. However, it is difficult to extrapolate a tokamak to a reactor, since the expected achievable plasma beta is low [6] and considerable engineering problems are involved in designing the thermal and radiation shielding required for the toroidal field coils.

Alternative methods of field reversal have been achieved in experiment. Plasma guns [7] and conical pinches [8] have produced plasma rings in field-reversed configurations that have lasted several microseconds. To date there has not been a reliable way to trap such plasma rings for long-time confinement. Work at Livermore's Beta II facility [9] is directed at this problem. In addition, reversed-field $\theta$-pinch experiments by Eberhagen [10], Kurtmullaev [11], Linford [12] and others have clearly demonstrated that long-lived (50–100 $\mu$s) field-reversed configurations can be established and are stable (at least until plasma rotation becomes sufficiently rapid to drive rotational instability).

Recently, the desire of researchers in the tokamak field to design higher-beta systems, free of the engineering difficulties associated with the toroidal field, has given rise to the study of the Spheromak [13]. This is a reversed-field plasma containing the poloidal plasma currents that give rise to plasma toroidal fields, but where there is no external toroidal field. Preliminary experiments that produce this configuration have been reported by Goldenbaum [14]. The plasma ring expected from the Livermore Marshall gun will produce toroidal fields [9].

Hence, for a variety of reasons, the interest of the plasma community is converging on plasma rings of many Larmor radii whose self-current reverses an externally applied mirror field and can produce toroidal field.

Once the field is reversed, some of the more serious mirror-machine problems are solved and others are ameliorated, but new problems arise. The principal problem solved is that the loss-cone drive disappears in a field-reversed, closed field line configuration. A caveat should be mentioned that a reversed-field configuration with open field lines is possible and hence some loss-cone drive still needs to be considered. Other microinstabilities associated with drift waves are still potentially present, and although the high-beta operation of the field-reversed configuration is a stabilizing factor, detailed calculations are needed in this area. In open-ended configurations, the maintenance of an electrostatic sheath at the ends, to inhibit thermal conduction losses, is of great concern. To the extent the field lines are closed in a reversed-field configuration, one expects thermal electron conduction along field lines to be less of a problem. The great advantage of a mirror machine is its intrinsic MHD stability, which is achieved by having stabilizing curvature everywhere. By closing field lines, we sacrifice this advantage. In theoretical equilibria analysed so far, MHD stability can be achieved only by shear. The stability observed in experiment is believed to be due to finite Larmor radius.

With so many uncertainties, it is clear that further experimental and theoretical investigations are needed to ascertain whether a reversed-field configuration can be
developed into a reactor. In the next section the status and research directions of theories of the reversed-field configuration are summarized.

2. START-UP

The high beta achieved by the 2XIIB experiment [15], where a central beta of about unity was achieved, has renewed interest at Livermore in field reversal. It was believed that currents created by ionizing neutral beams could increase the plasma currents up to the point of achieving reversal. Calculations with the superlayer code indicated that this was feasible with neutral beam currents of 400–600 A [16]. A serious objection arose to modelling a plasma in the superlayer code. The code does not have electron dynamics; it is only assumed that electrons provide a neutralizing background. For open field lines this is a good approximation, since an electron reservoir can exist at the wall, and charge neutralization can be provided from outside the plasma. However, in a closed field-line configuration, electrons can only neutralize by moving together with ions across field lines. The consequence of such dynamics leads to the principle of flux conservation of ideal MHD and the implication that neutral beams (with the same atomic number of the background plasma) cannot provide toroidal currents in a toroidally symmetric system. Consequently, modelling of a plasma should drastically change as the plasma attempts to reverse its field. Doubt was therefore cast upon the predictions of the superlayer code after reversal.

Baldwin and Fowler [17] have used conventional MHD and transport arguments to study field reversal. They include Ohkawa currents [18] arising from neutral beams injected in a plasma with an atomic number differing from that of the neutral beam (the difference in Z is due to impurities). They estimate that neutral beam currents in excess of 1 kA are needed to achieve reversal. In the 2XIIB experiment the neutral beam current never exceeded 500 A, and this may explain the failure of reversal.

To avoid the difficulties of achieving field reversal with a relatively slow start-up phase, a rapid start-up phase was envisaged which creates a reversed-field ring. The neutral beam would then be used to heat the ring after reversal is achieved [9]. Successful methods of achieving field reversal are known from other experiments.

Before discussing new pulsed methods of start-up, re-examination of past interpretations is in order. A major discrepancy between superlayer simulations and 2XIIB data was the axial length of the plasma [16]. The experiment has twice the predicted value for conditions where field reversal is not predicted. The superlayer code is azimuthally symmetric, which the experiment is not. One major discrepancy is that the pitch-angle distribution of the input beam is not properly accounted for in the simulation, which leads to greater lengths. It would also
lead to higher current being necessary to achieve reversal. Another discrepancy is that only in a toroidally symmetric system is a neutral beam of the same atomic number as the plasma background not a current source only for a toroidally symmetric system. The ideal MHD model predicts current neutralization because radial electric fields cause electrons to flow at the same velocity as the ions. However, these electric fields can be shorted by various effects such as stray open-field lines, magnetic field perturbations (in three dimensions), viscosity, etc. So long as these shorting mechanisms are on a fine enough scale they do not effect ion dynamics as strongly as electron dynamics. The best way to model the complicated dynamics of electrons might be to treat electrons as a neutralizing background fluid which are able to cross field lines more readily than ions. There is considerable evidence for such behaviour in tokamaks, where electron cross-field thermal conduction determines the lifetime [18], although classical transport predicts that ion diffusion should determine the lifetime. Consequently, if the lack of symmetry due to quadrupole fields is taken into account, the superlayer code may be a good way to simulate field reversal.

Successful start-up of a field-reversed plasma has been achieved in \( \theta \)-pinch experiments. The most efficient production has been achieved by Kurtmullaev, who devised an innovative technique using octupole barrier fields to allow a relatively slow (~10 \( \mu \)s) reversed-field implosion to produce a field-reversed plasma. He claims that the energy of the final state is equally divided between thermal and magnetic field energy. A problem with this method of formation is access to the plasma for subsequent heating, e.g., neutral beam heating. The plasma lifetime is about 100 \( \mu \)s, but it is claimed that this limitation is due only to external circuitry rather than the plasma properties. To improve access, the plasma should be transported to a different region. This experiment should be tried, and a paper by Kurtmullaev indicates that it is being attempted. If transport is successful this may be a viable way to proceed.

In Linford's experiments with \( \theta \)-pinch reversal, he observes a rotational instability developing from 30 \( \mu \)s to 100 \( \mu \)s after reversal formation. It is conjectured that the rotation is induced by transport as the ions lost from the ends leave with a preferential angular momentum, allowing the ring to rotate with the opposite angular momentum. At a critical rotation speed, instability is induced that destroys the plasma. The threshold for instability has been predicted, using a finite Larmor radius theory, by Pearlstein and Friedberg [19] together with a kinetic model developed by Seyler [20]. The uncertainties of experimental diagnostics make it impossible to confirm whether theory and experiment are in agreement. An attempt is now being made to improve the modelling of the theory by proper treatment of the dynamics near the vortex region of the plasma.

A modification of the field-reversed pinch approach has recently been performed by Goldenbaum [14]. Before imploding the reversed field, he draws an axial current with end electrodes. Upon implosion, oppositely directed field
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lines tear and reconnect at the ends, while the original plasma currents along the field lines continue to follow the reversed-field lines, causing a toroidal field to develop. The present plasma is short-lived as the circuitry limits the lifetime to less than $10 \mu s$.

An alternative approach to plasma reversal is to produce a plasma ring with a Marshall gun. This was successfully achieved in 1958 by Lindberg and Jacobsen [7]. The guns form a toroidal field, while poloidal fields are induced by forcing the ring to shield external imposed fields which the plasma tries to cut through. A variant of this method is to be used in the Livermore plasma-gun experiment at the Beta II facility. One of the surprises of the 1958 experiment was the measurement of poloidal flux enhancement, i.e. the poloidal flux of the final ring exceeded the external flux it has to shield. A prevalent interpretation of this response is related to a mechanism propounded by J.B. Taylor [21]. Given the total flux input into the plasma, which is

$$K = \int d^3r \mathbf{A} \cdot \mathbf{B}$$

where $\mathbf{A}$ is the vector potential and $\mathbf{B}$ the magnetic field, the plasma will find its lowest energy state by shifting the toroidal and poloidal fluxes keeping $K$ constant. The 1958 experiment reported that the plasma appeared to kink early in the formation and then relax to a symmetric state with the poloidal flux enhanced. More data and more theoretical analysis of this mechanism will arise when the Beta II experiment becomes operational.

In an attempt to understand the plasma-gun formation stage, an MHD code is being used that describes the ring formation in the gun, the motion out of the muzzle and, finally, transport in a mirror field. The first two parts of the calculation have been performed relatively successfully, while the motion in the mirror field was hindered by numerical time step difficulties. This is at present being fixed somewhat by solving the MHD code with higher-order accuracy, but it remains an expensive calculation and extremely short time steps have to be taken to avoid numerical instability.

Calculation of the plasma characteristics in the gun have been compared favourably with experiments without externally imposed poloidal fields, and the calculation has been extrapolated to the mode of operation desired in the Beta II facility. A sensitive point of the calculation is the determination of what is required to force the tearing necessary to close the field lines of the ring. The code predicts (perhaps unrealistically) a high momentum flux of low-energy high-density particles that directly follows the initial high-energy pulse through the muzzle. The effect is to inhibit tearing, and a high-field snipper coil producing a field of $\sim 50 \text{ kG}$ must then be used to achieve tearing in the code. Such high fields have not been required in other experiments, such as those of Kurtmullaev, Linford and Goldenbaum, and the code prediction is subject to doubt. Besides the concern that the calculation predicts too much low-energy density behind the
pulse, there is a suspicion that the calculation is numerically inaccurate in predicting tearing, which requires fine grid resolution for an accurate description. The tearing phase of the experiment is crucial and will have to be studied in detail if tearing does not readily occur in the experiment.

Researchers at Princeton recently proposed another innovative method in an attempt to produce a reversed-field configuration [22]. Results of investigations using this method will be followed closely.

3. THEORETICAL DESCRIPTIONS OF FIELD-REVERSED EQUILIBRIA

The modelling of field-reversed configurations is probably best described in an MHD framework, but this may still be inaccurate as current experiments produce plasmas of only a few Larmor radii (some reactor designs assume only a few Larmor radii). For such cases an MHD description ignores important finite Larmor radius effects. (Nevertheless, we shall begin with an MHD description.)

Prototype equilibria can be studied, using simple analytic models of a toroidally symmetric problem which are solutions of the Grad-Shafranov equations for pressure isotropy. In cylindrical coordinates, the equation takes the form

$$\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} = - \frac{\partial p(\psi)}{\partial \psi} + \frac{1}{2\tau^2} \frac{\partial}{\partial \psi} g^2(\psi)$$

Where \( \psi \) is the poloidal flux, \( p(\psi) \) is the pressure, and \( g(\psi) = rB_\phi \), where \( B_\phi \) is the toroidal magnetic field. Specific analytic solutions of the reversed-field type are Hill's vortex, where \( B_\phi = 0 \), and

$$\psi = \psi_0 r^2 (r^2 - r_0^2 - z^2)$$

as well as the Spheromak solution [23] for \( p(\psi) = 0 \), and

$$\psi = \psi_0 (1 - \mu^2) \rho^{1/2} J_{3/2} (\gamma \rho)$$

where

$$\rho = (r^2 + z^2)^{1/2}, \ \mu = z/\rho$$

In addition, we have constructed a wide class of analytic solutions where

$$\frac{\partial p}{\partial \psi} = \alpha \psi + \beta \ \ \text{and} \ \ \frac{1}{2} \frac{\partial g^2}{\partial \psi} = \gamma^2 \psi$$
Hammer [24] has constructed a code that determines the vacuum currents needed for a given analytic equilibrium. Anderson [25] has developed a code that calculates equilibria given $p(\psi)$, $g(\psi)$ and the external currents and/or conductors.

Positional instability to rigid motion is important. Analytic equilibria in a homogeneous magnetic field are spherically shaped and are neutrally stable to rigid motions if no external conductors are present [26]. Neutral stability is obvious for motion along and across a uniform external field as the new configuration is equivalent to the original one. Rotational motion about the axis of symmetry is likewise neutrally stable as it does not change the configuration. Another marginal perturbation is a tilting motion with respect to the external field [23]. This result is not obvious from symmetry; it relies on the ideal conduction properties of the plasma which produce surface currents to keep the internal currents in their original configuration (with respect to Lagrangian coordinates). These surface currents, together with the perturbed tilted currents, cause fields that exactly cancel outside the sphere. However, since non-ideal MHD effects are to be expected at the edge of a plasma, further assessment of the tilting mode is needed.

Placing mirrors on a reversed-field configuration gives the equilibrium an oblate shape and guarantees axial stability. Cross-field stability will arise from a conductor that is fairly distant from the plasma. If $a$ is the axial length, $b$ the cross-field length and $1 + \varepsilon = b/a$, then the shell can be a distance $a/e$ from the plasma. However, tilting stability requires a much more stringent condition, i.e. the shell needs to be $0.2 ea$ from the plasma. Hence reasonably large oblateness is needed for practical consideration of conductor placement.

If the external magnetic field is min-B, then cross-field stability would be guaranteed without a conducting shell. However, external min.-B fields are only formed with a breakdown of toroidal symmetry. Besides being difficult to analyse, this symmetry breakdown may have some profound consequences. For example, the separatrix that exists in the symmetric case becomes an ergodic open field line region when the symmetry-breaking external fields are present. Hence the edge plasma must be considered on open field lines, and the surface current response of the plasma can be quite different. It is thus conceivable that the tilting mode behaves very differently in a min.-B field.

To obtain a prolate shape, the external fields must form an antimirror well (at least near the midplane of the ring). Hence we have axial instability when conductors are not present (it may be possible to prevent axial instability by field shaping where we have an antimirror well at the centre and a mirror well further out). The stability picture can be improved with conductors whose image current can produce stability. It may also be possible to maintain stability by feedback stabilization. It is significant that the most successful reversed-field configurations produced to date are prolate-shaped. They are plasmas that produce image currents on cylindrical shells. If the plasma moves, the coils producing the antiwell
move with it, since the coils are the image currents. Overall positional stability is achieved by placing small external mirror fields at the ends.

Internal tilting mode instability has been predicted for prolate-shaped rings (more precisely, the analysis is limited to the force-free zero-pressure case) [23]. However, experiments have not indicated this instability. To obtain such instability, non-rigid perturbations, which may be sensitive to finite Larmor radius effects, are required. Higher-wavelength kink modes also require for stabilization the proximity of a conducting shell. It is hoped that future investigations will show that finite Larmor radius or cold plasma on open field lines can make conducting walls close to the plasma unnecessary.

Pressure-driven modes have been investigated. In the presence of toroidal fields, shear supplies a stabilizing influence, and the Mercier-Suydam criterion indicates that a relatively high beta can be achieved, \( \sim 10-50\% \) with respect to the external field [22]. Without toroidal field, an analysis by Newcomb [27] on Hill's vortex equilibrium indicates that several unstable modes are present which are similar to the kinks of the Z-pinch. The growth rates are of the order of the Alfvén transit time, and one would not expect much improvement with finite Larmor radius. An exception is for the highly prolate Hill's vortex, where the growth rate is inversely proportional to the long axis of plasma. From simple scaling arguments, finite Larmor radius effects would be expected to be important when \( \frac{a_i}{a} \gg \frac{a}{b} \), where \( a_i \) is the ion Larmor radius, \( a \) the radial scale length, \( b \) the axial length and \( \ell \) the azimuthal mode number. This scaling may be the reason why these modes have not been observed in field-reversed pinch experiments.

4. EQUILIBRIUM AND TRANSPORT

Simultaneous solution of the equilibrium and transport of a reversed-field ring is a major theoretical undertaking. The simplest self-consistent formulation of the problem is to consider the Grad-Shafranov equation for a symmetric equilibrium together with suitable transport properties that depend on local flux surfaces. Such a code is now being developed at Livermore. The prototype case will use fluid transport coefficients for resistivity and thermal conduction and the energetics of neutral beam deposition, but more sophisticated transport needs to be developed. Neoclassical coefficients have to be calculated since particle orbits are significantly different in a tokamak. Experience with tokamaks has shown that transport is likely to be governed by anomalous processes. Drift-wave instabilities are quite possible in field-reversed configurations and the transport coefficients due to them can be self-consistently included in the code.

Other transport effects should also be considered. Orbits may be large enough to require a non-local description, and a satisfactory description of such effects will probably require Monte-Carlo techniques. One such example is the
code developed by Driemeyer et al. [28] who assume a Hill's-vortex equilibrium
and follow the energetics of $\alpha$-particle deposition. They find, using a rough
classical model for transport of the background plasma, that a substantial $Q$ is
obtained in the field-reversed configuration. The superlayer code can be seen
as a selfconsistent Monte-Carlo code, and this will be expanded to include self-
consistent toroidal fields. Considerable success was recently achieved using
orbit-averaging techniques to obtain the selfconsistent fields [29]. The superlayer
code can potentially describe extremely complicated nonlocal effects. Whether it
can be a realistic simulation depends on whether there is toroidal symmetry and
whether it is correct to assume that electrons can be considered simply as a
neutralizing background. It should be noted that microturbulence on an electron
scale can cause such a simple picture of electrons to be relatively accurate, as
electrons would then be able to cross field lines faster than ions and neutralize any
macroscopic electric field. It is also possible that without toroidal symmetry the
superlayer code can be adjusted to account for the symmetry-breaking pertur-
bations in the orbits. Without a toroidal field, Boozer [30] has given arguments
to show that field lines will be open if the external fields do not have toroidal
symmetry. If this is so, electrons can be expected to mix readily in the body of
the plasma, and to this extent the neutralizing assumption of the superlayer code
is justified.

Other directions of equilibrium and transport need development. As neutral
beams are injected, the plasma rotates. Generalizations of the Grad-Shafranov
equation have been developed [31] to take the rotation into account. Viscosity
must be included to find the level of rotation that a plasma might be expected to
reach. By taking viscosity into account, Hammer [32] has shown that the neutral
beam can support a toroidal plasma current in a field that weakly breaks toroidal
symmetry. Rotation can induce destructive instability, as dramatically observed in
the field-reversed pinch, and an accurate prediction of the rotation speed needs to
be reliably obtained.

For final applications, a reactor concept has to be envisioned. Two different
states are being considered at present. First, a transient state [33] using ideas of
Fleischmann [34], where rings are continually generated and transported down a
cylindrical shell for a burn-time less than the magnetic diffusion time of the plasma.
This design has the pleasant feature of not having to maintain a steady state and
yet allowing the heat load on the wall to be essentially steady state. It remains to
be shown that such a system is compatible with equilibrium and stability require-
ments of field-reversed plasmas. The second is a steady-state concept. Since
current cannot now be maintained with external transformers, the design rests on
whether a practical drive can be obtained from neutral beams (with the help of
Ohkawa currents) or viscosity effects arising from symmetry-breaking pertur-
bations, from external fields or internal fluctuations. Alternatively, RF current
drives [35] can be considered.
Other reactor concepts are also feasible, e.g. multiple connected mirrors with a reversed-field core in each cell [36] and high-field compression scenarios with a liner wall [37]. Realistic designs will appear as our knowledge of the physics of reversed-field plasmas increases.

REFERENCES

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PHYSICS OF ELECTRON CYCLOTRON ABSORPTION

M. BORNATICI
Istituto di Fisica Applicata,
Facoltà di Ingegneria,
Università di Pavia,
Pavia, Italy

Abstract

PHYSICS OF ELECTRON CYCLOTRON ABSORPTION.

The electron cyclotron absorption of electromagnetic waves propagating in a weakly relativistic Maxwellian plasma at angles with respect to the applied magnetic field such that the relativistic effects due to the dependence of the electron mass on velocity are dominant is investigated on the basis of the energy balance equation. Both dispersion and polarization effects, including the finiteness of the electron Larmor radius, are evaluated along with the contribution to the absorption from both the Poynting vector and the flux of the sloshing energy. It is found that (a) the absorption of the first harmonic extraordinary mode is strongly affected by polarization effects, yielding an anomalous behaviour with respect to plasma density; (b) for $\omega_p^2 \approx \omega_c^2$ ($\omega_p$ and $\omega_c$ being, respectively, the plasma and the cyclotron frequency of the electrons), finite density and finite Larmor radius effects combined enter both the propagation and absorption of the second-harmonic extraordinary mode; (c) for $\omega_p^2 \leq \omega_c^2$, dispersion and polarization effects have a weak influence on the propagation and absorption of both the ordinary mode and the $n^{th}$ harmonic extraordinary mode, $n \gg 3$.

The optical thicknesses of a tokamak plasma are evaluated and the potentialities of the electron cyclotron absorption and emission, respectively, as a heating and diagnostic method are discussed.

1. INTRODUCTION

The theory of the propagation and absorption (emission) of electromagnetic waves at frequencies close to the $n$-th harmonic of the electron cyclotron frequency $\omega_c$, in a plasma in thermodynamic equilibrium, has been well established for quite a long time within the framework of the single-particle approximation, i.e. in the limit in which the dispersion, polarization and correlation effects are negligible [1]. For plasma parameters and magnetic fields of practical interest, however, such an approximation is, in general, not valid for the first harmonic extraordinary mode [2-10], the combined effect of finite density and temperature entering the dielectric tensor via terms large compared to 1, this being the case even for $(v_T/c)^2 \ll 1$ ($v_T = (T/m)^{1/2}$).
is the electron thermal velocity and $c$ the speed of light. Plasma effects are, therefore, expected to strongly affect the absorption of the extraordinary mode close to the fundamental frequency $\omega_c$.

To assess the potentiality of the electron cyclotron heating as a method for supplementary heating in a tokamak plasma [3, 11, 12] and to exploit the radiation around $n\omega_c$ as a diagnostic tool [13-19] require that both the absorption (emission) coefficient be evaluated accounting for finite-density effects and the physics of the absorption (emission) process be understood.

In this paper we will discuss systematically and comprehensively the different regimes appearing in the interaction of a weakly relativistic ($v^2 \ll c^2$) Maxwellian plasma with electromagnetic waves having frequencies around $n\omega_c$ ($> \omega_p$, with $\omega_p$ the electron plasma frequency), concentrating on directions of propagation so close to perpendicular with respect to the external magnetic field $B_0$ that the relativistic variation of the electron mass matters. The case of the non-overlapping harmonics is considered and particular attention is paid to the physics of the absorption process, the absorption coefficient $\alpha$ being evaluated by means of the energy balance equation [10].

We note that, for a thermal plasma, one can take advantage of the simplicity of the evaluation of the absorption coefficient, deriving from the use of the energy balance equation, to obtain the emission coefficient $\eta$ by making use of Kirchhoff's law [1],

$$
\frac{1}{N_r} \frac{\eta}{\alpha} = \frac{\omega^2 T}{8\pi^2 c^2}
$$

thus avoiding the direct derivation of $\eta$ [20-22], which is much more laborious than that of $\alpha$. In (1), $N_r$ is the ray refractive index [1] and the right-hand side is the vacuum blackbody intensity.

The paper is organized as follows. In Sec. 2 we examine the dielectric tensor appropriate to describe the propagation and absorption of both the ordinary and extraordinary mode for propagation in directions close to $90^\circ$ with respect to $B_0$. The wave dispersion and polarization close to the harmonic frequencies $n\omega_c$ are evaluated in Sec. 3, together with the discussion of finite Larmor radius effects in the polarization of both the first and second harmonic extraordinary mode. The absorption coefficient is obtained in Sec. 4 on the basis of the energy balance equation, whereas
the evaluation of the line width and optical thickness for a tokamak plasma is performed in Sec. 5. The results are summarized in Sec. 6.

2. THE WEAKLY RELATIVISTIC DIELECTRIC TENSOR

Let us consider the dielectric tensor of a plasma in thermodynamic equilibrium for frequencies close to the harmonic frequencies $\omega_n$, $n \geq 1$ being the harmonic number. For propagation at an angle $\theta$ with respect to the static magnetic field $B_0 = \frac{2}{|B_0|}$ such that

$$N \cos \theta < \frac{v_t}{c}$$

the cyclotron interaction between electrons and radiation is dominated by the relativistic effects associated with the relativistic dependence of the electron mass on velocity. Such a case will be referred to as "perpendicular" propagation, to lowest significant order in $(N \cos \theta / c v_t)^2$, the wave dynamics being identical with that for exactly perpendicular ($\theta = \pi/2$) propagation. In this case, for a weakly relativistic $(T/mc^2 \equiv (v_t/c)^2 << 1)$ Maxwellian plasma and for wavelengths perpendicular to $B_0$ which are larger than the electron Larmor radius $\lambda \equiv \left(\frac{N v_t}{c}\right)^2 << 1$, the appropriate components of the dielectric tensor for $k_y = 0$ are given in Table I for frequencies close to the fundamental frequency $\omega_c$ and in Table II for frequencies close to the harmonic frequencies $\omega_n$, $n \geq 2 [9,23]$. The functions $\varepsilon_q(z_n)$ and $F_q(z_n)$ are defined as follows:

$$\varepsilon_q(z_n) = -\frac{2q-3}{2(q-\frac{5}{2})} \left(\frac{\omega}{\omega_c}\right)^2 \left(\frac{\omega}{\omega_c}\right)^{(2q-7)} \left(\frac{v_t}{c}\right)^{(2q-7)} N_l (2q-5) F_q(z_n)$$

$$\varepsilon' = \text{Re}(\varepsilon_q), \quad \varepsilon'' = \text{Im}(\varepsilon_q); N_l \equiv \frac{k^l c}{\omega} \text{ is the perpendicular (real) refractive index and [2,3],}$$

$$F_q(z_n) = -i \int_0^\infty \frac{e^{-t z_n}}{(1 - i t) q} dt$$

$$= \frac{q^{2q/2}}{\rho_{\xi_0}} (-z_n)^p \Gamma(q-1-p) \Gamma(q) + \frac{\sqrt{\pi}}{\Gamma(q)} (-z_n)^{q-3/2} [\sqrt{2/\pi} (\xi_n) z (\sqrt{2/\pi} \xi_n)]$$
TABLE I. NONZERO COMPONENTS OF THE DIELECTRIC TENSOR $\epsilon_{ij}$ NEAR THE FIRST HARMONIC ($\omega \approx \omega_c$) FOR PERPENDICULAR ($N \cos \theta < v_t/c$) PROPAGATION IN A WEAKLY RELATIVISTIC ($v_t^2 \ll c^2$) MAXWELLIAN PLASMA

<table>
<thead>
<tr>
<th>$\epsilon_{ij}$ = $\epsilon_{ij,h} + ie_{ij,a}$</th>
<th>Hermitian part, $\epsilon_{ij,h}$</th>
<th>Anti-Hermitian part, $\epsilon_{ij,a}$</th>
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<tbody>
<tr>
<td>$\epsilon_{xx}$</td>
<td>1 - $\frac{\omega^2}{2\omega(\omega + \omega_c)}$</td>
<td>$\epsilon_{\frac{3}{2}}^\text{H} (z_1)$ $\left( 1 - \frac{4\lambda}{3} \frac{F_{\frac{3}{2}}^\prime (z_1)}{F_{\frac{3}{2}}^\prime (z_1)} \right)$</td>
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<td>$+ \epsilon_{\frac{1}{2}} (z_1) \left[ \frac{F_{\frac{1}{2}} (z_1)}{F_{\frac{3}{2}} (z_1)} \right]$</td>
<td>$\frac{c^2}{8} \lambda^2 \frac{F_{\frac{3}{2}}^\prime (z_1)}{F_{\frac{3}{2}}^\prime (z_1)}$</td>
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<tr>
<td>$\epsilon_{yy}$</td>
<td>1 - $\frac{\omega^2}{2\omega(\omega + \omega_c)}$</td>
<td>$\epsilon_{\frac{3}{2}}^\text{H} (z_1)$ $\left( 1 - \frac{4\lambda}{3} \frac{F_{\frac{3}{2}}^\prime (z_1)}{F_{\frac{3}{2}}^\prime (z_1)} \right)$</td>
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<td>$+ \epsilon_{\frac{1}{2}} (z_1) \left[ \frac{F_{\frac{1}{2}} (z_1)}{F_{\frac{3}{2}} (z_1)} \right]$</td>
<td>$+ \frac{37}{8} \lambda^2 \frac{F_{\frac{3}{2}}^\prime (z_1)}{F_{\frac{3}{2}}^\prime (z_1)}$</td>
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<tr>
<td>$\epsilon_{xy} = -\epsilon_{yx}$</td>
<td>$- \frac{1}{2} \frac{\omega^2}{2\omega(\omega + \omega_c)}$</td>
<td>$- i \epsilon_{\frac{3}{2}}^\text{H} (z_1)$ $\left( 1 - \frac{4\lambda}{3} \frac{F_{\frac{3}{2}}^\prime (z_1)}{F_{\frac{3}{2}}^\prime (z_1)} \right)$</td>
</tr>
<tr>
<td></td>
<td>$+ \epsilon_{\frac{1}{2}} (z_1) \left[ \frac{F_{\frac{1}{2}} (z_1)}{F_{\frac{3}{2}} (z_1)} \right]$</td>
<td>$+ \frac{15}{8} \lambda^2 \frac{F_{\frac{3}{2}}^\prime (z_1)}{F_{\frac{3}{2}}^\prime (z_1)}$</td>
</tr>
<tr>
<td>$\epsilon_{zz}$</td>
<td>1 - $\frac{\omega^2}{2\omega_c}$</td>
<td>$\epsilon_{\frac{3}{2}}^\text{H} (z_1)$ $\frac{F_{\frac{3}{2}}^\prime (z_1)}{F_{\frac{3}{2}} (z_1)}$</td>
</tr>
<tr>
<td></td>
<td>$+ \epsilon_{\frac{1}{2}} (z_1) \lambda \frac{F_{\frac{1}{2}} (z_1)}{F_{\frac{3}{2}} (z_1)}$</td>
<td>$\frac{c^2}{8} \lambda^2 \frac{F_{\frac{3}{2}}^\prime (z_1)}{F_{\frac{3}{2}}^\prime (z_1)}$</td>
</tr>
</tbody>
</table>

with $z_n \equiv (c/v_c)^2 (\omega - n\omega_c)/\omega$, $Z(i\sqrt{z_n})$ being the familiar plasma dispersion function. For $z_n < 0$, i.e. for $\omega < n\omega_c$, the function $F_q(z_n)$ has an imaginary part given by

$$F''_q(z_n) = \frac{\pi}{\Gamma(q)} |z_n|^{q-1} e^{-|z_n|}$$

(5)

$$\int_{-\infty}^{\infty} (-F''_q) dz_n = \pi$$

Let us first examine the dielectric tensor for frequencies close to the fundamental frequency $\omega_c$ (see Table I). In the Hermitian part $\epsilon_{ij,h}$ corrections of order $(v_t/c)^2$ have been neglected, the terms proportional to $\omega_p^2$ being the (cold) contributions
TABLE II. NONZERO COMPONENTS OF THE DIELECTRIC TENSOR $\varepsilon_{ij}$ NEAR THE $n$-th HARMONIC ($\omega \approx n\omega_c$, $n \geq 2$) FOR PERPENDICULAR ($N \cos \theta < v_t/c$) PROPAGATION IN A WEAKLY RELATIVISTIC ($v_t^2 \ll c^2$) MAXWELLIAN PLASMA

<table>
<thead>
<tr>
<th>$\varepsilon_{ij}$</th>
<th>$\omega \approx 2\omega_c$</th>
<th>$\omega \approx n\omega_c$, $n &gt; 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_{xx,h} = \varepsilon_{yy,h}$</td>
<td>$1 - \frac{\omega^2}{\omega^2 - \omega_c^2} + i\varepsilon_{\frac{1}{2}}(z_2)$</td>
<td>$1 - \frac{\omega^2}{\omega^2 - \omega_c^2}$</td>
</tr>
<tr>
<td>$\varepsilon_{xy,h} = -\varepsilon_{yx,h}$</td>
<td>$\frac{\omega^2\omega_c^2}{\omega(\omega^2 - \omega_c^2)} - i\varepsilon_{\frac{1}{2}}(z_2)$</td>
<td>$i\frac{\omega^2\omega_c^2}{\omega(\omega^2 - \omega_c^2)}$</td>
</tr>
<tr>
<td>$\varepsilon_{zz,h}$</td>
<td>$1 - \frac{\omega^2}{\omega_c^2}$</td>
<td>$1 - \frac{\omega^2}{\omega_c^2}$</td>
</tr>
</tbody>
</table>

Hermitian part, $\varepsilon_{ij,h}$

| $\varepsilon_{xx,a} = \varepsilon_{yy,a}$ | $\varepsilon_{\frac{3}{2}}(z_2)$ | $\varepsilon_{n+\frac{3}{2}}(z_n)$ |
| $\varepsilon_{xy,a} = -\varepsilon_{yx,a}$ | $-i\varepsilon_{\frac{1}{2}}(z_2)$ | $-i\varepsilon_{n+\frac{1}{2}}(z_n)$ |
| $\varepsilon_{zz,a}$ | $\frac{2}{3}\varepsilon_{\frac{3}{2}}(z_2)$ | $\frac{2}{n+1}\varepsilon_{n+\frac{3}{2}}(z_n)$ |

Anti-Hermitian part, $\varepsilon_{ij,a}$

From the harmonic $n = 0$ to $\varepsilon_{zz,h}$ and from the harmonic $n = -1$ to the other components of $\varepsilon_{ij,h}$. Of particular interest are the terms associated with the resonant harmonic $n = 1$ and proportional to $\varepsilon_{\frac{5}{2}}(z_1)$, to be referred to as resonant terms. Noting that, cf. Eq. (3),

$$\varepsilon_{\frac{5}{2}}(z_1) \equiv -\frac{1}{2} \left[ \frac{\omega}{\omega_c} \right]^2 \left[ \frac{c}{v_t} \right]^2 F_{\frac{5}{2}}(z_1) \tag{6}$$

near resonance, i.e. for $|z_1| \approx 1$, one has $|F_{\frac{5}{2}}(z_1)| \approx 1$, so that $|\varepsilon_{\frac{5}{2}}(z_1)| > 1$ for

$$\left( \frac{\omega}{\omega_c} \right)^2 > 2 \left( \frac{v_t}{c} \right)^2 \tag{7}$$
which is indeed the case for most plasma parameters and magnetic fields of practical interest. Moreover, \(|\varepsilon_{5/2}(z_1)| \approx (c/v_t)^2 \gg 1\) near resonance and for plasma densities and magnetic fields for which \((\omega_p/\omega_c)^2 \approx 1\). Under condition (7) ("finite density" regime) the elements of the electric susceptibility of the plasma electrons proportional to \(\varepsilon_{5/2}(z_1)\) are large compared to 1 so that important plasma effects are to be expected, dispersion, polarization and correlation effects contributing in generating a non-linear dependence on plasma density. On the other hand, when condition (7) is reversed ("tenuous plasma" regime) (e.g. for the runaway electrons of a tokamak plasma), the magnitude of all the elements of the susceptibility tensor is small compared to 1 and plasma effects are weak and essentially proportional to the plasma density. This regime has been treated exhaustively already a long time ago, mostly using the single-particle approximation to calculate the plasma emissivity \([1,13]\). In both these regimes a simplified analytical treatment is possible, as in the first the inverse of the large contributions to the electron susceptibility may be taken as an expansion parameter while, in the second, one may expand in the electron susceptibility. This is no longer possible in the transition regime in which the susceptibility regime is of order 1.

In the anti-Hermitian part \(\varepsilon_{ij,a}\), corrections due to finite Larmor radius (FLR) effects to the lowest-order contribution \(\varepsilon_{5/2}\) have been retained up to order \(\varepsilon_{5/2}^\prime \), \((v_t/c)^2\), for \((\omega_p/\omega_c)^2 \approx 1\) and \(N \approx 1\). Note that, since \((v_t/c)^2 \ll 1\), the non-relativistic Maxwellian distribution can be used for the electrons. In fact, for a relativistic Maxwellian distribution, to lowest significant order in \((v_t/c)^2\), the elements of the dielectric tensor \(\varepsilon_{ij}\) are formally equal to the ones given in Table I with the \(F_q\)-functions replaced by new functions which differ from the \(F_q\)'s only by terms of order \((v_t/c)^2\), or higher \([25]\). Changes of order \((v_t/c)^2\) of the \(F_q\)'s, however, do not enter the dispersion relation to lowest significant order in \((v_t/c)^2\) \([9]\).

Turning now to the dielectric tensor of the plasma for frequencies close to the harmonic frequencies \(n\omega_c\), \(n \geq 2\), given in Table II, it appears that near the second harmonic a contribution to the Hermitian part \(\varepsilon_{ij,h}\) proportional to

\[
\varepsilon_{ij,h}'(z_2) = \varepsilon_{ij,h}'(z_2) \cdot \frac{F_{ij,h}'(z_2)}{F_{ij}'(z_2)}
\]
i.e. a combination of resonant and FLR effects, is present together with the cold contributions from the harmonics \( n = \pm 1 \), all these contributions being of order 1 for \( \omega_P^2 \approx \omega_c^2 \). For higher harmonics \( n \geq 3 \), instead, it is only the cold contributions which enter \( \varepsilon_{ij,h} \). As far as the anti-Hermitian part \( \varepsilon_{ij,a} \) is concerned, one has \( \varepsilon_{ij,a} \sim (v_t/c)^2(n-1) \), \( n \geq 2 \), and \( \varepsilon_{ij,a} \sim (v_t/c)^2(n-2)^2 \), \( n \geq 2 \), for the other elements.

3. DISPERSION AND POLARIZATION

The wave propagation in a plasma such that the spatial and temporal variations are characterized by lengths and times long compared with the local wavelength and period can be described in the framework of geometric optics (WKB approximation), which to lowest order yields the local dispersion relation

\[
N^2 \left[ \frac{k'k'}{(k' \cdot \hat{r})^2} - \frac{1}{\varepsilon_h} \right] + \frac{\varepsilon_h}{\varepsilon_h} \cdot E = 0
\]

where \( N = k'c/\omega \) is the (real) refractive index; \( k' \equiv \text{Re}k \) is the real part of the wave vector \( k \), (\( |k'| \gg |k''| \), \( k'' \equiv \text{Im}k \) being the imaginary part of \( k \)); \( \varepsilon_h \equiv \varepsilon_h(k',\omega,\xi,t) \) is the (slowly varying) Hermitian part of the dielectric tensor and \( E = E(k',\omega,\xi,t) \) is the (slowly varying) Fourier component of the wave electric field.

As is well known [1], for perpendicular propagation, that is, when condition (2) is satisfied, the two fundamental modes near the extraordinary (X) and ordinary (O) mode, are decoupled from one another and can, therefore, be discussed separately.

3.1. Extraordinary mode

For \( k_y' = 0 \) and \( B_0 = 2|B_0| \), for the X-mode, which is elliptically polarized in a plane perpendicular to \( B_0 \), Eq. (8) yields

\[
N^2 = \varepsilon_{yy,h} + \frac{\varepsilon_{xy,h}^2}{\varepsilon_{xx,h}}
\]
with \( N_1^2 \equiv (k_x c/\omega)^2 \), whereas for the wave amplitudes

\[
\frac{E_x}{E_y} = \frac{\varepsilon_{xy,h}}{\varepsilon_{xx,h}}
\]

holds.

**Fundamental frequency:** The (real) refractive index of the X-mode near the fundamental frequency is obtained from (9) by making use of the expressions for \( \varepsilon_{ij,h} \) given in Table I,

\[
N_1^2 = \begin{cases} 
1 - \frac{1}{2} \left( \frac{\omega_p}{\omega_c} \right)^2 \frac{\varepsilon_e}{\varepsilon_t}^2 F_{\gamma / 2}^\ast (z_1), & \left( \frac{\omega_p}{\omega_c} \right)^2 < 2 \left( \frac{\nu_t}{c} \right)^2 \\
2 - \left( \frac{\omega_p}{\omega_c} \right)^2, & \left( \frac{\omega_p}{\omega_c} \right)^2 > 2 \left( \frac{\nu_t}{c} \right)^2 
\end{cases}
\]

For the propagation of the first harmonic X-mode in a finite density plasma, Eq. (lib) requires that \( \omega_p^2 < 2 \omega_c^2 \).

As far as the polarization is concerned, from (10) one finds that in the tenuous plasma limit the wave is linearly polarized in the \( y \)-direction, i.e. \( E = \psi E_y \), whereas in the finite-density regime

\[
\frac{E_x - iE_y}{E_y} \approx iN_1^2 \left( \frac{\nu_t}{c} \right)^2 \frac{1}{F_{\gamma / 2}^\ast (z_1)} \left[ \left( \frac{\omega_c}{\omega_p} \right)^2 - F_{\gamma / 2}^\ast (z_1) \right]
\]

with \( N_1^2 \) given by (lib). From (12) it appears that the mode polarization deviates from circular by terms of order \((\nu_t/c)^2\), FLR effects, proportional to \( F_{\gamma / 2}^\ast \), contributing as well as the term, proportional to \((\omega_c/\omega_p)^2\), that contains the cross-effect of finite density and temperature. Near resonance, \( |F_{\gamma / 2}^\ast (z_1)| \approx 1 \), so that FLR effects are important only for \( \omega_p^2 \approx \omega_c^2 \). It is interesting to note that \( (E_x - iE_y)/E_y \approx (\varepsilon_{\gamma / 2}^\ast)^{-1} \), with \( \varepsilon_{\gamma / 2}^\ast \) the Hermitian part of the dielectric tensor to lowest significant order in \((\nu_t/c)^2\). Since \( |\varepsilon_{\gamma / 2}^\ast| >> 1 \) for finite densities, i.e. for \((\omega_p/\omega_c)^2 >> 2(\nu_t/c)^2\), the polarization of the X-mode near the fundamental frequency is almost circular (in the direction opposite to the gyration of the electrons) and hence the resulting absorption is expected to be much weaker than that corresponding to the tenuous plasma limit.
ELECTRON CYCLOTRON ABSORPTION

2nd harmonic: With the expressions for \( \varepsilon_{ij,l} \) given in Table II for \( \omega \approx 2\omega_c \), Eq. (9) leads to a biquadratic equation for \( N_{l,\pm}^2 \) which yields two branches [24]:

\[
N_{l,\pm}^2 = \frac{-(1 + b) \pm [(1 + b)^2 + 4aN_{l,c}^2]^{1/2}}{2a}
\]  
(13)

with

\[
a = -\frac{1}{2} \left[ \frac{\omega_p}{\omega_c} \right]^2 \frac{\omega_2 - \omega_c^2}{\omega_c} - \frac{\omega_2}{\omega_c} P'_{1/2}(z_2)
\]  
(14)

\[
b = -2 \left( 1 - \frac{\omega_p^2}{\omega_c(\omega + \omega_c)} \right) a
\]  
(15)

and

\[
N_{l,c}^2 = \frac{(\omega_2^2 - \omega_c^2)^2 - \omega_2^2 \omega_c^2}{\omega_2(\omega_2 - \omega_c^2 - \omega_c^2)}
\]  
(16)

Far from resonance, i.e. for \( |z_2| >> 1 \), \( |P'_{1/2}(z_2)| << 1 \) and from (13) - (15) one gets \( N_{l,+}^2 \approx N_{l,c}^2 \), whereas \( N_{l,-}^2 \approx (-1/a) \), requiring \( a < 0 \), i.e. \( P'_{1/2}(z_2) > 0 \), for propagation to occur; furthermore, the condition \( \lambda = \left( (k_1 v_t) / \omega_c \right)^2 << 1 \) requires that \( (1 <) N_{l,c}^2 << (\omega_c / \omega_c)^2 (c/v_t)^2 \). Near resonance, however, and for densities and magnetic fields such that \( \omega_2^2 \approx \omega_c^2 \), significant deviations of the branch \( N_{l,+}^2 \) from the Appleton-Hartree solution (16) can occur. Note that for \( \omega = \omega_c \), (16) yields the same result as (11b).

For the mode polarization, from (10) and the explicit expressions of \( \varepsilon_{xx,l} \) and \( \varepsilon_{xy,l} \), one obtains

\[
\frac{E_x}{E_y} = -\frac{i}{2} \left( \frac{\omega_p}{\omega_c} \right)^2 \frac{1 + 3N_{l,c}^2 P'_{1/2}(z_2)}{3 - (\omega_p^2/\omega_c^2)[1 - 3N_{l,c}^2 P'_{1/2}(z_2)]} \equiv -i a_2
\]  
(17)

with \( N_{l,c}^2 \) given by (13). It appears that effects due to both finite density and the combination of finite density and FLR, the latter ones being proportional to \( P'_{1/2} \), make the polarization deviate from linear. Moreover, for \( \omega_2^2 \approx \omega_c^2 \) and \( N_{l,c}^2 |P'_{1/2}(z_2)| > 1 \), the latter condition being possible for the large wavenumber.
branch only, (17) yields $E_x \approx -i E_y$, i.e. a right-handed (with respect to the confining magnetic field $B_o$) circular polarization.

Higher harmonics ($\omega \approx n \omega_c$, $n \geq 3$): Equation (9) together with the expressions for $\epsilon_{ij,h}$ given in Table II yields

$$N_1^2 = N_{1,c}^2 = \frac{[n^2 - (\omega_p/\omega_c)^2]^2 - n^2}{n^4 - n^2 - 1 - (\omega_p/\omega_c)^2}$$

(18)

where (16) has been used for $\omega = n \omega_c (> \omega_p)$. To have propagation one has to require $N_1^2 > 0$, i.e.

$$\omega_p^2 < n(n-1)\omega_c^2$$

(19a)

or

$$(n^2 - 1)\omega_c^2 < \omega_p^2 < n^2\omega_c^2$$

(19b)

For the mode polarization, Eq. (10) leads to

$$\frac{E_x}{E_y} = -i \left(\frac{\omega_p}{\omega_c}\right)^2 \frac{1}{n[n^2 - 1 - (\omega_p/\omega_c)^2]} \equiv -i a_n$$

(20)

3.2. Ordinary mode

The O-mode propagating perpendicular to $B_o$ is much simpler to treat. For $k'_y = 0$ and $B_o = \frac{2}{3}|B_o|$, the dispersion relation which follows from (8) is

$$N_1^2 = \epsilon_{zz,h}$$

(21)

the mode being linearly polarized with the wave electric field along $B_o$, i.e. $E = \hat{z}E_z$.

Fundamental frequency: Inserting the expression for $\epsilon_{zz,h}$ given in Table I into (21) yields

$$N_1^2 = \frac{1 - \left(\frac{\omega_p}{\omega_c}\right)^2}{1 + \frac{1}{2}\left(\frac{\omega_p}{\omega_c}\right)^2 F_{1/2}(z_1)}$$

(22)
from which it appears that, notwithstanding the perpendicular propagation, the dynamics of the O-mode is affected by the magnetic field as a result of the combined effect of the (relativistic) cyclotron resonance and the finiteness of the Larmor radius.

Harmonic frequencies ($\omega \approx n\omega_c$, $n \geq 2$): From (21) and the expression for $\varepsilon_{zz,h}$ given in Table II one obtains

$$N_1^2 = 1 - \frac{d^2}{\omega^2}$$

(23)

so that, for $\omega = n\omega_c$, $N_1^2 > 0$ for

$$\frac{d^2}{n^2\omega_c^2}, \ n \geq 2$$

(24)

4. ELECTRON CYCLOTRON ABSORPTION

To clarify the physics of the wave absorption around the electron cyclotron harmonics it is convenient to evaluate the absorption coefficient by making use of the energy conservation law [10]. To first order of the WKB approximation, under steady-state conditions and for the case in which the (slow) spatial variations associated with the energy absorption are dominant with respect to the variations due to the plasma inhomogeneities, the conservation law of electromagnetic energy (Poynting theorem) has the form (see, e.g., [1])

$$2k'' \cdot S = \frac{\omega}{4\pi} E^* \cdot \varepsilon_a \cdot E$$

(25)

which yields the absorption coefficient

$$\alpha = 2k'' \cdot S = \frac{\omega}{4\pi} \frac{E^* \cdot \varepsilon_a \cdot E}{|S|}$$

(26)

as the ratio between the absorbed power, $\frac{\omega}{4\pi} E^* \cdot \varepsilon_a \cdot E$, and the total flux of electromagnetic energy

$$S \equiv \frac{C}{4\pi} \text{Re} (E \times B^*) - \frac{\omega}{8\pi} \frac{2\varepsilon_{ij,k}^*}{\kappa} E^*_i E^*_j$$

(27)
where the first and second term on the right-hand side of (27) represent respectively the Poynting vector and the flux of the "sloshing" energy. In (25) - (27), \( s \equiv \Re \frac{S}{|S|} \); \( \kappa'' = \Im \kappa \) and \( \kappa' \equiv \Re \kappa \) denote respectively the imaginary and real part of the wave vector; \( \omega \) is the (real) wave frequency; \( \mathbf{E} \) and \( \mathbf{B} \) are the Fourier components of the wave electric and magnetic field (the asterisk denotes the complex conjugate); \( \mathbf{g} = \mathbf{g}_h + i \mathbf{g}_a \) is the dielectric tensor, \( \mathbf{g}_h \) and \( \mathbf{g}_a \) being the Hermitian and anti-Hermitian part of \( \mathbf{g} \), respectively. Note that a further condition for the validity of (25) is \( |\kappa''| < |\kappa'| \), but no smallness assumption is required for \( \mathbf{g} \) or \( \mathbf{g}_a \). We will make use of (26) together with the dielectric tensor given in Tables I and II to evaluate the absorption coefficient of both the X- and O-mode around the electron cyclotron harmonics for propagation perpendicular to the applied magnetic field \( \mathbf{B}_0 \) in a weakly relativistic \( (v^2_t << c^2) \) Maxwellian plasma.

4.1. Extraordinary mode

**Fundamental frequency:** By making use of the expressions for \( \mathbf{g}_a \) given in Table I and noting that \( \mathbf{E} = \hat{\mathbf{x}} \mathbf{E}_x + \hat{\mathbf{y}} \mathbf{E}_y \), one gets the absorbed power around the fundamental frequency, to lowest significant order in \( (v^2_t/c^2) \),

\[
\frac{\omega}{4\pi} \mathbf{E} \cdot \mathbf{g}_a \cdot \mathbf{E} = \frac{\omega}{4\pi} \mathbf{g}_a^{\nu\frac{1}{2}}(z_1) \left[ \left| \mathbf{E}_x - i \mathbf{E}_y \right|^2 - \frac{\mathbf{F}_{\mathbf{g}_a}^{\nu\frac{1}{2}}(z_1)}{\mathbf{F}_{\mathbf{g}_a}^{\nu\frac{1}{2}}(z_1)} \left( |\mathbf{E}_y|^2 - |\mathbf{E}_x|^2 \right) \right]
\]

\[
+ \frac{3}{2} \lambda^2 \frac{\mathbf{F}_{\mathbf{g}_a}^{\nu\frac{1}{2}}(z_1)}{\mathbf{F}_{\mathbf{g}_a}^{\nu\frac{1}{2}}(z_1)} \left( |\mathbf{E}_y|^2 \right)
\]

(28)

It appears that the absorption process has to do both the deviation from circular of the polarizations of the mode and the finiteness of the electron Larmor radius. In fact, the first and second term on the right-hand side of (28) are zero in the limit of (left-handed) circular polarization, whereas the second and the third vanish for zero Larmor radius.

To lowest order in the tenuous-plasma limit the wave is linearly polarized in the \( y \)-direction, \( \mathbf{E} = \hat{\mathbf{y}} \mathbf{E}_y \) (cf. Sec. 3.1), and the absorbed power, to zero order in the Larmor radius, is

\[
\frac{\omega}{4\pi} \mathbf{E} \cdot \mathbf{g}_a \cdot \mathbf{E} = \omega \mathbf{g}_a^{\nu\frac{1}{2}}(z_1) \frac{|\mathbf{E}_y|^2}{4\pi}
\]

(29)
for \((\omega_p/\omega_c)^2 < 2(v_e/c)^2\). Making use of (29) and (27) in (26) yields the absorption coefficient of the X-mode around the fundamental frequency

\[
\alpha_1^{(X)} = \frac{1}{2} \left(\frac{\omega_p}{\omega_c}\right)^2 \left(\frac{\omega_c}{v_t}\right)^2 \frac{\omega_c}{v} \left(-F_{5/2}^{0}(z_1)\right), \quad \left(\frac{\omega_p}{\omega_c}\right)^2 < 2\left(\frac{v_e}{c}\right)^2
\]

(30)

In contrast to the tenuous plasma limit, close to the fundamental resonance and for finite density, i.e. \((\omega_p/\omega_c)^2 > 2(v_e/c)^2\), the mode polarization is given by (12), so that all terms on the right-hand side of (28) are of the same order, i.e. \(O(v_e^2/c^2)\), and hence the FLR terms must be consistently retained. Inserting (12) into (26) - (28) yields

\[
\frac{\omega}{4\pi} \mathbf{E} \cdot \mathbf{E} = B \alpha^{(0)}(N_1c) \frac{|E_y|^2}{4\pi}
\]

(31)

\[
B = B \left(z_1, \frac{\omega_p^2}{\omega_c^2}\right) = \left[1 + \left(\frac{\omega_p}{\omega_c}\right)^2 \left(\frac{2}{3} |z_1| F_{5/2}'(z_1) - F_{5/2}'(z_1)\right)\right]^2
\]

\[
+ \frac{2}{2\sqrt{3}} \left(\frac{\omega_p}{\omega_c}\right)^4 (z_1 F_{5/2}'(z_1))^2
\]

(32)

\[
\alpha^{(0)} = \sqrt{\frac{\omega_p^2}{2\omega_c^2}} \left(\frac{\omega_c}{\omega_p}\right)^2 \left(\frac{v}{c}\right)^2 \frac{\omega_c}{v} \left(-F_{5/2}^{0}(z_1)\right)
\]

(33)

Noting that to lowest order in \((v_e/c)^2\) the flux of electromagnetic energy of the X-mode is simply the Poynting vector, \(P = (N_1c)|E_y|^2/4\pi\left(k_1'/k_1\right)\), Eq. (26) combined with (31) yields the absorption coefficient [9]

\[
\alpha_1^{(X)} = B \alpha^{(0)} \quad \text{for} \quad \left(\frac{\omega_p}{\omega_c}\right)^2 > 2\left(\frac{v_e}{c}\right)^2
\]

(34)

In the limit of zero Larmor radius, \(B = 1\) and the absorption coefficient is given by (33). Comparing \(\alpha^{(0)}\) with the absorption coefficient valid in the tenuous-plasma limit, given by (30), it appears that [6]:

1) the absorption is by a factor \(\left(\frac{\omega_p^2}{\omega_c^2}(c^2/v_e^2)F_{5/2}'(z_1)\right)^2\) (>> 1), typically, smaller than would follow from applying the result (30);
$a_{1}^{(X)}$ is normalized to its maximum value in the tenuous-plasma regime \[ \left( a_{1}^{(X)} \right)_{\text{Max}} = \frac{(3\pi)^{1/2}}{(2e)^{1/2} \omega_{p}^{1/2} c v_{t}} \frac{c^{2}}{v_{t}^{2}} \] and $z_{1} = \left( \frac{c^{2} \omega - \omega_{p}^{2}}{v_{t}^{2}} \right)^{1/2}$. Furthermore, the maximum $M_{1}$ of the finite-density profile is of the order of $\frac{\omega_{p}^{2} c z_{2}^{1}}{\omega_{c}^{2} v_{t}^{1}}$. The finite-density profile is proportional to

\[
\frac{|z_{1}|^{1/2} e^{-|z_{1}|}}{(z_{1})^{2}}
\]

whereas for the tenuous-plasma regime

\[
\frac{|z_{1}|^{1/2} e^{-|z_{1}|}}{(z_{1})^{2}}
\]
is proportional to $|z_{1}|^{1/2} e^{-|z_{1}|}$.

**FIG. 1.** Schematic plot of the line profile for the first harmonic extraordinary mode for perpendicular propagation ($N \cos \theta \leq v_{t}/c$), for tenuous plasma, \( \frac{1}{2} (\omega_{p}/\omega_{c})^{2} (c/v_{t})^{2} < 1 \) (solid curve), and for finite density, \( \frac{1}{2} (\omega_{p}/\omega_{c})^{2} (c/v_{t})^{2} > 1 \) (dashed curve).

ii) $a^{(0)} \sim (\omega_{c}/\omega_{p})^{2} (v_{t}/c)^{2}$, in contrast to the $(\omega_{p}/\omega_{c})^{2} (c/v_{t})^{2}$-scaling characteristic of the tenuous-plasma regime;

iii) the absorption-line shape is described by the function

\[
\frac{|z_{1}|^{1/2} e^{-|z_{1}|}}{(z_{1})^{2}} \quad \text{with} \quad z_{1} = \left( \frac{c^{2} \omega - \omega_{p}^{2}}{v_{t}^{2}} \right)^{1/2} \frac{c^{2}}{\omega_{c}^{2} v_{t}^{1}} < 0
\]

which is somewhat broader than the profile \( |z_{1}|^{1/2} e^{-|z_{1}|} \) of the tenuous-plasma limit, as qualitatively shown in Fig. 1.

As a result of the finiteness of the Larmor radius, the quantity $B$ deviates from $1$. From (32) one sees that it is, in fact, the value of $(\omega_{p}/\omega_{c})^{2}$ which weighs the importance of FLR effects in the absorption of the X-mode. In particular, $B > 1$ for $(\omega_{p}/\omega_{c})^{2} << 1$. The numerical analysis of $B$ given in Ref. 9 shows that:

i) $B < 1$, i.e. the absorption is reduced with respect to its value corresponding to the limit of the zero Larmor radius, $\lambda = 0$, the absorption being smaller than that in the limit of no Larmor radius effects by more than 50% when
(ω_P/ω_c)^2 > 1; ii) B is a monotonically decreasing function of (ω_P/ω_c)^2, so that the anomalous scaling of the absorption (i.e. the scaling of α^(0) like (ω_c/ω_P)^2) is further enhanced by FLR effects; iii) for (ω_P/ω_c)^2 ≤ 0.8, B is independent of z to within a few percent, i.e. the profile of the absorption is essentially the same as that for λ = 0, while for larger (ω_P/ω_c)^2 the profile tends to be flatter than in this reference case. In conclusion, since B ≈ 1, FLR effects, although modifying the scaling of absorption with plasma density and magnetic field as well as the line profile for ω_P ≈ ω_c, do not change the order of magnitude of the absorption coefficient.

Harmonics (ω ≈ nω_c, n ≥ 2): Near the n-th harmonic of the X-mode, the absorbed power is

$$\frac{\omega}{4\pi} E^* \cdot \varepsilon_a \cdot E = \frac{\omega}{4\pi} \varepsilon_{n+\frac{1}{2}}(z_n) |E_x - iE_y|^2$$

with

$$z_n = \left[ \frac{c}{v_t} \right]^2 \frac{\omega - n\omega_c}{\omega} (< 0), \ n \geq 2$$

Close to the second harmonic, the Poynting vector as well as the flux of sloshing energy

$$Q_s = \frac{1}{2} \left( \frac{\omega_P}{\omega_c} \right)^2 \left( N_1 C \right) \frac{\gamma_2}{\gamma_1} (z_2)^2 \frac{|E_x - iE_y|^2}{4\pi} \frac{K_2'}{K_1'}$$

contribute to the total flux (27); whereas for higher harmonics, n ≥ 3, the flux of the sloshing energy can be neglected, to lowest order in (v_t/c)^2. Equation (26), combined with (35) and (36), yields the absorption coefficient for ω ≈ nω_c,

$$a_n(X) = A_n \left[ a_n(X) \right] \omega_p^2 < \omega_c^2, \ n \geq 2$$

with

$$\left[ a_n(X) \right] \omega_p^2 < \omega_c^2, \ 2n^2 \left( \frac{v_t}{\omega_c} \right)^2 \left( \frac{\omega_c}{c} \right)^2 \left[ -P_{n+\frac{1}{2}}(z_n) \right]$$

$$A_n = N_1^{(2n-3)} \left( 1 + a_n \right)^2 b_n$$
The first factor on the right-hand side of (39) accounts for dispersion effects, \( N_l \) being given by (13) and (18), respectively, for the second harmonic and the harmonics with \( n \geq 3 \); the factor in (39) containing \( a_n \) describes polarization effects, \( a_n \) being defined in (17) for \( n = 2 \) and in (20) for \( n \geq 3 \). Finally, the quantity \( b_n \) is related to the energy flux entering the absorption process, cf. (25), the deviation from 1 being due to the flux of sloshing energy. Note that \( A_n \approx 1 \) for \( \omega_p^2 \ll \omega_c^2 \), and the absorption coefficient is given by (38), i.e. \( a_n(X) = \omega_p^2 \sim \omega_c^2 \) is the absorption coefficient of the \( n \)-th harmonic, \( n \geq 2 \), in the limit in which both dispersion and polarization effects are negligible. Note that (38) reduces to (30) for \( n = 1 \), the latter result, however, being valid under the condition \( (\omega_p/\omega_c)^2 < 2(v_t/c)^2 \).

4.2. Ordinary mode

Accounting for the linear polarization of the O-mode, the power absorbed near the \( n \)-th harmonic is

\[
\frac{w}{4\pi} E^* \cdot E = \frac{2}{n+1} \omega E_n^{\prime \prime} s_{1/2} (z_n) \frac{|E_n|^2}{4\pi}
\]

with \( n \geq 1 \).

In evaluating the absorption coefficient (26) near the first harmonic, one has to include the flux of sloshing energy together with the Poynting vector. Using also (41) with \( n = 1 \) yields

\[
a_1^{(O)} = \frac{1}{2} k_1 \left[ \frac{\omega_p}{\omega_c} \right]^2 \frac{\left| - E_{1/2} (z_1) \right|^2}{\left| 1 + \frac{1}{2} \left[ \frac{\omega_p}{\omega_c} \right]^2 \left( z_1 \right) \right|^2}
\]

From (42) it appears that for finite density \( a_1^{(O)} \) deviates somewhat from the corresponding absorption coefficient obtained in
The absorption coefficient \( \alpha_n^{(0)} \) is normalized to its maximum value at the fundamental frequency \( \left( \frac{\alpha_1^{(0)}}{\alpha_1^{(0)}} \right)_{\text{Max}} \sim \frac{2\pi}{3c} \left( \frac{\omega}{\omega_c} \right)^{3/2} k_1 \left( \frac{\omega}{\omega_c} \right) \), and \( z_n = \left( \frac{\omega}{\omega_c} \right)^{3/2} \frac{\omega - \nu_c}{\omega} \). The maximum \( M_2 \) of the second harmonic profile is of the order of \( \left( \frac{\nu_1}{c} \right)^2 \). The line profile of the second harmonic is proportional to \( |z_2|^{1/2} e^{-|z_2|^2} \), whereas for \( \omega_p^2 < \omega_c^2 \) the line profile of the first harmonic is approximately proportional to \( |z_1| \).\( \frac{1}{2} e^{-|z_1|^2} \).

**FIG.2.** Schematic plot of the line profile for the first and second harmonic ordinary mode for perpendicular propagation (\( N \cos \theta \lesssim \nu_t/c \)).

the low-density limit \( \omega_p^2 \ll \omega_c^2 \), for which \( 1 + \frac{1}{2} \left( \frac{\omega_p}{\omega_c} \right)^2 F_2' (z_1) \simeq 1 \), this deviation being related to the non-negligible contribution of the flux of sloshing energy to the total energy flux. The scaling and the order of magnitude of the absorption, however, are the same in the two cases. The profile of \( \alpha_1^{(0)} \) is essentially given by the function \( F_2' (z_1) \) and is shown schematically in Fig.2. Note that the condition for the validity of (42) is \( |k''| < |k'| \), which implies \( \omega_p^2 < \omega_c^2 \) near resonance.

Close to the harmonic frequencies \( \omega_c \), the Poynting vector is the only contribution to the energy flux, to lowest order in \( (\nu_t/c)^2 \), and (41) together with (26) yields the absorption coefficient

\[
\alpha_n^{(0)} = \frac{n (2n-1)}{z_n n!} \left( 1 - \left( \frac{\omega_p}{\omega_c} \right)^2 \frac{1}{\omega_c} \frac{\omega - \nu_c}{\omega} \right) \left( \frac{\nu_t}{c} \right)^{2(n-1)} \frac{\omega_c}{c} (-F''_{n+1/2} (z_n)) \quad (43)
\]

with \( n \geq 2 \). The absorption coefficient \( \alpha_2^{(0)} \) for the second-harmonic 0-mode is schematically shown in Fig. 2.
From a comparison of the different absorption coefficients, cf. (34), (37), (42) and (43), one sees that:

i) for $\omega_p^2 < \omega_c^2$, the first harmonic ordinary mode and the second harmonic extraordinary mode are the most strongly absorbed modes, the absorption coefficient being of the same order,

$$a_1(0) \approx a_2(0) \approx \frac{\omega_p^2}{c^2} \frac{\omega_c}{c}$$

(44)

ii) for $\omega_p^2 \lesssim \omega_c^2$, the absorption of the first and third harmonic extraordinary mode is of the same order as that of the second harmonic ordinary mode, and such that

$$a_1(X) \approx a_3(X) \approx a_2(0) \approx \left(\frac{\omega_p}{\omega_c}\right)^2 \left(\frac{v_t}{c}\right)^2 \frac{\omega_c}{c}$$

(45)

iii) for the other harmonics, $n \geq 3$ ordinary mode and $n \geq 4$ extraordinary mode, the absorption coefficient is of order $(v_t/c)^k$ or higher.

5. LINE WIDTH AND OPTICAL THICKNESS

5.1. Line width

As already mentioned, cf. also Figs 1 and 2, the cyclotron absorption has a profile which is given by the function $[-P_n(z_n)]$ defined in (5). By defining as line width, related to relativistic broadening, the quantity

$$\int_{\alpha_n}^{\omega_c} \frac{a_n}{\omega} d\omega$$

(46)

with $a_{n,max}$ the maximum (with respect to $\omega$) of the absorption coefficient, one gets

$$\frac{(\Delta\omega)_n}{\omega} = c_n \left(\frac{v_t}{c}\right)^2 , \quad n \geq 1$$

(47)
with

\[ c_n = \begin{cases} \sqrt{\frac{\pi}{2}} \left( \frac{e}{2n+1} \right)^{n+\frac{1}{2}} (2n+1)!! & \text{for the X-mode} \\ \sqrt{\frac{\pi}{2}} \left( \frac{e}{2n+3} \right)^{n+\frac{3}{2}} (2n+3)!! & \text{for the O-mode} \end{cases} \] (48a)

\[ (2n+1)!! = (2n+1)(2n-1)\cdots5\cdot3\cdot1 \]. The relativistic broadening (47) does not lead to line overlapping if \((\Delta\omega)_n < \omega_c\), that is to say, if \(n\sigma_n < (c/v_t)^2\).

5.2. Optical thickness

From the point of view of radiation transport, which is described by the equation of transfer [1], the interesting quantity is the optical thickness of the plasma, defined by integrating the absorption coefficient over the optical path [1], i.e.

\[ \tau = \int_{0}^{l} \sigma(x) \, dx \] (49)

where \(l\) is the corresponding path length. The factor \(e^{-\tau}\) describes the attenuation the wave suffers in traversing the medium. E.g., a simple solution of the equation of transfer is [1]

\[ I = I_{bb}(1 - e^{-\tau}) \] (50)

where \(I\) is the intensity of radiation and \(I_{bb} = N^2(\omega^2\pi/8\pi^3c^2)\) is the blackbody intensity in the medium. The factor \(e^{-\tau}\) in (50) accounts for the self-absorption of the radiation. Note that for \(\tau \gg 1\), an optically thick line, (50) yields \(I \approx I_{bb}\), i.e. the medium emits as a blackbody; for \(\tau \ll 1\), an optically thin line, from (50), \(I \approx I_{bb}\), and the medium is said to be optically transparent to the radiation.

As an example, let us consider the tokamak configuration, for which the (toroidal) magnetic field has a spatial variation of the form \(B_0(x) = B(0)/(1 + x/R_0)\), with \(R_0\) the major radius of the torus, \(B(0)\) the magnetic field on the axis of the plasma column and \(x\) a coordinate perpendicular to the torus axis. The corresponding effective spatial width of the cyclotron resonance...
\[ \omega = n \omega_c \] is, e.g., for light paths perpendicular to the torus axis, \( \Delta x \simeq (\Delta \omega/\omega)_n R_0 \), so that, from (47),

\[
\Delta x/a = c_n \left( \frac{v_t}{c} \right)^2 \left( \frac{R_0}{a} \right)
\]

(51)

with \( a \) the minor radius of the torus. It appears that the spatial width of the region where the resonance takes place is quite narrow; typically, \( \Delta x = (1/20)a \). One should note that \( \Delta x > \lambda \) (\( \equiv 2\pi/Re_k \), the wavelength) must be fulfilled for geometrical optics to be applicable.

For the tokamak configuration and light paths perpendicular to the torus axis, making use of the results for the absorption coefficient obtained in Sec. (4) yields the optical thickness given in Table III.

In Table III, \( \lambda_o \equiv 2\pi c/\omega_c \);

\[
\frac{B(z_1)}{[F_{\text{s/2}}(z_1)]^2} \equiv \frac{1}{\pi} \int_0^\infty \frac{B(z_1)}{[F_{\text{s/2}}(z_1)]^2} \left[ -F_{\text{s1/2}}(z_1) \right] \, dz_1 \approx 1
\]

with \( B(z_1) \) given by (32); for \( n \geq 3 \), \( \langle A_n \rangle = A_n \), defined by (39), whereas, for \( n = 2 \),

\[
\langle A_2 \rangle = \frac{1}{\pi} \int_0^\infty N_1 (1 + a_2)^2 b_2 \left[ -F_{\text{s1/2}}(z_2) \right] \, dz_2 \approx 1
\]

with \( N_1, a_2 \) and \( b_2 \) given, respectively, by (13), (17) and (40); for \( n \geq 2 \), \( \langle D_n \rangle = 1 \), whereas, for \( n = 1 \),

\[
\langle D_1 \rangle = \frac{1}{\pi} \int_0^\infty \left[ 1 + \left( \frac{\omega}{\omega_c} \right)^2 F_{\text{s1/2}}(z_1) \right]^{-\frac{3}{2}} \left[ -F_{\text{s1/2}}(z_1) \right] \, dz_1 \approx 1
\]

With regard to the scaling with the plasma density \( N_o \), temperature \( T \) and confining magnetic field \( B_o \), one has \( \tau^{(X)}_1 \sim N_o^{-1} B_o^3 T^2 \) for the case \( (\omega_c/\omega)^2 > 2(\nu_t/c)^2 \) relevant to a tokamak plasma, \( \tau^{(X)}_n \sim N_o^{-1} B_o^{-1} T^{n-1} \), \( n \geq 2 \), whereas \( \tau^{(O)}_n \sim N_o^{-1} B_o^{-1} T^n \), \( n \geq 1 \). For parameters typical of a tokamak plasma, one has \( \tau^{(X)}_1 \approx \tau^{(X)}_3 \ll 1 \), i.e. the first and third harmonic extraordinary mode are optically thin, from the diagnostic point of view the third harmonic being particularly appropriate for measurements of the plasma density. On the other hand, \( \tau^{(O)}_1 \approx \tau^{(O)}_2 \), i.e. the first harmonic ordinary mode and the second harmonic extraordinary
### TABLE III. OPTICAL THICKNESS OF BOTH EXTRAORDINARY AND ORDINARY MODES FOR A TOKAMAK PLASMA

<table>
<thead>
<tr>
<th>Mode</th>
<th>Optical Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>First harmonic</strong>&lt;br&gt;( \omega = \omega_C )</td>
<td>( \tau_1^{(X)} = \pi^2 \left( \frac{\omega_C}{\omega} \right)^2 \frac{R_0}{\lambda_0} ) for ( \left( \frac{\omega_C}{\omega} \right)^2 &lt; 2 \left( \frac{V_t}{c} \right)^2 )</td>
</tr>
<tr>
<td><strong>n-th harmonic</strong>&lt;br&gt;( \omega = n\omega_C, \ n \geq 2 )</td>
<td>( \tau_n^{(X)} = \frac{\pi^2 n^2}{2^{n+1}(n-1)!} \left( 1 - \left( \frac{\omega_C}{\omega} \right)^2 \right)^{n/2} \left( \frac{\omega}{\omega_C} \right)^{2(n-1)} \frac{R_0}{\lambda_0} )</td>
</tr>
<tr>
<td><strong>Ordinary Mode</strong>&lt;br&gt;( \omega = n_\omega_C, \ n \geq 1 )</td>
<td>( \tau_n^{(O)} = \frac{\pi^2 n^2}{2^{n+1}(n-1)!} \left( 1 - \left( \frac{\omega_C}{\omega} \right)^2 \right)^{n/2} \left( \frac{\omega}{\omega_C} \right)^{2n} \frac{R_0}{\lambda_0} )</td>
</tr>
</tbody>
</table>

Mode tend to be optically thick, the first harmonic ordinary mode appearing attractive for plasma heating, and the second harmonic extraordinary mode being appropriate for the diagnostics of the electron temperature.

### 6. CONCLUSIONS

The electron cyclotron absorption of electromagnetic waves propagating in a weakly relativistic \( (v^2 \ll c^2) \) Maxwellian plasma has been investigated on the basis of the energy balance equation for the case in which the relativistic effects due to the dependence of the electron mass on velocity are dominant \( (N \cos \theta < V_t/c, \ \text{propagation "perpendicular" with respect to the applied magnetic field}) \). We note that, in general, the evaluation of the absorption coefficient by means of the energy balance equation is much simpler than the corresponding evaluation of the emission coefficient \([20-22]\); moreover, the physics of the absorption process appears to be simpler than that of the emission process, e.g., particle correlation effects do not enter the
absorption process, whereas they are important in the emission process[20]. More specifically, it is found that:

i) for plasma parameters of practical interest \((\omega_p/\omega_c)^2 > 2(v_\perp/c)^2\) the absorption of the first harmonic extraordinary mode is strongly affected by polarization effects, resulting in both an anomalous scaling with respect to the plasma density (i.e. the absorption coefficient decreases with increasing density) and a drastic reduction of the absorption with respect to that predicted by the tenuous-plasma approximation \((\omega_p/\omega_c)^2 < 2(v_\perp/c)^2\);

ii) for \(\omega_p^2 \approx \omega_c^2\), finite density effects combined with finite Larmor radius effects enter the dispersion relation of the second harmonic extraordinary mode producing a splitting of the (cold) Appleton-Hartree branch; moreover, effects due to dispersion, polarization and the flux of sloshing energy have to be consistently accounted for in the evaluation of the absorption coefficient;

iii) for \(\omega_p^2 \approx \omega_c^2\), the propagation and absorption of both the ordinary mode and the n-th harmonic extraordinary mode, \(n \geq 3\), are only slightly affected by dispersion and polarization effects, the absorption coefficients derived here being equal to the ones following from Kirchhoff's law and the emission coefficients evaluated by Audenaerde [20].

The optical thicknesses of a tokamak plasma have been evaluated with the result that:

i) the first harmonic ordinary mode and the second harmonic extraordinary mode tend to be optically thick and are, respectively, suitable for plasma heating and the diagnostics of the electron temperature;

ii) the first and third harmonic extraordinary mode are optically thin, the third harmonic being particularly appropriate for the diagnostics of the plasma density.

Finally, let us note that the relativistic regime \((N \cos \theta < v_\perp/c)\) of electron cyclotron absorption considered here matches with continuity to the regime in which the effects due to finite \(k_\parallel \equiv k \cos \theta\) are dominant over the relativistic effects \((N \cos \theta > v_\perp/c)\) [6].
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INTERNAL KINK MODES IN TOROIDAL PLASMAS WITH CIRCULAR OR NON-CIRCULAR CROSS-SECTION

M.N. BUSSAC, R. PELLAT
Centre de physique théorique,
Ecole polytechnique,
Palaiseau

D. EDERY, J.L. SOULE
Association Euratom-CEA,
Centre d’études nucléaires,
Fontenay-aux-Roses,
France

Abstract

INTERNAL KINK MODES IN TOROIDAL PLASMAS WITH CIRCULAR OR NON-CIRCULAR CROSS-SECTION.

The stability criterion and linear growth rate of the internal kink modes are given for different magnetic configurations. A method is introduced which allows a simple computation of the internal kink potential energy so long as the modulation of the poloidal magnetic field remains small.

INTRODUCTION

Internal and main disruptions are observed to be an important macroscopic behaviour of the tokamak configuration. The main disruption is not fully understood and it is not yet known if it may be avoided in long discharges of future installations like JET. Internal disruptions, at least in the available plasma conditions of actual experiments, are in fact favourable to plasma confinement: by their non-linear relaxation (‘sawtooth’ behaviour) they prevent current peaking on the magnetic axis and help to keep a more or less round current profile. In tokamak experiments, the disruptions are, up to now, accepted as closely related to the resistive tearing modes [1]. In thermonuclear plasmas of higher electronic temperature, the plasma resistivity will become negligible and the standard tearing theory breaks. The $m = 2,3$ modes ($m$ is the poloidal wavenumber) have no ideal MHD counterpart and kinetic effects remain important.

1 And see also Ref. [7].
The \( m = 1 \) mode associated with the internal disruption will become essentially an ideal MHD internal kink mode. The results obtained for this mode are reviewed in this paper. The interesting feature of the subject is that the internal localized modes are likely be harmless for small enough plasma pressure (the poloidal \( \beta_p = 2 \mu_0 p / B_p^2 \) will remain smaller than 1 in the JET device). If the internal kink may be stable in toroidal-shaped geometry for values of the safety factor \( q < 1 \), then an important improvement of the confinement properties is expected (smaller \( q \) implies larger \( \beta_p \)).

The main difficulty in studying the internal kink in the ideal MHD limit is the smallness of its MHD potential energy [1]. Recall that external kink modes have a potential energy of order \( B^2 T / 2 \mu_0 (\xi / R_0)^2 \) if \( \xi \) is the displacement of the linearized plasma motion, \( B_T \) the toroidal field and \( R_0 \) the main toroidal radius. The internal kink has a potential energy of order

\[
(r_0 / R_0)^2 \frac{B^2}{2 \mu_0} \left( \frac{\xi}{R_0} \right)^2
\]

i.e. smaller by the square of the inverse of the aspect ratio. This is the main reason why its stability may be strongly affected by toroidal or by shaping effects (non-circular cross-sections). To be more precise and to give more insight into these effects, let us take a Fourier component of an Alfvén mode \( \bar{\xi}_{m,n} \exp (imx - iv) \) in toroidal geometry (\( x \) is a poloidal angle defined in an intrinsic system of coordinates and \( v \) is the toroidal angle). Then this mode is coupled by the toroidal modulation \( r_0 / R_0 \cos x \) of the main magnetic field \( B_T \bar{\varphi} \) to the modes \( \xi_{m \pm n+1,n} \). Let us symbolically call \( \bar{\xi}_{mn} \xi_{mn} \) the system of equations for \( \xi_{mn} \) in cylindrical geometry. In toroidal geometry, we obtain a coupled system:

\[
\bar{\xi}_{mn} \xi_{m,n} = \Phi \partial_x \left( \frac{r}{R_0} \right) \cos x \xi_{m \pm 1,n},
\]

and an extra contribution of order \( (r/R_0)^2 \) to the cylindrical dispersion relation. A similar coupling is expected if we include modulations of the poloidal field in straight geometry. The coupling is now between \( \xi_{m,n} \) and \( \xi_{m \pm 1,n} \) (\( \ell \) is the Fourier modulation of the poloidal field; \( \ell = 2 \) for an elliptical cross-section; \( \ell = 3 \) for a triangular cross-section). It is clear that these effects will also lead to corrections of order \( \delta^2_2, \delta^2_3 \) in \( \delta W \) (\( \delta_{2,3} \) are the corresponding amplitudes of the poloidal field modulations). These effects, \( (r/R_0)^2, \delta^2_2, \delta^2_3 \), are additives and are independent corrections to the cylindrical geometry (for small enough modulations). This is the main simplification introduced in our work.
Another important remark concerns the growth rate computation. As is well known in cylindrical geometry, the solutions of the Euler equations for the energy principle are singular on a magnetic surface when the helicoidal perturbation has locally the pitch of the equilibrium magnetic field. This singularity may be removed by including the plasma inertia in a layer radially localized in the vicinity of the singularity. Here, to compute the growth rate, we first minimize the volume energy outside the singular layers, taking into account the matching conditions across the singular layers. In the layers we include the inertia and compute the growth rates as a function of the volume potential energy. This method can be used for any geometry, provided the departure from the cylindrical geometry with circular cross-section can be expanded in small parameters.

The paper is organized as follows. In Section 1 we introduce our general method and recover the result of the cylindrical pinch [1]. In Section 2 the internal kink for non-circular cross-sections in the cylindrical pinch is computed. Section 3 discusses the toroidal pinch with circular cross-sections. The main result of Sections 2 and 3 is the possibility of stabilizing the cylindrical internal mode in toroidal geometry or by poloidal shaping, which shows that high-temperature installations may achieve \( q < 1 \) on the magnetic axis, at least in the ideal MHD limit. The lowest value of \( q \) has to be computed case by case and comes from the fact that if \( q = 1/n \) the cylindrical result is expected to hold — for large values of \( n \). Numerically, a combination of elliptical and triangular shaping appears to be very efficient for suppressing the internal modes. On the other hand, the stabilizing toroidal effects already become negligible for \( q \approx 1/2 \).

1. INTERNAL KINK IN CYLINDRICAL GEOMETRY FOR CIRCULAR CROSS-SECTIONS

We recall Shafranov's result [1] for the internal kink of a cylindrical pinch with a circular cross-section. We include the corresponding demonstration for completeness and because we have a different approach which is more convenient for complex geometries.

The classical tool in MHD stability theory is the 'energy principle' [2]. One constructs the potential energy \( \delta W \) of the linearized motion in the vicinity of a given equilibrium. Then if \( \delta W > 0 \) for any displacement \( \xi \), the equilibrium is stable. If \( \delta W < 0 \) for one displacement \( \xi \), the equilibrium is unstable [2].

For an equilibrium with no vacuum where the plasma is bounded by a perfectly conducting shell (coincident with a magnetic surface) \( \delta W \) reads:

\[
\delta W = -\frac{1}{2} \int \left( \nabla \cdot (\vec{\xi}, F^{\vec{\xi}}) \right) \, d\tau
\]
where $\tau$ is the plasma volume and

$$F(\xi) \equiv -\nabla(\nabla \cdot \mathbf{P}_0 - \gamma \rho_0 (\nabla \cdot \xi)) + \frac{1}{\mu_0} (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 + \mathbf{J}_0 \times \mathbf{Q}$$

(1)

$\mathbf{Q}$, the perturbed magnetic field, is given by $\mathbf{Q} = \nabla \times (\mathbf{E} \times \mathbf{B}_0)$; $\rho_0$, $\mathbf{B}_0$, $\mathbf{J}_0$ are respectively equilibrium pressure, magnetic field and current density; $\gamma$ is the compressibility factor. The solution of the set of equations $F(\xi) = 0$ provides the minimization of $\delta W$. For arbitrary equilibria this set can only be solved approximately, the internal kink being a good example of such a procedure. Let us finally recall that an unstable displacement, if any, has a growth rate $\Gamma$ given by $\delta H \equiv \frac{1}{2} \int \rho_0(\xi, \xi) \, d\tau + \delta W = 0$ where $\rho_0$ is the plasma equilibrium density and $\delta H$ the Hamiltonian of the linearized motion. A very useful property of $\delta W$ is its self-adjointness, i.e. for two displacements $\xi_1$, $\xi_2$ one has $\delta W(\xi_1, \xi_2) = \delta W(\xi_2, \xi_1)$. It is convenient to make a first integration by part and write:

$$\delta W = \delta W_1 + \delta W_2$$

where

$$\delta W_1 = \frac{1}{2} \int d\tau \left\{ \mathbf{Q} \cdot \left( \frac{3}{\mu_0} + \mathbf{J}_0 \times \xi \right) + \mathbf{P}_0 (\nabla \cdot \xi) \right\}$$

$$\delta W_2 = \frac{1}{2} \int d\tau \left\{ \gamma \rho_0 (\nabla \cdot \xi)^2 \right\}$$

A second integration by part in the subset $\tau_1$ of the volume $\tau$ is convenient for $\delta W_1$, which can be written:

$$\delta W_1(\tau_1) = \frac{1}{2} \int_{\tau_1} d\tau \left\{ \mathbf{B}_0 \times \nabla \times \left( \frac{3}{\mu_0} + \mathbf{J}_0 \times \xi \right) \right\} - \frac{1}{2} \int_{S_1} (\mathbf{B}_0 \cdot \mathbf{B}_0) \cdot \left( \frac{3}{\mu_0} + \mathbf{J}_0 \times \xi \right)$$

(2)

where $S_1$ is the magnetic surface giving the bounds of the volume $\tau_1$. On this last expression it is easily seen that $\delta W_1$ is independent of the component $\xi_1$ of $\xi$ along $\mathbf{B}_0$. As is well known, one of the Euler equations $\mathbf{B} \cdot F(\xi) = 0$ implies $\nabla \cdot \xi = 0$ if the equilibrium has a finite shear of magnetic field lines. Since $\delta W_1$ is independent of $\xi_1$, this constraint can be ignored and the minimization achieved by choosing $\xi_1$ such that $\nabla \cdot \xi$ vanishes. The last expression of $\delta W_1(2)$ also shows the necessity of the continuity of $\mathbf{B}_0 \cdot ((\mathbf{Q}/\mu_0) + \mathbf{J}_0 \wedge \xi)$ on a magnetic surface for a solution of the Euler equation. This property completes the continuity of the normal component $\mathbf{Q} \cdot \mathbf{n}$ and is in fact equivalent to the continuity of the first-order perturbation of the total pressure $p + B^2/2\mu_0$. 
As we are dealing with the internal kink, we take the safety factor \( q \) of order unity and limit our investigation to poloidal \( \beta_p \) also of order unity. (The precise definition of \( \beta_p \) suitable for the internal kink will be given later.) We now summarize what we know from previous work about the internal kink and then explain our systematic approach.

In \( \delta W_1 \) the first term is, a priori, of order \( B_T^4 \) if \( B_T \) is the main magnetic field component (axial in cylindrical geometry, toroidal in a real tokamak). The method for minimizing \( \delta W \) is equivalent to a choice of \( \xi \) such that this largest term is reduced by four orders of magnitude to order \( (r_0/R_0)^4 B_x \) where \( r_0 \) is the radius of the plasma column, \( R_0 \) the radius of the magnetic axis. In cylindrical geometry, the displacement is Fourier-analysed in \( \theta \) and \( z \), \( \xi(r) \exp(i(m\theta-kz)) \), the longitudinal wave number being equivalent to \( n/R_0 \) where \( n \) is the toroidal wave number, \( (r_0/R_0)^2 \) is replaced by \( (kzr_0)^2/n^2 \), and the cylindrical geometry is a good approximation of a tokamak if \( n \gg 1 \). (In the final section the complete result is given for any value of \( n \).)

For the minimizing displacement we also know [3] that for \( \beta_p \leq 1 \) the pressure term in \( \delta W_1 \) is of order \( (r_0/R_0)^4 B_T^4 \) because the perpendicular motion of the plasma is almost divergence-free and the axial component of \( \xi \) is of order \( (r_0/R_0)^2 \). Let us now summarize Shafranov's result for the displacement corresponding to the internal kink in cylindrical geometry. We consider a Fourier component \( m = 1 \) such that \( rkBz/mB_T = 1 \) in \( r = r_1 \) (in toroidal geometry this is equivalent to taking \( nq = 1 \) on a magnetic surface). To simplify the presentation we assume \( q \) to be monotonically increasing in \( 0 < r < r_0 \). After an exact minimization due to Newcomb [4], Shafranov finds the radial component \( \xi_r \) of \( \xi \) has the following properties: \( \xi_r = X_1 \), constant for \( 0 \leq r < r_1 - \epsilon \) and vanishes for \( r_1 + \epsilon < r < r_0 \). This solution allows an explicit computation of the \( \delta W_1 \). In fact, this solution is only the leading term of an expansion in the inverse of the aspect ratio. If one computes with the Newcomb equations \( \xi_\theta \) and \( \xi_z \), a divergence is found which can be removed when including the second-order term in \( \xi_r \) (it will be given later). Shafranov's solution has also to be completed in the layer \( r_1 - \epsilon < r < r_1 + \epsilon \) by a \( \delta \xi_r \) computed in the following way [5]. In this boundary layer one can make a plane sheet approximation, \( \nabla \cdot \delta \xi = 0 \), \( \delta \xi_\theta \sim ir_1 d\delta \xi_r/dr \), the kinetic energy is included and one obtains for the residual \( \delta H \):

\[
\delta H = \frac{1}{2} \int_{r_1^-}^{r_1^+} r^2 \frac{d}{dr} \delta \xi_r^2 \left. \right|_{r_1^-}^{r_1^+} + \int_{r_1^-}^{r_1^+} r^3 \frac{d}{dr} \left( \frac{d}{dr} \delta \xi_r^2 \right) + \delta W_1(X_1, X_1) = 0
\]

with the boundary conditions \( \delta \xi_r \rightarrow 1,0 \) for \( r \rightarrow r_1 \pm \epsilon \). This result assumes implicitly that \( \delta W \) has been minimized with respect to \( \xi_\parallel \) for \( \Gamma = 0 \) and neglects
the volume contribution of $\xi_r$ in the inertial term. We shall come back to a more systematic computation of the growth rate at the end of this section. Shafranov's method for evaluating $\delta W_i(X_1, X_1)$ cannot be generalized to more complex geometries because the Euler equation cannot be reduced to one equation on $\xi_r$.

We now illustrate our general algebra for the cylindrical result. We start from the last expression (2) of $\delta W_i(\tau_1)$ where $\tau_1$ now includes the subsets $(0, r_1 - \epsilon)$ and $(r_1 + \epsilon, r_0)$. We deduce from (2) an interesting form of the Euler equations:

$$\bar{F}_1(\xi) = \bar{F}_0 \times [\nabla \times \left( \frac{\partial}{\mu_0} + \bar{J} \times \xi \right) - \bar{J} \nabla \cdot \xi] = 0$$

Let us remember that the parallel component $\xi_l$ can be minimized independently. Then, as usual, if $dq/dr \neq 0$, this equation is minimized as

$$v \times \left[ \frac{\partial}{\mu_0} + \bar{J} \times \xi \right] = \bar{J} v \cdot \xi = 0$$

For the displacement $\xi$ which minimizes $\delta W$, the second term will be of the right order of smallness with respect to $r_0/R_0$. Hence it can be ignored. The Euler equation is then reduced to

$$\frac{\partial}{\mu_0} + \bar{J} \times \xi = v \xi$$

(4)

Even this simple equation cannot be solved exactly for the internal kink but we know that it is sufficient to find an expansion of $\xi$ in $r_0/R_0$ which reduces $\delta W_i$ to an order $B_T r_0/R_0^4$. This will be done now independently of the explicit equilibrium current profile and, consequently, of the magnetic field gradient. To illustrate this property of the equation we write Eq.(4) in a very illuminating form:

$$v(\xi - \frac{\bar{B}}{\mu_0} - \bar{D} \xi) = - \frac{\bar{B}}{\mu_0} \nabla \xi$$

(5)

where $\bar{D} \xi$ is a symmetric tensor:

$$\bar{D} \xi = \nabla \xi + (v^T \xi - \bar{F} \nabla \xi)$$

This form of the Euler equations shows that it is, a priori, possible to choose $\xi$ independently of the equilibrium properties as a solution of $\bar{D} \xi = 0$. In fact, as already said, it is sufficient to find a systematic expansion procedure which reduces
δW_t to the necessary order. The expression for $\tilde{D}$ suggests that one should first decouple the poloidal and toroidal components, which is obviously achieved with

$$\tilde{D} = \nabla \cdot \nabla - \nabla \cdot \nabla$$

With this choice we simplify the expression for $\tilde{D} \xi$ if we introduce the complex coordinates $X + iY = \alpha, \ X - iY = \bar{\alpha}$:

$$\tilde{D} \xi = 2i \tilde{\sigma}_z \left[ \frac{\delta^2 \xi}{\delta \alpha^2} + \frac{\delta^2 \xi}{\delta \alpha^2} \right] + 2i \tilde{\sigma}_x \left[ \frac{\delta^2 \xi}{\delta \alpha^2} - \frac{\delta^2 \xi}{\delta \alpha^2} \right] + \tilde{I} \frac{\delta^2 \xi}{\delta z^2} - \epsilon \epsilon_z \Delta \xi$$

(6)

$\tilde{\sigma}_x, \tilde{\sigma}_z$ are the Pauli matrices; $\tilde{I}$ is the unit poloidal tensor.

The largest contribution in $B \cdot \tilde{D} \xi$ comes from the $B_z$ component, which suggests limiting the family of solutions for $\xi$ by assuming

$$\Delta \xi \equiv 4 \frac{\delta^2 \xi}{\delta \alpha \delta \bar{\alpha}} - \kappa^2 \xi = 0$$

Then we solve this equation by expansion in powers of $k^2r^2$, starting from Shafranov’s solution which in these coordinates is $V = \alpha X_1$. We obtain

$$V = X_1 [\alpha + \frac{k^2}{8} \alpha^2 \bar{\alpha} + \ldots] \ldots e^{-ikz}$$

and our final result for $\tilde{D} \xi$:

$$\tilde{D} \xi = X_1 \left\{ \frac{k^2}{2} \alpha (\tilde{\sigma}_z + i \tilde{\sigma}_x) - \kappa^2 \alpha \tilde{I} \right\} e^{ikz}$$

or in cylindrical coordinates:

$$\tilde{D} \xi = X_1 \left\{ \frac{k^2}{2} r e^{i \theta} (\tilde{\sigma}_z + i \tilde{\sigma}_x) - \kappa^2 r e^{i \theta} \tilde{I} \right\} e^{-ikz}$$

Now after straightforward algebra we obtain

$$\tilde{B}_o \cdot \tilde{D} \xi = X_1 \left\{ \frac{k^2 r B_\theta}{2} e^{i(\theta - kz)} \right\} \left\{ i e_r - 3 e_\theta \right\}$$

(7)
It is sufficient to replace $\Phi$ by its expression in $\delta W_1$ to recover the final result of Shafranov! The first part of the volume integral reads:

$$\frac{1}{2} \int_0^{r_1-\varepsilon} \mathrm{d} r \cdot \left\{ \vec{B}_o \times \nabla \times \left( \frac{\vec{B}_o}{\mu_o} + \vec{J} \times \vec{B}_o \right) \right\} = \frac{1}{2} \int_0^{r_1-\varepsilon} \mathrm{d} r \cdot \left\{ \vec{B}_o \times \nabla \times \left( \frac{\vec{B}_o}{\mu_o} \right) \right\}$$

In this integral, using the self-adjointness of $\delta W_1$, it is sufficient in the scalar product to take $\vec{B}$ at its lowest order in $kr$. We obtain by unit length $dz$:

$$\frac{1}{2} \int_0^{r_1-\varepsilon} \mathrm{d} r \cdot \left\{ \vec{B}_o \times \nabla \times \left( \frac{\vec{B}_o}{\mu_o} \right) \right\} = \frac{\pi}{\mu_o} \int_0^{r_1-\varepsilon} \mathrm{d} r \left[ k^2 B^2_0 r^2 \right] + \frac{3}{4} \frac{d}{dr} \left( k^2 B^2_0 r^2 \right) + 2B_z B_0 k^3 r X_1^2$$

The pressure term is easily computed if the value of $\nabla \cdot \vec{B} = 2k^2 r X_1$ is deduced from our result:

$$+ \frac{1}{2} \int_0^{r_1-\varepsilon} \mathrm{d} r \cdot \left\{ \nabla \cdot \vec{B}_o \right\} = 2\pi \int_0^{r_1-\varepsilon} \mathrm{d} r \frac{dp}{dr} k^2 r^2 X_1^2$$

Let us now consider the last contribution, the surface integral on $r = r_1 - \varepsilon$:

$$- \frac{1}{2} \int_{S_1} r \mathrm{d}s \left( \nabla \cdot \vec{B}_o \right) \left\{ \vec{v} \cdot \left[ \frac{\vec{B}_o}{\mu_o} \right] - \frac{\vec{B}_o}{\mu_o} \vec{B}(\phi) \right\}$$

By substituting (7) we obtain

$$\frac{3\pi}{2\mu_o} k^2 B^2_0 \left| X_1 \right|^2$$

If we collect the three terms (8), (9), (10) of $\delta W_1$ we obtain Shafranov’s result:

$$\delta W = \pi \frac{B^2_1(r_1)}{\mu_o} \left( \frac{B^2_1(r_1)}{B_z} \right)^2 \left| X_1 \right|^2 \delta W_{\text{cycl}}$$

with

$$\delta W_{\text{cycl}} = -\beta_p(r_1) - \int_0^{r_1} \mathrm{d} r \left( \frac{r^3}{r_1^4} \right) (Q^{-2} + 2Q^{-1} - 3)$$

$$\beta_p(r_1) = -2\mu_o B^2_1(r_1) \int_0^{r_1} \left( \frac{r}{r_1} \right)^2 \left( \frac{dp}{dr} \right) \mathrm{d} r \quad \text{and} \quad Q = \frac{rB^2_1(r_1)}{r_1 B^2_1(r)}$$
\( \beta_p(r_1) \) appears as a definition of the poloidal \( \beta \), which usually remains finite on the magnetic axis. \( B_1 \) is the poloidal magnetic field.

We end this section by computing the growth rate. We have first to achieve the minimization with respect to \( \xi_1 \). Starting with \( \delta W \), we obtain

\[
\Gamma^2 \rho \frac{\delta}{\partial t} \frac{\partial \xi}{\partial t} = \nabla P_\parallel \nabla \cdot \frac{\partial}{\partial t} \left( \nabla \cdot \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial t} \right)
\]

where

\[
\frac{\delta}{\delta t} = \frac{\delta}{\left| \delta \right|} \quad \text{and} \quad \nabla \cdot \frac{\partial \xi}{\partial t} = 2k^2 r^2 x_1
\]

Our final result is that for \( \beta_p \leq 1 \) the growth rate is bounded by \( (r_1/R_0) \sqrt{\gamma P_\parallel /\rho_0} \). In this case the minimization of \( \xi_1 \) gives a residual contribution in \( \delta H \) of order \( (r_1/R_0)^2 \) as compared with the other terms and can be safely neglected.

Two contributions to \( \delta H \) remain: a boundary-layer integral and a volume integral, together given by

\[
\Gamma^2 \left\{ \frac{r_1^{-2}}{2} \int_0^{r_1^{-\varepsilon}} \rho \, d\tau \left( x_1^2 + \frac{1}{2} \int_{r_1^{-\varepsilon}}^{r_1^{+\varepsilon}} r \, dr \, \frac{d}{dr} \delta \xi_r^2 \right) \right\} + \int_{r_1^{-\varepsilon}}^{r_1^{+\varepsilon}} r \, dr \, \frac{\nabla \delta \xi_r}{\nabla \xi_r} \right|_{\partial \Omega} \left( \frac{\nabla \delta \xi_r}{\nabla \xi_r} \right)^2 + \delta W_1 = 0
\]

After the standard minimization with respect to \( \delta \xi [5] \) we obtain

\[
dz \rho \, r_1^2 \left\{ \frac{\pi}{2} - \left( \frac{k^2 (q'r_1)^2 B_0^2}{\mu_0 \rho_0} \right)^{1/2} \right\} x_1^2 + \delta W_1 = 0
\]

For large values of shear, \( q'r_1 > r_1/R_0 \), we obtain the usual result:

\[
\Gamma \sim \pi (r_1/R_0)^2 (V_A/R_0) \delta \tilde{w} c/(q'r_1)
\]

and for small values of shear, \( q'r_1/R_0 \), we obtain

\[
\Gamma \sim (r_1/R_0) (V_A/R_0) \delta \tilde{w} c / 2 \delta \tilde{w} c^{1/2}
\]

where \( V_A = (B_0^2/\mu_0 \rho_0)^{1/2} \) is the Alfvén velocity.
2. INTERNAL KINK IN CYLINDRICAL GEOMETRY FOR NON-CIRCULAR CROSS-SECTIONS

The interest in studying other configurations lies in the fact that tokamaks with non-circular cross-sections are under construction (JET, PDx, T11). All the following results may be found elsewhere [6] apart from the modification of the method introduced in Section 1. Here, we limit ourselves to small poloidal departures from the cylindrical circular case.

In cylindrical geometry, the equilibrium magnetic field is defined by

\[ \vec{B} = B_T \hat{e}_z + \hat{e}_z \Lambda \nabla F \]

where the poloidal flux \( F(r, \theta) \), will be taken as

\[ F = F_o(r) + \sum_{n \geq 1} F_n(r) \cos n\theta \]

with \( F_n/F_o = \varepsilon_n < 1 \) (\( R, \theta, z \) are the usual cylindrical coordinates).

The equilibrium equation \( \Delta F = J_z(F) \) can be solved by expansion for a given explicit \( J_z(F) \). The magnetic surfaces \( \rho(F) \) are easily found; we set \( r = \rho + \Delta r \) and find

\[ \Delta r = - \sum_n \frac{F_n}{dF_o/dr} \cos n\theta + \ldots \]

Then in the case of non-circular cross-sections the equilibrium depends on two small parameters:

\[ \varepsilon_n = \frac{F_n}{F_o} \quad \text{and} \quad \varepsilon = a/R_o \sim -\frac{B_\perp(a)}{B_T} \]

(\( a \) measures the radius of the plasma column).

In Section 1 we have seen that the plasma's potential energy \( \delta W \) is, a priori, \( \geq 0 \), so that instability arises only for displacements which cancel the lowest-order term in \( \delta W \): \( Q_z^2 \cong B_\perp^2 (\xi/a)^2 \). Moreover, in the circular cross-section case, the internal kink is also marginally stable at order \( \varepsilon^2 B^2_\perp \), so that the available energy is of order \( \varepsilon^4 B^2_\perp \).

Here, because of the departure from circularity, the internal kink is no longer marginal at order \( \varepsilon^2 B^2_\perp \) providing that \( 1 \gg \varepsilon_n \gg \varepsilon \). And we expect the available energy to be of order \( \varepsilon^2 \varepsilon_n^2 B^2_\perp \). Then, if we assume \( \varepsilon_n = F_n/F_o \gg \varepsilon \sim a/R_o \), we may
neglect the contribution of the pressure terms in the plasma potential energy as we know that $\text{div} \xi$ is of order $\varepsilon_t^2 (\xi/a)$ to avoid positive $\delta W_i$. We then take

\[
\delta W \simeq \delta W_1 = \frac{1}{2\mu_0} \int dz \left\{ \xi \cdot \overrightarrow{B}_0 \wedge \nabla \Lambda \left( \frac{\partial}{\partial \mu_0} + \overrightarrow{j} \wedge \xi \right) \right\}
- \frac{1}{2\mu_0} \int dS(z, \overrightarrow{n}) \overrightarrow{B} \cdot \left( \frac{\partial}{\partial \mu_0} + \overrightarrow{j} \wedge \xi \right)
\]

and the Euler equation reduces to

\[
\nabla \left( \frac{\varepsilon_t^2 \xi}{2} \right) = - \frac{\overrightarrow{B}}{\mu_0} \overrightarrow{B}(\xi)
\]

where $\overrightarrow{B}(\xi)$ is given by (5). As we want to reduce $\delta W_i$ to order $\varepsilon_t^2 \xi^2$, the displacement $\xi$ has to be solution of the Euler equation at order 0 and $\varepsilon_t$.

We now show that the Shafranov solution is such a solution (apart from the boundary conditions). Indeed if we set $\xi = \xi_0 + \xi_D = \nabla \cdot \nabla \cdot \nabla \cdot \nabla \cdot \overrightarrow{V}_D - \nabla \cdot \nabla \cdot \nabla \cdot \nabla \cdot \overrightarrow{V}_D$, with $\overrightarrow{V}_D = (x + iy) e^{-i k z}$

\[
\overrightarrow{B}(\xi) = -\left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right) k^2 \overrightarrow{V}_D \sim \varepsilon_t^2 (\overrightarrow{V}_D / r^2)
\]

Then after integration

\[
\xi_D = \overrightarrow{B}_0 \cdot \nabla \overrightarrow{V}_D - \overrightarrow{B}_0 \cdot \nabla \overrightarrow{V}_D \quad \text{and} \quad \delta W_i = \frac{1}{2\mu_0} \int dS \left( \xi \cdot \overrightarrow{n} \right) \overrightarrow{B} \cdot \nabla \xi_D
\]

where $S_1$ is the surface where the factor

\[
q(F) = \frac{k B_T}{2\pi} \oint_{B_\perp} dt / B_\perp
\]

is equal to unity. When the cross-section is circular, $\delta W_i$ vanishes as $\overrightarrow{B} \cdot \nabla \Phi_D$ vanishes. This is not the case when the cross-section is non-circular, because of the modulation of $B_\perp$ on $S_1$. As this is also one of the difficulties of toroidal geometry, we shall explain it in detail.
Let us first define an intrinsic system of coordinates \((\rho, \chi, z)\)
\[ \chi = \rho - \sum_{n=1}^{\infty} \frac{1}{r} \frac{d}{dr} \left[ \frac{r F_n}{d F_0 / dr} \right] \frac{\sin n\rho}{n} \]
and
\[ \vec{B} \cdot \nabla \equiv k B \cdot \left( \frac{\lambda}{\delta \chi} - i \right) \text{ for } q(\rho_1) = 1 \]

With this choice, one easily obtains
\[ V_D = \rho_1 X_1 \exp i (\chi - kz) \left\{ 1 - \sum_{n=1}^{\infty} \frac{1}{2n} a_{1-n} \exp - in\chi + n \to n \right\} \]
with
\[ a_{1-n} = \frac{1}{r^{1+n}} \frac{d}{dr} \left( \frac{r^{1+n} F_n}{d F_0 / dr} \right) \]
and
\[ \vec{\xi}_D = i X_1 (k \rho_1 B_T) \exp i (\chi - kz) \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{2n} a_{1-n} \exp - in\chi + n \to n \right\} \]
\[ \vec{\xi}_D \cdot \nabla \rho = \xi_{Ni\rho} = X_1 \exp i (\chi - kz) \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{2n} a_{1-n} \exp - in\chi + n \to -n \right\} \]

It appears that \(\Phi_D\) has small but finite components \(m = 1 \pm n\) on \(\rho = \rho_1\). Although this means that the displacement \(\vec{\xi}_D\) is no longer an acceptable solution, we can easily remove this difficulty. The correct conditions of continuity for a solution of the MHD equations are the continuity of \(\vec{B} \cdot \nabla (\xi_N)\) and \(\vec{B} \cdot \nabla \Phi\) (see Section 1).

We must consequently complete the particular solution \((\vec{\xi}_D, \Phi_D)\) by a solution of the Euler equation, to fulfill the boundary conditions and the conditions of continuity on \(\rho\). We shall comment later on the boundary conditions and now concentrate on the conditions in \(\rho_1\), which are

\[ \xi_{Ni} + \xi_{ND} = \xi_{Ne} \] for the harmonics \(m = 1 \pm n\) on \(\rho = \rho_1\)

(12)
The components $m = 1 \pm n$ of $\xi_{ND}$ and $\Phi_D$ being of first order in $\xi_n$, it is sufficient to solve the cylindrical Euler equation with circular cross-section to compute $\xi_{i,e}$ and $\Phi_{i,e}$ ($\xi_{i,e} \sim \xi_n$):

$$\frac{d}{d\rho} \left( \rho^3 (1 - \frac{m}{q}) \frac{d}{d\rho} \xi^m \right) - \left( m^2 - 1 \right) \left( 1 - \frac{m}{q} \right)^2 \rho \xi^m = 0$$

(13)

when $\xi_{N,i}^m$ and $\xi_{N,e}^m$ are, respectively, solutions of (13) inside and outside $\rho = \rho_1$ ($\xi_N^m$ is the harmonic m of $\xi_N$).

In the circular cylinder, $\Phi_{i,e}$ can be easily expressed in terms of $\xi_{N,i,e}$:

$$\Phi^m = \frac{ikB_n \rho_1}{m} \left\{ (m + 1) \xi^m_N - (m - 1) \rho \frac{d\xi^m_N}{d\rho} \right\}$$

(14)

The numerical integration of the Euler equation will give the values of

$$A_{i,e} = \left[ \frac{c}{\xi_{N,i,e}} \frac{d\xi_{N,i,e}}{d\rho} \right]_{\rho = \rho_1}$$

We easily deduce

$$\xi_{N,i}^{1-n} = - \frac{n-1}{2n} \frac{[n - 2 - A_{i,e}^{1-n}]}{A_{i}^{1-n} - A_{e}^{1-n} a^{1-n} x_1}$$

Let us now return to the $\delta W$. If $\bar{\xi}_D + \bar{\xi}$ is the complete solution ($\bar{\xi}_D + \bar{\xi}_i$ inside and $\bar{\xi}_e$ outside) the potential energy reads:

$$\delta W(\bar{\xi}_D + \bar{\xi}, \bar{\xi}_D + \bar{\xi}) = \delta W(\bar{\xi}_D^*, \bar{\xi}_D) + 2\delta W(\bar{\xi}_D^*, \bar{\xi}_e) + \delta W(\bar{\xi}_e, \bar{\xi}_e)$$

Integrating by part, using the properties of $\bar{\xi}$ and $\bar{\xi}_D$, we obtain, respectively,

$$\delta W(\bar{\xi}_D^*, \bar{\xi}_D) = \frac{1}{2\mu_o} \int dz \, d\chi \rho_1 [\bar{\xi}_{DN}^* \vec{B} \cdot \nabla \bar{\xi}_D]$$

$$\delta W(\bar{\xi}_D^*, \bar{\xi}_e) = \frac{1}{2\mu_o} \int dz \, d\chi \rho_1 [\bar{\xi}_{iN}^* \vec{B} \cdot \nabla \bar{\xi}_e]$$

$$\delta W(\bar{\xi}_e, \bar{\xi}_e) = \frac{1}{2\mu_o} (dz \, d\chi \rho_1 [\bar{\xi}_{Ni}^* \vec{B} \cdot \nabla \bar{\xi}_i - \bar{\xi}_{Ne}^* \vec{B} \cdot \nabla \bar{\xi}_e])$$
After some straightforward algebra we obtain

\[ \delta W = \int dz \, \delta W_S (\xi, \xi_D) = \frac{1}{2\mu_0} \int dz \, \delta \chi \otimes \left[ \varepsilon_{Ni}^* \mathbf{B} \cdot \nabla \mathbf{B}_D - \varepsilon_{ND}^* \mathbf{B} \cdot \nabla \mathbf{B}_i \right] \]

and consequently

\[ \frac{d}{dz} \delta W = \delta W_S (\xi, \xi_D) = - \frac{\pi}{4\mu_0} (kB_n \rho_1)^2 \times \sum_{n>1} \left( \frac{2}{a_1^{-n}} \frac{(n-2 - A_i^{-1-n})(n_2 - A_e^{-1-n})}{A_i^{-1-n} - A_e^{-1-n}} \chi_1^2 + n \to n \right) \]

(15)

We still have to solve explicitly the Euler equation for \( \xi_{i,c} \) (13), with the given matching condition on \( \rho = \rho_1 \) (12). The displacement \( \xi_i \) must vanish on the magnetic axis; the displacement \( \xi_e \) must be finite between \( \rho = \rho_1 \) and \( a \), and vanish at the boundary of the plasma \( \rho = a \) if the plasma is confined by a perfectly conducting wall. Two situations may occur:

(a) If \( q(a) \) is smaller than \( n + 1 \), \( \xi_e \) is regular in \( (\rho_1, a) \);
(b) If \( q(a) \) is larger than \( n + 1 \), \( \xi_e \) is singular on \( \rho_{n+1} \) where \( q(\rho_{n+1}) = n + 1 \) and one must take the regular solution of the Euler equation, which is finite for \( \rho \to \rho_{n+1}^- \) and vanishes for \( \rho > \rho_{n+1} \).

This analysis closely follows the classical Newcomb analysis: it is sufficient to minimize \( \delta W \) and look for a condition of stability but this has to be somewhat modified if one is interested in the growth rate of an unstable mode. We must include the inertia inside each singular layer \( \rho = \rho_1, \rho = \rho_{n+1} \) (if any) and also remove the constraint of taking the small solution, \( (\xi_n^{n+1}) \) finite, in the vicinity of \( \rho = \rho_{n+1} \).

Let us consider the Hamiltonian \( \delta H \) of the linearized motion:

\[ \delta H = \frac{1}{2} \Gamma^2 \int \rho \, d\tau \, |\dot{\gamma}|^2 + \delta W(\eta, \eta) = 0 \]

We have correctly minimized this expression for \( \Gamma \to 0 \), i.e. when \( \delta W \to 0 \). If \( \Gamma = 0 \) we must add to the previous result the kinetic energy and potential energy inside the singular layers. We work out successively the contribution of each layer. The contribution of the layer \( \rho = \rho_1 \) is the usual one:

\[ \frac{1}{\mu_0} \int dz \, \Gamma \, kV_A \rho_1^2 \left( \frac{\partial}{q} \frac{\partial \Delta}{d\rho} \right) \rho = \rho_1 \]
If \( q(a) > n + 1 \), with a monotonie profile of \( q \) and \( \Gamma \neq 0 \), we have to correct the solution of the Euler equation (13) corresponding to \( m = n + 1 \). If \( \Gamma = 0 \), Eq. (13) has two regular solutions instead of one when \( \Gamma = 0 \). To clarify this point it is more convenient to introduce

\[
\psi^{n+1} = -kB_T \rho \left( \frac{1}{p} - \frac{1}{n+1} \right) \xi^{n+1}
\]

The small solution taken previously corresponds to \( \psi^{n+1} = 0 \) for \( \rho \geq \rho_{n+1} \) and has a discontinuity of slope for \( \rho = \rho_{n+1} \). To construct the complete solution, we add to \( \psi^{n+1}_c \) (which corresponds to \( \xi_c^{n+1} \)) a solution \( (\psi_c, \xi_c) \) of the homogeneous Euler equation for \( \delta H \). But that solution, being a \( m = n+1 \) mode, is coupled to the component \( m = n+1 \) of the displacement \( \xi_D \) on \( \rho = \rho_1 \). To compute its contribution to \( \delta H \) we consider successively the volume integral outside and inside the singular layer \( \rho = \rho_{n+1} \). Outside \( \delta H \sim \delta W \) and after integration by part we obtain

\[
\frac{d}{dz} \delta W(\xi_D, \xi^{n+1}_c, e + \xi^c)
\]

\[
= \delta W_S(\xi^{n+1}_e, e, \xi^c) + \delta W_S(\xi^c, \xi^c) + \frac{\pi}{\mu_o} \left( \rho \Delta', \psi^2 \right)_{\rho = \rho_{n+1}}
\]

To obtain the last term it is easier to start from an equivalent form of \( \delta W \) [7]:

\[
\delta W = \frac{1}{2\mu_o} \int dz \int_{\rho_{n+1}} \delta e \{ (\psi_c \cdot \nabla \psi)_c \rho_{n+1} - (\psi_c \cdot \nabla \psi)_c \rho_{n+1} - \}
\]

\( \rho \Delta' \) has the meaning that is familiar in the resistive mode theory:

\[
\rho \Delta' = -\frac{\psi_c}{\psi_c} \left\{ \left( \frac{d\psi}{d\rho} \right)_c \rho - \left( \frac{d\psi}{d\rho} \right)_c \rho - \right\}
\]

Inside the singular layer \( \rho = \rho_{n+1} \) we proceed as for \( \rho = \rho_1 \). We minimize \( \delta H \) with respect to \( \xi_c, N \). The boundary condition is now

\[
\psi_c = -(kB_T \rho \frac{d\alpha}{d\rho})_{\rho_{n+1}} (\rho - \rho_{n+1}) \delta_{\xi_c, N} \quad \text{for} \quad \frac{\rho - \rho_{n+1}}{\Delta \rho_{n+1}} \gg 1
\]
We obtain

\[ \frac{\Delta^2}{\mu_0} \int dz \left[ \frac{\psi^2}{\Gamma'} \right]_0 = c_{n+1} \]

with \( \Gamma' = \frac{\Gamma a^2}{kV_A \rho \frac{dq}{d\rho}} \).

The total \((d/dz) (\delta H)\) Hamiltonian per unit longitudinal length is now given by

\[ \frac{d}{dz}(\delta H) = \delta W_S(\xi_c^i, \xi_c^D) + \frac{1}{\mu_0} \left[ X_{\frac{2}{3}} 1 \Gamma' (kB_T \rho \frac{dq}{d\rho} \right]_0 + \delta W_S(\xi_c^i, \xi_c^D) + \frac{\pi}{\mu_0} (\psi^2 (\rho_1 - \frac{\pi}{\Gamma'}) \rho_{n+1} \]

(16)

with the new volume contribution \(\delta W_S(\xi_c^i, \xi_c^D)\). This contribution defined by

\[ \delta W_S(\xi_c^i, \xi_c^D) = \frac{1}{2\mu_0} \int_{\rho_1}^{\rho_0} d\rho \left[ \xi_c^N, \frac{B_\rho B_\rho}{\rho_1} \xi_c^D - \xi_c^N, \frac{B_\rho B_\rho}{\rho_1} \xi_c^D \right] \]

is easily computed and given by

\[ \delta W_S(\xi_c^i, \xi_c^D) = \frac{\pi}{2\mu_0} (kB_T \rho_1)^2 \frac{n a}{(n+1) (n+2) A_i^{n+1}} x_1 \xi_c^N (\rho_1) \]

It is more convenient to replace \(\xi_c^N (\rho_1)\) in this expression by

\[ \xi_c^{n+1} (\rho_1) = - \frac{\psi_c (\rho_1)}{\psi_c (\rho_{n+1})} \frac{\psi_c (\rho_{n+1})}{\rho_1 kB_T} \frac{(n+1)}{n} \]

A final minimization of the bilinear form \(\delta H (15)\) upon \(X_1\) and \(\psi_c (\rho_{n+1})\) gives the dispersion relation:

\[ 2(\rho_1 + \frac{\pi}{\Gamma'}) \left\{ \frac{4}{\pi} \Gamma' (\rho \frac{dq}{d\rho} \right]_0^{n+1} \]

\[ \sum_{n>1} \frac{a^{n+1}}{n+1} \frac{(n+2) A_i^{n+1}}{(n+2) A_i^{n+1}} + n \rightarrow -n \]

\[ = \left\{ a^{n+1} (n+2 + A_i^{n+1}) \frac{\psi_c (\rho_1)}{\psi_c (\rho_{n+1})} \right\}^2 \]
When $\Gamma \to 0$, the condition for instability reads $\delta W_0(\vec{E}, \vec{E}_D) \geq 0$, as already established, but the expression for the growth rate is somewhat unusual and has not yet been correctly computed [6]. The explicit values of $A_{ie}$ need an integration of the cylindrical Euler equation. The integration can be done analytically only if the radial current profile is a step function, but the results have been shown numerically [6] to be representative of the general case. To summarize, a triangular ($n = 3$) and a quadrangular ($n = 4$) deformation are stabilizing and their effect is larger than the (destabilizing) effect of an elliptical deformation ($n = 2$).

To conclude this section, let us recall that plasmas with elliptical cross-sections are unstable [3] with regard to axisymmetric displacements ($n = 0, m = 1$) in which case the displacement must be controlled by external windings.

3. INTERNAL KINK IN TOROIDAL CONFIGURATION

In this section, we study the stability, the structure of the eigenfunction and the growth rate of the internal kink mode in a toroidal plasma with circular cross-section. We limit ourselves to large aspect ratio $\varepsilon^{-1} = R_0/a \gg 1$. Even with this restriction, the result differs greatly from Shafranov's, essentially because the internal kink is nearly marginal for any circular cross-section configuration. Hence it is very dependent on any departure from circularity of the magnetic surfaces. Here, the modulation of the poloidal field and the displacement of the magnetic surfaces couple the neighbouring harmonics of the displacement to the main one, $m = 1$. Although the amplitudes of the harmonics $m = 0, 2$ are of order $\varepsilon$, they contribute to the energy at the same order as $m = 1$.

When computing the growth rate of the internal kink, it will be necessary to consider at the same time the three harmonics and the dynamics in each resonant layer $n_q = 1, n_q = 2$ (if any) of the $m = 1$ and $m = 2$ components. First, if $n_q$ is smaller than 1 at the centre, and smaller than 2 at the plasma edge, there is only one resonant surface for the $m = 1$ mode ($n_q(r_1) = 1$). The harmonics $m = 0$ and $m = 2$ are completely defined by their boundary conditions and the Euler equation. The growth rate of all components is obtained from the (negative) value of $\delta W$ and from the treatment of the layer around $r = r_1$.

In the second case, when $n_q$ is larger than 2 at the boundary, the $m = 2$ harmonic, which would be stable alone, is pumped by the $m = 1$. Hence it is necessary to consider both amplitudes of the $m = 1$ and $m = 2$ as unknown and to derive two coupled algebraic equations. If the amplitude of the pumped $m = 2$ mode vanishes, the energy of the internal kink mode reduces to $\delta W_1$ without any coupling term. Similarly, if the amplitude of the kink mode vanishes, the energy for the $m = 2$ mode is $(-r \Delta')$, as in the tearing theory. The new coefficient (coupling term) is identical in both equations (because the energy expression is quadratic). The last quantities to be derived are the inertias of each layer.
We conclude that the fundamental problem to solve is the following. Find the perturbation which (approximately) minimizes $\delta W$ and properly satisfies the boundary conditions and the continuity conditions on the singular surfaces. To get this result, we expand the equilibrium and the perturbation in terms of $\varepsilon$ up to the significant order, $\varepsilon^2$. This being high and the expansion being tedious, we shall take several short cuts.

We use both the energy principle and the Euler equation $\overline{F}(\xi) \equiv 0$ (without inertia outside the layers) written as

$$
\delta W(\xi^*, \xi) = \frac{1}{2} \int \tau Q^* \cdot G, \quad \delta(\xi) = \xi + \xi \times \xi - (B\varepsilon)^{-1} (\text{div} \ \xi \ \varepsilon p)
$$

The $\gamma p$ term playing no role outside the layers, it has been dropped, and $\overline{F}(\xi) = 0$ is equivalent to

$$
\begin{align*}
\left\{ \begin{array}{l}
\varepsilon + \varepsilon \times \xi = \nabla \psi \\
\text{div} \ \xi = 0
\end{array} \right.
\end{align*}
$$

(17)

The $\xi$ term has to be known modulo a multiple of $B$. So it is sufficient to solve (18) instead of (17):

$$
\delta(\xi) = \nabla \phi
$$

(18)

Let us recall that, for axisymmetric configurations

$$
\bar{B} = T(F) \ \nabla \psi + \nabla F \times \nabla \psi
$$

where the poloidal flux $F$ is the solution of

$$
- R^2 \text{div} \left[ \frac{1}{R^2} \nabla F \right] = R J_\phi = T \frac{dF}{dF} + R^2 \frac{dp}{dF}
$$

$R$ being the major radius, $\phi$ the toroidal angle, $J_\phi$ the toroidal current density, and $p$ the plasma pressure. We limit ourselves to the case of circular cross-section, large aspect ratio $1/\varepsilon$, but arbitrary current and pressure profiles (poloidal $\beta$ of order unity).
As seen in Section 1, we expect $\delta W$ to be of order $\epsilon^2 (\int dr B^2_1 (\xi/a)^2)$. The a priori lowest-order term in $\delta W$, $Q^2_\varphi$, is of order $\epsilon^{-2} (B^2_1 (\xi/a)^2)$, and has to be reduced to order $\epsilon^2$. Then to get instability, $Q_\varphi$ must be of order $\epsilon$. To achieve minimization of $\delta W$, we have to determine $Q_\varphi$, $\xi_\varphi$ at order $\epsilon$, $Q_m$, $G_m$, $\xi_m$ and $\text{div} \vec{\xi}$ at order $\epsilon^2$. So we compute first an approximate solution $(\xi, \Phi)$ of (18):

$$G = v_\Phi + O(\epsilon^2 B_\perp \xi/a)$$

with $G_\varphi = 0(\epsilon B_\perp \xi/a)$ and get the corresponding values of $G_m$, $Q_m$ at order $\epsilon^2$.

As $\xi$ is constant at the lowest order, $\text{div} \vec{\xi}$ is of order $\epsilon$ and constant at this order. Thus if $\beta = (B \cdot \nabla)^{-1} (\text{div} \vec{\xi})$, $p\nabla \beta$ is of order $\epsilon^2$. So we can neglect the pressure term when solving the Euler equation and only reintroduce it when computing $\delta W$. Then Eq.(18) can be written:

$$\nabla \cdot \vec{D} = \nabla (\xi^2) = 0$$

or equivalently

$$\vec{B} \cdot \vec{D} (\xi) - \nabla (\vec{\xi} \cdot \vec{B}) = \nabla \xi$$

(19)

where

$$\vec{D} = \nabla \xi - \nabla \cdot \vec{\xi} \cdot \vec{\xi} \cdot \text{div} \vec{\xi}$$

Again, as in Section 2, we first look for a particular solution of Eq.(19), which minimizes $\delta W$ at order $\epsilon^2 \times \int dr (B_1 \xi/a)^2$. At lowest order $\epsilon$, the particular solution coincides with Shafranov's displacement, but at order $\epsilon$ the particular solution will not generally satisfy the continuity conditions. We therefore complete this solution, at order $\epsilon$, by general solution of the Euler equation in order to satisfy the continuity conditions. We choose the particular approximate solution of (19) (which is independent of the equilibrium) among the exact solutions of the Euler equation in the vacuum:

$$\vec{\xi}_\perp = R^2 \nabla_\perp V, \quad \xi_\varphi \text{ arbitrary (where } \perp \text{ means in a plane } \varphi = \text{cst})$$

$$\Delta V = 0$$

(20)
It is convenient to write the tensor $\mathbf{D}$ in the $R, \phi, Z$ basis:

$$
\begin{align*}
D_{RR} &= 2 \frac{d\xi_R}{dR} - \text{div} \xi \\
D_{R\phi} &= \frac{1}{R} \frac{d\xi_R}{d\phi} + R \frac{d}{dR} \left( \frac{\xi_\phi}{R} \right) \\
D_{RZ} &= \frac{d\xi_R}{dZ} + \frac{d\xi_Z}{dR} \\
D_{ZZ} &= 2 \frac{d\xi_Z}{dZ} - \text{div} \xi \\
D_{\phi\phi} &= \frac{1}{R} \frac{d\xi_\phi}{d\phi} - R^2 \text{div} \left( \frac{\xi_\phi}{R^2} \right)
\end{align*}
$$

As the component $\xi_\phi$ is arbitrary, we choose $\xi_\phi$ in order to cancel the elements $D_{R\phi}$ and $D_{Z\phi}$ of the tensor $\mathbf{D}$:

$$
\frac{\xi_\phi}{R} = \text{Cst} \ e^{-i\nu} + i n V = -\frac{i}{nR} e^{-i\nu} + i n V
$$

With this choice of constant, the elements of $\mathbf{D}_1 - D_{RR}, D_{ZZ}, D_{RZ}$ are null at lowest order for Shafranov's displacement: $\xi_R + i \xi_Z = (1 + i) e^{-i\nu}$.

$$
V = \frac{R - R_o + i Z}{R_o^2} e^{-i\nu}
$$

and $G_\phi(\xi)$ is of order $\mathcal{O}_B(1/\alpha)$.

It remains now to show the precise expansion of $V$ in $\mathcal{O}$. We choose $V$ a solution of $\Delta V = 0$, which at lowest order coincides with (21) and which cancels the elements of $\mathbf{D}_1$ at order 0 and $\mathcal{O}$. Then $V$ satisfies:

$$
\begin{align*}
\frac{\Delta}{\alpha^2} (\xi_R + i \xi_Z) &= 0 + \mathcal{O}(\alpha^2) \\
\frac{\Delta}{\alpha^2} (\xi_R - i \xi_Z) &= 0 + \mathcal{O}(\alpha^2) \\
\frac{\xi_R}{R} + \frac{e^{-i\nu}}{R_o} - n^2 V &= 0 + \mathcal{O}(\alpha^2)
\end{align*}
$$

(22)
with

\[ \vec{\xi}_1 = \frac{R}{2} \Delta \vec{\xi} \quad \text{and} \quad \alpha = \frac{R - R + \imath Z}{2}, \quad \tilde{\alpha} = \frac{R - R - \imath Z}{2} \]

Solving (20) and (22) order by order in \( \alpha/R_0, \tilde{\alpha}/R_0 \), we get

\[ RV = \frac{2\alpha}{R_0} + \frac{\alpha\bar{\alpha}}{R_0} + (n^2 - 5/4) \frac{\alpha^2}{R_0^3} - \frac{\alpha\bar{\alpha}}{4R_0^3} \ldots \quad (23) \]

For this particular solution \( D \varphi \varphi = -e^{-\imath n \varphi}/R_0 \). However, its contribution for any equilibrium is nearly a gradient. Finally, for \( \vec{D}_1 \) we get at order \( \varepsilon^2 \):

\[ \vec{D}_1 = \frac{1}{4R_0} \left\{ (4n^2 - 5) \left[ \begin{array}{c} 1 \\ i \\ 1 - 1 \end{array} \right] - \alpha \left[ \begin{array}{c} 1 - 1 \\ -i \\ -1 \end{array} \right] - [(4n^2 - 3) 2\alpha + 2\bar{\alpha}] \left[ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right] \right\} \]

and

\[ \text{div} \vec{\xi} = \frac{1}{R_0} + (4n^2 - 3) \frac{\alpha}{R_0} + \frac{\bar{\alpha}}{R_0} \]

From \( G_{\varphi}(\vec{\xi}) = \nabla \varphi \) we deduce \( \vec{\xi} + \vec{\xi} \varphi \vec{B} = \frac{T}{R_0} e^{-\imath n \varphi} \) and \( \vec{G}(\vec{\xi}) = \nabla T e^{-\imath n \varphi}/\imath R_0 \).

However, although \( \nabla \varphi (\vec{\xi} + \vec{\xi} \varphi \vec{B}) \) is of order \( \varepsilon (B_1 \xi/a) \), it does not contribute to the energy as

\[ \delta W = \frac{1}{2} \int d\tau \left( \vec{Q} \varphi (\vec{Q} + B_1 \vec{\xi}) \right) \]

\[ = \frac{1}{2} \int d\tau \frac{T_0}{R_0} \frac{T^2 - T_0}{R^2} R^2 \text{div} \left( \frac{\vec{\xi}_1}{R^2} \right) d\tau \sim \varepsilon^3 \int d\tau B_1 \frac{\vec{\xi}}{a} \]

where \( T_0 = T (r = 0) \). Then the particular solution

\[ \vec{\xi}_p = R^2 \varphi V - R^2 \varphi \varphi V - \frac{iR}{nR_0} e^{-\imath n \varphi} e\varphi \]

minimizes \( \delta W_1 \) at order \( \varepsilon^2 \).
At this point, we compute the value of $\delta W$ for $\xi_p$, at the significant order, in the domain inside the surface $nq(r_1) = 1$:

$$
\delta W = \frac{1}{2} \int d\tau \, \delta \mathbf{Q} \cdot \left[ \mathbf{B} \cdot \frac{\mathbf{B}}{R} - \mathbf{v} \left( \frac{\partial \mathbf{B}}{\partial t} \right) \right] + \mathbf{v} \mathbf{v} \cdot \left( \frac{T_o}{R_o} \right) e^{-i\phi}
$$

The guideline is to separate in $\mathbf{Q}$ terms which are gradients from terms which are of order $\xi^2$. Then, after integration by parts of the gradients, only surface terms and small volume contribution remain. Noticing that

$$
\mathbf{B} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{B} - \frac{T - T_o}{R_o} e^{-i\phi} \mathbf{v} \left( \frac{T_o}{R_o} \right) e^{-i\phi}
$$

we get $\delta W = \delta W_V + \delta W_S$ with

$$
\delta W_V = \frac{1}{2} \int d\tau \, \delta \mathbf{Q} \cdot \left[ \mathbf{B} \cdot \mathbf{B} - (p - p_1) \mathbf{v} - \frac{T - T_o}{R_o} e^{-i\phi} \mathbf{v} \right]
$$

$$
\delta W_S = \frac{1}{2} \int dS \left( \frac{\delta \mathbf{Q}}{nq} \right) \cdot \mathbf{v} \left( \frac{T_o}{R_o} \right) e^{-i\phi}
$$

where $p_1 = p(r_1)$

In $\delta W_V$ all the terms inside the bracket are of order $\xi^2$. The last one does not contribute as it is in the $\phi$ direction. Then to compute $\delta W_V$ we need only to know $Q^*$ at the lowest order, i.e. in the cylindrical approximation:

$$
\delta W_V = \frac{2\pi^2 T_o^2 n^2}{\mu_o R_o} \left| \xi_R \right|^2 \int_0^\infty r^2 dr \left( \frac{1}{2n} \frac{2}{2} \left( \frac{3}{4} - n^2 \right) \right)
$$

$$
+ \frac{2}{nq} \left( 1 - n^2 \right) + \frac{12n^2 - 11}{4} \left( \frac{1}{nq} - 1 \right)
$$

$$
+ 2\pi^2 \left| \xi_R \right|^2 \int_0^\infty r dr \left( p - p_1 \right) \frac{1 - 4n}{2}
$$

We now consider the complete solution of the Euler equation for a given equilibrium. Since the normal displacement must vanish at the plasma edge, the particular solution $\xi_p$, previously computed inside the resonant surface $nq = 1$, has to vanish in the whole domain outside $nq(r_1) = 1$. At order $\xi$, $\xi_p$ does not satisfy the continuity conditions for $\mathbf{B} \cdot \mathbf{v}(\xi_p \cdot \mathbf{v})$ and $\mathbf{B} \cdot \mathbf{v} \Phi_p$. Hence we have to
complete $\xi_e$ by another solution $\xi_{i,e}$ of the Euler equation. The amplitude of $\xi_{i,e}$ being of order $\mathcal{E}$, $\xi_{i,e}$ is the solution of the cylindrical Euler equation (13).

An important feature of the method is that the coupling between $\xi_p$ and $\xi_{i,e}$ involves only $\xi_p$ at order $\mathcal{E}$, when computing the energy. Indeed we write

$$
\delta W(\xi_p, \xi_c) = \frac{1}{2} \int d\tau \, \mathcal{E}(\xi_p) \left[ \mathcal{B}_1 \mathcal{B}_1(\xi_p) - (p - p_1) \mathcal{V} \right] + \frac{1}{2} \int dS \left( \frac{\mathcal{E}^2}{r_1} \right) \mathcal{B}_1 \cdot \mathcal{V} \left[ \mathcal{B}_p^2 - \frac{r_1}{R_o} \varepsilon - \text{in} \varphi \right]
$$

(26)

$\xi_{i,e} = \xi_1$ inside $\Omega(r_1) = 1$, and $\xi_{i,e} = \xi_e$ outside $\Omega(r_1) = 1$. In (26) the volume term is negligible ($\sim \mathcal{E}^3$). The surface term is of order $\mathcal{E}^2$ because $\xi_{i,e}$ is of order $\mathcal{E}$, and because on the surface $\Omega(r_1) = 1$ the contribution of the lowest-order term ($\mathcal{E}^0$) in $\xi_p$ cancels out. Moreover, as $\xi_{i,e}$ minimizes $\delta W$, the term $\delta W(\xi_c, \xi_c)$ can be reduced to a surface integral and is of order $\mathcal{E}^2$.

To take the surface contribution properly into account, we now introduce the intrinsic coordinate system $(\rho, \varphi, \chi)$ defined by

$$
\frac{2}{R_o} = \int_0^F \frac{a(F)}{T(F)} dF, \quad \chi = \frac{T}{q} \int_0^{\ell} \frac{d\ell}{R^2 B_1}
$$

In this system

$$
\mathcal{B}_1 \cdot \mathcal{V} = \frac{T}{R^2} \left( \frac{1}{q} \frac{\partial}{\partial \chi} + \frac{\partial}{\partial \varphi} \right), \quad d\tau = \frac{R^2}{R_o} \rho \, d\rho \, d\chi \, d\varphi
$$

For an equilibrium with circular cross-section, the centre of magnetic surfaces is displaced by $\Delta(\rho)$ and the poloidal magnetic field $B_1$ is modulated as

$$
1 + \lambda \frac{\rho}{R_o} \cos \chi \quad \text{with} \quad \frac{d\Delta}{d\rho} = -\frac{\rho}{R_o}(1 - \lambda)
$$

$\lambda$ can be expressed in terms of $q$ and $p$ as

$$
\lambda + 1 = \frac{1}{\rho} \int_0^\rho \frac{2 \frac{d\rho}{R_o}}{n q^2} - \frac{2}{\rho} \frac{R^4}{T^2} \int_0^\rho \frac{2 \frac{d\rho}{R_o}}{n q^2} d\rho
$$
The definition of $\lambda$ results in the relation between intrinsic and cylindrical coordinates:

$$R - R_0 + iZ = \rho e^{i\chi} + \frac{\rho}{2R_0} (\lambda + 2) (e^{2i\chi} - 1) + \Delta$$  \hspace{1cm} (27)

This system $(\rho, \varphi, \chi)$ coincides with the polar system $r, \varphi, \theta$ in the limit $\delta \to 0$. As the representation of the operator $R^2 \vec{B} \cdot \vec{V}$ is diagonal in this system, it is appropriate to Fourier-analyse the displacement as $e^{im\chi}e^{i\varphi}$. Finally, the location of the resonant surface is defined by $q(\rho)$ and the singularities of the perturbation can be consistently described.

In Fourier-analysing $\Xi = \hat{\vec{r}} \cdot \vec{V} \rho$ and $\Phi$, the condition of continuity must be fulfilled harmonic by harmonic. As the modulation of $B_j$ is proportional to $\cos \chi$ at the first order, we have to compute $\Xi_p$ and $\Phi_p$ for the harmonics $m = 0$, $m = 2$, on the surface $\rho = r_1$, $nq(r_1) = 1$. From (23) and (27), we get

$$\Phi_p = \frac{T}{\ln R_0} - \frac{\hat{\vec{r}} \cdot \vec{B}}{\hat{\vec{r}} \cdot \vec{V}} = T \left( \frac{\partial V_p}{\partial \varphi} - \frac{1}{q} \frac{\partial V_p}{\partial \chi} \right)$$

$$\Xi_p = R^2 (V_p \cdot V_p)$$

or, in terms of intrinsic coordinates,

$$V_p = \rho e^{i\chi} + \frac{\rho}{2R_0} (\lambda + 1) e^{2i\chi} - \frac{\rho}{2R_0} (\lambda + 5/2) + \Delta$$

$$\Xi_p = e^{i\chi} + \frac{\rho}{R_0} (\lambda + 3/2) e^{2i\chi}$$

$$\Phi_p = \frac{\ln T}{R_0} 2\rho e^{i\chi} + \frac{3}{2} \ln T \frac{\rho}{R_0} (\lambda + 1) e^{2i\chi} + (\xi_m^{m=0} \frac{T}{R} - \frac{T}{\ln R_0})$$

Now $\xi_m^{m=0}$ is chosen so that $\Phi_p^{m=0}$ vanishes on the resonance $\rho = r_1$ as $\Xi_p^{m=0}$. Hence $\xi_{i,e}$ does not contain any $m = 0$ harmonic. Noticing that knowledge of $\Xi_{i,e}$ and $\Phi_e$ is equivalent to knowledge of $\Xi_{i,e}$ and

$$A = \frac{c}{\Xi_{i,e}} \frac{d}{d_0} \Xi_{i,e}$$
(Eq.(14)), when \( \Xi_c \) satisfies the cylindrical Euler equation (13), we can express the continuity condition for \( \Xi_c \) in terms of \( \Xi_p \) and \( \Phi_p \):

\[
\Xi_p + \Xi_i = \Xi_e \quad \text{on } nq(r_1) = 1 \text{ for the } m = 2 \text{ harmonic}
\]

\[
\Phi_p + \Phi_i = \Phi_e
\]

If \( \Xi_i \) is the solution of the Euler equation which vanishes at the origin and which is equal to 1 on \( \rho = r_1 \), and if \( \Xi_e \) is the solution which vanishes at the edge of the plasma (and which is finite on \( nq(r_2) = 2 \)) and which is equal to 1 on \( \rho = r_1 \), then

\[
\Xi_i = \frac{r_1}{R_0} \left[ \frac{3(\lambda + 1/2) + A_i(\lambda + 3/2)}{A_i - A_e} \right] \Xi_e \quad \rho = r_1
\]

\[
\Xi_e = \frac{r_1}{R_0} \left[ \frac{3(\lambda + 1/2) + A_i(\lambda + 3/2)}{A_i - A_e} \right] \Xi_e \quad \rho = r_1
\]

Now having in hand the complete solution \( (\Xi_p + \Xi_{i,e}) \), which minimizes \( \delta W \), with the proper boundary and continuity conditions, we achieve the computation of \( \delta W \) by the contribution of \( \Xi_{i,e} \):

\[
\delta W(\Xi_i, e, \Xi_{i,e}) + 2\delta W(\Xi_i, e, \Xi_p)
\]

\[= \frac{1}{2} \int_{r_1} ds_i \Xi_i \vec{B} \cdot \nabla(2\Phi_p + \Phi_i) - \frac{1}{2} \int_{r_1} ds_e \Xi_e \vec{B} \cdot \nabla \Phi_e \]  

(28)

Replacing \( \Xi_{i,e} \) and \( \Phi_p, \Phi_{i,e} \) by their expressions and adding (28) to \( \delta W_S \) and \( \delta W_V \ ((24 \text{ and } 25)) \), we get

\[
\frac{\delta W_{IK}}{2\pi R_o} = n \pi 2 \frac{r_1^2 B_0^2(r_1)}{R_o^2} \left[ (1 - \frac{1}{n^2}) \frac{\delta W_{cyl}}{2} + \frac{\delta W_{T}}{n^2} \right]
\]
with

$$\delta W_T = - \frac{3}{4} \beta_p$$

$$- \frac{1}{4} s - \frac{[3 (\lambda + 1/2) + (\lambda + 3/2) A_i]}{4 (A_i - A_e)}$$

where

$$\beta_p = - 2 \left( \frac{R_o}{r_m B_T} \right)^2 \int_0^{r_1} \frac{p^2}{r_1^2} \frac{dp}{d\theta} d\theta$$

$$s = \int_0^{r_1} \frac{\beta_p}{r_1^4} \left( \frac{1}{n q} - 1 \right)$$

$$s = \frac{1}{n q} - 1$$

$$\delta W_{cycl}$$ is the potential energy for the internal kink in cylindrical geometry. In this result we observe that for \( n = 1 \) (and consequently \( q = 1 \)) there is a special property: the cylindrical contribution vanishes and the stability is driven by the other small contributions [8], which have been numerically verified [9]. The main result for \( n = 1 \) is that the internal kink is relatively easy to stabilize even for \( q(0) \geq 1/2 \) and poloidal values of \( \beta_p \) of order unity. If now \( q(0) < 1/2 \), the mode \( (m = 1, n = 2) \) remains cylindrical and unstable. So far, we have correctly minimized the Hamiltonian:

$$\delta H \equiv 1/2 \int d\tau \rho_0 |\vec{\xi}|^2 + \delta W(\vec{r}, \vec{\xi}) = 0$$

in the limit where the growth rate \( \Gamma \to 0 \). If \( \Gamma \neq 0 \) we must add to the previous result the kinetic energy and the potential energy inside the singular layers

$$n q(r_m) = m.$$  

The toroidal geometry introduces a modification of the correction \( \delta \xi \) in the layers \( r = r_m \). Minimizing \( \delta H \) with respect to \( \vec{r} \cdot B_T \), we find that \( \text{div}(\delta \xi) \) has to vanish. On the other hand, we know that \( R^2 \text{div}(\delta \xi_i / R^2) \sim 0(\zeta^2) \); then \( \delta \xi \) is given by:

$$\xi \phi = a B_T$$

with

$$\vec{B} \cdot \nabla a = - 2 \frac{\vec{B} \cdot \nabla R}{R}$$

This component of the displacement modifies the inertia by a factor \( (1 + 2q^2) \) in each singular layer. If at the plasma boundary \( n q(a) < 2 \), the contribution of the resonant layer \( r = r_1 \) is the usual one and the growth rate can be written:

$$\Gamma = - \pi \left( \frac{r_1}{R_o} \right)^2 \frac{V A}{R_o \sqrt{1 + 2q^2}} \left( \frac{r dq}{dr} \right)^{-1} \delta W_T$$

(30)
If $nq(a) > 2$, we must correct the solution of the Euler equation corresponding to $m = 2$. If $\Gamma \neq 0$, we add to $\xi_{i,e}$ a solution $\xi_c$ of the homogeneous Euler equation for $\delta H$. This solution is coupled to the component $m = 2$ of the displacement $\xi_p$ on the surface $r = r_1$. To compute its contribution to $\delta H$, we consider (as in Section 2) successively the volume integral outside and inside the resonant surface $r_2$, outside $\delta H \approx \delta W$, and after integration by part we obtain

$$\delta W = \delta W_{I,K} + \delta W(\xi_p, \xi_c) + \frac{\pi}{\mu_0} (r \Delta' \psi_c)_{r_2}$$

where

$$\psi_c = -\frac{n B_T}{R_o} \left( \frac{1}{nq} - \frac{1}{2} \right) \Xi_c$$

and

$$\delta W(\xi_p, \xi_c) = \frac{1}{2 \mu_0} \int_{r_1} \partial \psi \left[ \Xi_c B \cdot \nabla \Xi_p - \Xi_p B \cdot \nabla \Xi_c \right]$$

$$= \frac{\pi}{2 \mu_0} \left( \frac{R_1}{R_o} \right)^2 B_T^2 \left[ 3(\lambda + 1/2) + A_1 (7 + 3/2) \right] r_1 \psi_c (r_1) \psi_c (r_2)$$

Inside the singular layer $r = r_2$, we proceed as in Section 2 and get

$$\frac{\pi^2}{\mu_0} 2 \pi R_2 \left( \frac{\psi_c^2}{\Gamma^r} \right)_{r_2}^2$$

with

$$\Gamma' = \Gamma \sqrt{1 + 2q^2 \left( \frac{n V_A}{R_o} \frac{1}{q} \frac{d q}{d \theta} \right)^{-1}}$$

Then the Hamiltonian is given by

$$\delta H = \delta W_{I,K} x_1^2 + \frac{1}{\mu_0} \left[ x_2^2 \Gamma (\frac{n B_T}{R_o} \frac{2 d q}{d \rho} \right]^2 \left( \frac{1}{r_1} \right)_{r_1}^2 + \frac{\pi}{\mu_0} \left[ \psi_c^2 (\rho \Delta' + \pi r') \right]_{r_2}$$

$$+ x_1 \psi_c (r_2) \left[ \frac{\pi}{2 \mu_0} \frac{R_1}{R_o} B_1 (r_1) \left[ 3(\lambda + 1/2) + A_1 (\lambda + 3/2) \right] \psi_c (r_1) \psi_c (r_2) \right]_{r_2}$$
A final minimization of the bilinear from $\delta H$ upon $X_j$ and $\psi_c(r_2)$ gives the dispersion relation:

$$
\left( r \frac{\Delta}{\Gamma'} + \frac{\pi}{\Gamma'} \right) \left[ \frac{1}{\pi} \left( \frac{d}{dr} \frac{r \psi_c(r_2)}{\psi_c(r_1)} \right) \right]_{r_1} + \left( \frac{r_1}{R_o} \right)^2 \delta W_T = \left( \frac{r_1}{R_o} \right)^2 \frac{\psi_c(r_1)}{\psi_c(r_2)}
$$

$$
\times \left[ \frac{3(\lambda + 1/2) + A_i(\lambda + 3/2)}{4} \right]_{r_1}
$$

When $\Gamma \to 0$, the condition for instability reads $\delta W_T < 0$, as already established. Here, in the limit of zero resistivity, the contribution of the layer $m = 2$ to the growth rate is negligible since $\Gamma'$ scales as $\delta^2$, the other root, $\Gamma' \sim (r_\Delta)^{-1} \approx 1$, being out of the range of validity of the boundary layer method. Hence the growth rate (for finite shear) is given by the expression (30).

We end with a comparison between toroidal effects and non-circular shaping of the magnetic surfaces. We found for their respective potential energies:

$$
\delta W_T \sim O \left( \frac{F_n}{F_0} \right)^2
$$

and

$$
\delta W_{Nc} \sim O \left( \frac{F_n}{F_0} \right)^2
$$

In fact, a non-circular shaping, by a triangular cross-section ($n = 3$), appears to be numerically one order of magnitude larger for $F_n/F_0 \sim r_1/R_0$. If such a shaping is experimentally achieved the internal kink will be suppressed even for $q < 1/2$.

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TRANSPORT IN TOROIDAL MAGNETOPLASMAS

F. ENGELMANN
FOM-Instituut voor Plasmafysica "Rijnhuizen",
Jutphaas,
Netherlands

Abstract

TRANSPORT IN TOROIDAL MAGNETOPLASMAS.

The basic features of transport in toroidal magnetoplasmas is discussed in a way that
displays the strong similarity of collisional and anomalous transport problems. Collisional
transport in the collision-dominated regime is treated as an example of application of the
general theory in which explicit results can be derived in a simple way. The basic equations
for treating anomalous transport are written down, and a general analysis of anomalous
transport effects is given.

1. INTRODUCTION

Transport theory aims at calculating the fluxes of particles,
charge and energy as well as the energy transfer between different par­
ticle species in terms of the acting "forces" such as pressure gradients,
temperature gradients and electric fields, the underlying mechanism being
fluctuating microfields related either to binary interactions between
particles or to collective effects ("plasma turbulence"). In the first
case, the transport is said to be induced by collisions, in the second
it is usually termed "anomalous".

The states to be considered are "equilibria", that is steady states
with respect to some fast time scale on which the system is, in principle,
able to move. In the case of magnetoplasmas, magnetohydrodynamic equili­
bria are to be adopted. The transport phenomena then either induce a slow
("adiabatic") time dependence of these equilibria or compensate the effect
of particle and energy sources present in the system. Of course, the trans­
port effects encountered do not depend on which of these cases is present.

For magnetically confined plasmas, it is of particular interest to
calculate the particle and energy fluxes across the magnetic field and
specifically in the direction of the total pressure gradient. In the fol­
lowing it will be discussed in which way this can be done, displaying the
basic analogy between collisional and anomalous transport problems.\(^1,2,3\) Energy transfer between particle species will also be touched upon.

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2. BASIC EQUATIONS

Since transport is induced by microscopic phenomena, the starting point for a theory of transport is a Boltzmann-like equation which, in the absence of particle sources, reads

\[
\frac{\partial f_j}{\partial t} + v \cdot \frac{\partial f_j}{\partial r} + K_j \cdot \frac{\partial f_j}{\partial v} = \left( \frac{\partial f_j}{\partial t} \right)_{\text{MF}}
\] (1)

where \( f_j(r,v,t) \) is the distribution function, in phase space, of any species \( j \) of particles present in the system, \( K_j(r,v) \) the acceleration of a particle of this species having the velocity \( v \) at the point \( r \), \( t \) the time, and \( (\partial f_j/\partial t)_{\text{MF}} \) the change in \( f_j \) induced by the microfields. In plasmas, \( K_j \) is due to electromagnetic effects, i.e.

\[
K_j = \frac{e_j}{m_j} \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right)
\] (2)

where \( E(r,t) \) and \( B(r,t) \) are the electric and magnetic fields, respectively, and \( e_j \) and \( m_j \) the charge and the mass of particles of species \( j \). In the special case of binary collisions being the dominant mechanism, \( (\partial f_j/\partial t)_{\text{MF}} \) is the Fokker-Planck collision term \(^4\). On the other hand, if plasma fluctuations prevail, one has

\[
\left( \frac{\partial f_j}{\partial t} \right)_{\text{MF}} = -\frac{\partial}{\partial v} \cdot \langle K_j f_j \rangle
\] (3)

where \( \tilde{K}_j \) is the fluctuating "force", \( \tilde{f}_j \) the fluctuating part of the distribution function, and \( \langle \rangle \) denotes a stochastic average.

In fact, the expression (3) can be derived from the collisionless kinetic equation (the Vlasov equation)

\[
\frac{\partial f_j}{\partial t} + v \cdot \frac{\partial f_j}{\partial r} + K_j \cdot \frac{\partial f_j}{\partial v} = 0
\] (4)

by splitting the distribution function and the force into an average and a fluctuating part, i.e.

\[
f_j = \bar{f}_j + \tilde{f}_j \quad \text{and} \quad K_j = \bar{K}_j + \tilde{K}_j
\]

where

\[
\bar{f}_j \equiv \langle f_j \rangle \quad \text{and} \quad \bar{K}_j \equiv \langle K_j \rangle
\]

so that

\[
\langle \tilde{f}_j \rangle = 0 \quad \text{and} \quad \langle \tilde{K}_j \rangle = 0
\]
and then taking the stochastic average $\langle \rangle$, which yields

$$\frac{\partial \bar{f}_j}{\partial t} + \mathbf{v} \cdot \frac{\partial \bar{f}_j}{\partial \mathbf{r}} + \nabla \cdot \frac{\partial \bar{f}_j}{\partial \mathbf{v}} = - \frac{\partial}{\partial \mathbf{v}} \langle \bar{k}_j \bar{f}_j \rangle$$  \hspace{1cm} (5)

This is just Eq. (1) with $(\partial f/\partial t)_m^f$ according to Eq. (3) if $f_j$ is identified with the averaged distribution function $\bar{f}_j$, and $\bar{k}$ with the averaged force $\bar{K}$.

This shows that the formal structure of the problem is indeed the same for collisional and anomalous effects. The difference consists only in the appearance of other "collision" terms in the kinetic equation. It is this fact which makes it possible to formulate a general framework for transport theory, comprising both collisional and anomalous transport phenomena, as long as the explicit form of $(\partial f/\partial t)_m^f$ does not matter. Of course, the special form of this term does essentially influence the quantitative results of the theory.

As transport has to do with macroscopic notions, like fluxes and fields of driving forces, the structure of a transport problem can be obtained from analysing the macroscopic equations of the medium under consideration. For a plasma, these are the equations for the densities of the different species,

$$n_j = \int d^3 v f_j$$  \hspace{1cm} (6)

the particle fluxes

$$\Gamma_j = \int d^3 v \mathbf{v} f_j$$  \hspace{1cm} (7)

the stress tensors

$$P_j = \int \rho d^3 v \mathbf{v} \mathbf{v} f_j$$  \hspace{1cm} (8)

the energy fluxes

$$Q_j = \int \rho d^3 v \mathbf{v} \frac{m_j}{2} \mathbf{v}^2 f_j$$, etc.  \hspace{1cm} (9)

They are obtained by taking the corresponding moments, with respect to velocity, of the kinetic equation (1) and read

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial r} \Gamma_j = 0$$  \hspace{1cm} (10)

("continuity equation").
\[
\frac{\partial \Gamma_j}{\partial t} + \frac{\partial}{\partial r} \cdot \frac{p_j}{m_j} = \frac{e_j}{m_j} \left( n_j \mathcal{E} + \frac{1}{c} \Gamma_j \cdot \mathbf{B} \right) + \left( \frac{\partial \Gamma_j}{\partial t} \right)_{\text{mf}}
\]

(11)

with \((\partial \Gamma_j/\partial t)_{\text{mf}} \equiv \int d^3v \, v \left( \partial f_j/\partial t \right)_{\text{mf}}
\]

("momentum balance"),

\[
\frac{\partial p_j}{\partial t} + \frac{\partial}{\partial r} \cdot \int d^3v \, m_j \, \mathbf{v} \cdot \mathbf{v} \, f_j = \frac{e_j}{m_j c} \left( \left[ \mathbf{e}_j \times \mathbf{B} \right] + \left( \mathbf{e}_j \times \mathbf{B} \right)^T \right) + \left( \frac{\partial p_j}{\partial t} \right)_{\text{mf}}
\]

(12)

with \((\partial p_j/\partial t)_{\text{mf}} \equiv \int d^3v \, v \cdot m_j \left( \partial f_j/\partial t \right)_{\text{mf}}
\]

("balance of stresses"),

\[
\frac{\partial q_j}{\partial t} + \frac{\partial}{\partial r} \cdot \int d^3v \, v \cdot \frac{m_j}{2} \, v^2 \, f_j = \frac{e_j}{m_j} \left[ \frac{1}{2} \left( \text{Tr} \mathbf{P}_j \right) \mathcal{E} + \mathbf{P}_j \cdot \mathbf{E} \right]
\]

\[
+ \frac{e_j}{m_j c} \left( \mathbf{Q}_j \times \mathbf{B} + \left( \frac{\partial q_j}{\partial t} \right)_{\text{mf}} \right)
\]

(13)

with \((\partial q_j/\partial t)_{\text{mf}} \equiv \int d^3v \, \frac{m_j}{2} \, v^2 \left( \partial f_j/\partial t \right)_{\text{mf}}
\]

("energy flux equation"). Note that in Eq. (10) conservation of particles of a given species in microscopic processes, \( \int d^3v \, \left( \partial f_j/\partial t \right)_{\text{mf}} = 0 \), has been assumed.

3. PARTICLE FLUXES

Relations for the particle fluxes \( \Gamma_j \) can be obtained from the momentum balance (11). Due to the slowness of the time dependences induced by transport effects, the term \( \partial \Gamma_j/\partial t \) is unimportant and can be neglected. Furthermore, in closed confinement geometries, only flux components perpendicular to magnetic surfaces (the \( \psi \)-components, say) imply transport. Therefore, it is useful to adopt a Cartesian reference system (cf. Fig. 1) based on the unit vectors \( \mathbf{e}_\psi \) perpendicular to a magnetic surface, \( \mathbf{e}_\parallel \) parallel to the (average) magnetic field \( \mathbf{B} \), and \( \mathbf{e}_\times = \mathbf{e}_\parallel \times \mathbf{e}_\psi \) within a magnetic surface, but perpendicular to \( \mathbf{B} \). On the other hand, and specifically in axisymmetric toroidal geometry, space dependencies are best described by the coordinate \( \psi \), the poloidal angle \( \theta \), and the toroidal angle \( \phi \).
From Eq. (11) one thus obtains under the very weak condition

|\partial/\partial t| \ll |\omega_{cj}| \equiv \frac{e_j B}{m_j c} \tag{14}

for the particle fluxes perpendicular to B

$$\Gamma_{j,\perp} = \frac{c}{B^2} \left[ n_j \mathbf{E} - \frac{1}{e_j} \frac{\partial}{\partial r} \cdot \mathbf{P}_j + \frac{m_j}{e_j} \left( \frac{\partial \Gamma_j}{\partial t} \right)_{\text{mf}} \right] \times \mathbf{B} \tag{15}$$

and specifically

$$\Gamma_{j,\psi} = \frac{c}{B} \left[ n_j \mathbf{E} - \frac{1}{e_j} \frac{\partial}{\partial r} \cdot \mathbf{P}_j + \frac{m_j}{e_j} \left( \frac{\partial \Gamma_j}{\partial t} \right)_{\text{mf}} \right] \tag{16}$$

Furthermore, if the inertia term $m_j (\partial \Gamma_j / \partial t)_{\parallel}$ in the component of Eq. (11) parallel to $\mathbf{B}$ is likewise unimportant, which is the case in most of the practical situations, one also has

$$e_j n_j \mathbf{E}_{\parallel} - \left( \frac{\partial}{\partial r} \cdot \mathbf{P}_j \right)_{\parallel} + m_j \left( \frac{\partial \Gamma_j}{\partial t} \right)_{\text{mf,\parallel}} = 0 \tag{17}$$

Now a further important simplification is possible when the induced part of the electric field can be taken to be known and determined by external sources, $\mathbf{E}^{(\text{ind})} = \mathbf{E}^{(\text{ext})}$, so that one can put

$$\mathbf{E} = \mathbf{E}^{(\text{ext})} - \nabla \phi \tag{18}$$
with $-\nabla \phi$ a purely electrostatic self-consistent field. In fact, in this case one can introduce combined force vectors

$$A_j \equiv - \nabla \phi - \frac{1}{e_j n_j} \frac{\partial}{\partial r} \cdot \mathbf{p}_j$$  \hspace{1cm} (19)$$

and write Eqs (16) and (17) as

$$\Gamma_j,\psi = \frac{n_j c}{B} \left[ E_{\chi} (\text{ext}) + A_j,\chi \right] + \frac{m_j c}{e_j B} \left[ \frac{\partial p_j}{\partial t} \right]_{mf,\chi}$$  \hspace{1cm} (20)$$

and

$$E_{\parallel} (\text{ext}) + A_j,\parallel + (m_j/e_j n_j) (\partial \Gamma_j/\partial t)_{mf,\parallel} = 0$$  \hspace{1cm} (21)$$

It is important to recognize that specifically toroidal effects in the particle fluxes appear via the $\chi$-components of the forces $A_j$. In fact, for axisymmetric ($\partial/\partial \phi = 0$) magnetic confinement geometries where the Larmor frequencies $|\omega_{c_j}|$ are large compared to all inverse time scales of macroscopic phenomena and in the absence of convective phenomena able to compete with thermal motions, one sees from the lowest order of Eq. (12) (Ref. 5) that $P_j$ is close to being diagonal and isotropic around $B$, i.e.

$$P_j = P_{j,\perp} \left( e_{\psi} e_{\psi} + e_{\chi} e_{\chi} \right) + P_{j,\parallel} e_{\parallel} e_{\parallel}$$  \hspace{1cm} (22)$$

so that

$$A_{j,\parallel} = |\mathbf{v}_\parallel| \frac{B_\parallel}{B} \left\{ - \frac{\partial \phi}{\partial n_j} - \frac{1}{e_j n_j} \left[ \frac{\partial P_{j,\parallel}}{\partial \phi} - (P_{j,\parallel} - P_{j,\perp}) \frac{1}{B} \frac{\partial B}{\partial \phi} \right] \right\}$$  \hspace{1cm} (23)$$

and

$$A_{j,\chi} = A_{j,\parallel} \frac{B_\phi}{B} + C_j^{(1)}$$  \hspace{1cm} (24)$$

with

$$C_j^{(1)} = \frac{|\mathbf{v}_\parallel|}{e_j n_j} \frac{|\mathbf{v}_\parallel|}{B} \frac{\partial}{\partial \phi} \left[ \frac{B_\phi}{B} \frac{P_{j,\parallel}}{\mathbf{v}_\parallel} - \frac{P_{j,\perp}}{B^2} \right]$$  \hspace{1cm} (25)$$

showing that $A_{j,\chi} = 0$ for one-dimensional geometries of which the cylindrical one ($\partial/\partial \theta = 0$) is a special case. It is also to be noted that it can readily be seen by inspection that the term $C_j^{(1)}$ deriving from anisotropy does not contribute to the average $\overline{\Gamma_j,\psi}$ of $\Gamma_j,\psi$ over a magnetic surface $\delta$, and therefore effectively does not yield a contribution to
TRANSPORT IN TOROIDAL MAGNETOPLASMAS

transport. In this sense one can say that the specifically toroidal transport effects are determined by the component of \( A_j \) parallel to \( B \) which in turn is given by the parallel momentum balance (see Eq. (21)).

Combining Eqs (20), (21) and (24) yields for the particle fluxes perpendicular to magnetic surfaces

\[
\Gamma_{j,\psi} = c \left[ n_j \frac{E^{(\text{ext})}}{e} + \frac{m_j}{e_j n_j} \left( \frac{\partial \Gamma_{j,1}}{\partial t} \right)_{mf,\psi} \right] \\
+ \frac{c B}{B_B} n_j A_{j,\parallel} + \frac{c}{B} n_j C_j^{(1)}
\]

with

\[
A_{j,\parallel} = \left[ E^{(\text{ext})}_{\parallel} + \frac{m_j}{e_j n_j} \left( \frac{\partial \Gamma_{j,1}}{\partial t} \right)_{mf,\parallel} \right]
\]

Equation (27) is to be supplemented by constraints for \( (\partial \Gamma_{j,1}/\partial t)_{mf,\parallel} \) reading

\[
\oint \frac{d\theta B}{|\nabla \theta|} B_\theta \left[ E^{(\text{ext})}_{\parallel} + E_{j,\parallel}^{(1)} + \frac{m_j}{e_j n_j} \left( \frac{\partial \Gamma_{j,1}}{\partial t} \right)_{mf,\parallel} \right] = 0
\]

where

\[
E_{j,\parallel}^{(1)} = \frac{1}{e_j n_j} \left( p_{j,\parallel} - p_{j,\perp} \right) \frac{B_\theta}{B^2} \frac{\partial B}{\partial \theta}
\]

represents a driving force due to anisotropy. These constraints appear because the electrostatic potential \( \phi \) must be a single-valued function of the poloidal angle \( \theta \) and hence can be eliminated from Eq. (21) by integration over \( \theta \) upon multiplying by \( B/(|\nabla \theta| B_\theta) \).

In relation (26) the first line describes "normal" transport effects, appearing also in one-dimensional geometry. The first term corresponds to the pinch effect due to the external electric field, whereas the second contains all cross-field diffusion effects. The terms in the second and third line describe toroidal transport effects but, as mentioned above, only the former yields a contribution to the average particle flux \( \Gamma_{j,\psi} \) across a magnetic surface. The physical reason for the appearance of the additional toroidal transport terms consists in the fact
that the fluxes within the magnetic surfaces driven by radial gradients, anisotropy, and external electric fields, and in particular the diamagnetic fluxes, are not divergence-free. They tend, in fact, to create a net particle transport in vertical direction (i.e. parallel to the major axis of the torus). Therefore, particle fluxes along the magnetic field have to appear which restore the steady-state particle balances. In the presence of (collisional or anomalous) dissipative effects such fluxes require forces along the magnetic field to drive them, which are provided by pressure gradients and a self-consistent electric field in poloidal direction. These forces, as well as anisotropy, induce diamagnetic and electric drift motions perpendicular to the magnetic surfaces which are the toroidal transport effects encountered. It must, however, be noted that these forces are completely determined only upon taking also the energy balances into account (cf. Sect. 4). Considering specifically the electric current, the one induced to flow parallel to $B$ by the radial gradient of the total pressure ($J$) is often called Pfirsch-Schlüter current, whereas the one driven by anisotropy ($J$) (cf. Eqs (21) and (23)) is termed bootstrap current. The explicit calculation of the latter, as well as of the corresponding transport effects, is usually performed adopting a kinetic approach, because anisotropy effects are difficult to calculate macroscopically, but, of course, all these effects can also be recovered in a macroscopic framework. They are, however, only important when there is no strong effect isotropizing the distribution functions. In particular, in the case of collisional transport, they play an essential role only in the weakly collisional ("banana" and "plateau") regimes. Note also that the average particle fluxes turn out to be ambipolar,

$$\sum J \nabla \Phi = 0$$

when the plasma is quasi-neutral,

$$\sum n_j = 0$$

and the microscopic interactions conserve momentum

$$\sum J (\partial n_j / \partial t)_{mf} = 0$$

as is generally the case.
4. ENERGY FLUXES

Energy fluxes can be calculated from Eq. (13) in much the same way as the particle fluxes from the momentum balance (11), since both relations have a completely analogous structure. This has, of course, to do with the fact that the underlying physics is largely the same.

The result obtained for the energy flux perpendicular to a magnetic surface is

\[ Q_{j,\psi} = \frac{c}{B} \left[ (2P_{j,\perp} + \frac{1}{2} P_{j,\parallel})E_{\chi} \right. \] 
\[ + \left. \frac{c}{B} n_{j} A_{j,\parallel}^{*} - \frac{c}{B} (P_{j,\perp} - P_{j,\parallel})(\nabla \phi)_{\chi} + \frac{c}{B} n_{j} C_{j}^{(2)} \right] \tag{30} \]

with

\[ A_{j,\parallel}^{*} = \frac{1}{n_{j}} (P_{j,\perp} + \frac{3}{2} P_{j,\parallel})(-\nabla \phi)_{\parallel} - \frac{m_{j}}{e_{j} n_{j}} \left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \rho} \right)_{\parallel} \] 
\[ = - \left[ \frac{1}{n_{j}} (P_{j,\perp} + \frac{3}{2} P_{j,\parallel})E_{\parallel}^{(\text{ext})} + \frac{m_{j}}{e_{j} n_{j}} \left( \frac{\partial Q_{j}}{\partial \tau} \right)_{\text{mf},\parallel} \right] \tag{31} \]

Here

\[ B_{j} = \int d^{3}v \left( \frac{1}{2} m_{j} v^{2} f_{j} \right) \approx n_{j} \left( e_{j} e_{\psi} + e_{\chi} e_{\chi} \right) + n_{j} e_{\parallel} e_{\parallel} \tag{32} \]

has been introduced, the latter form being the analogue of Eq. (22) for \( P_{j,\parallel} \).

Equation (31) is to be supplemented by constraints for \( \frac{\partial Q_{j}}{\partial \tau} \right)_{\text{mf},\parallel} \) reading

\[ \frac{d}{d\theta} B \right|_{\vartheta} \left[ E_{\parallel}^{(\text{ext})} + E_{j,\parallel}^{(2)} + \frac{m_{j}}{e_{j} (P_{j,\perp} + \frac{3}{2} P_{j,\parallel})} \left( \frac{\partial Q_{j}}{\partial \tau} \right)_{\text{mf},\parallel} \right] = 0 \tag{33} \]

where the contribution due to anisotropy is now given by

\[ E_{j,\parallel}^{(2)} = \frac{|\vartheta| m_{j}}{e_{j} P_{j,\perp} + \frac{3}{2} P_{j,\parallel}} B_{j} \frac{\partial B}{\partial \theta} \] 
\[ \frac{d}{d\theta} B \right|_{\vartheta} \frac{d}{d\theta} \] 
\[ \frac{d}{d\theta} B \right|_{\vartheta} \frac{d}{d\theta} \] 
\[ \frac{d}{d\theta} B \right|_{\vartheta} \]

These constraints follow from the parallel component of Eq. (13). Again, the first line of Eq. (30) describes the "normal" transport effects as
appearing in one-dimensional geometry, while the terms in the second and third line correspond to the toroidal contributions to energy transport. The term in the third line, related to anisotropy, which is not given explicitly, does not contribute to the energy flux averaged over a magnetic surface. Note that $Q_{\psi}$, as calculated here, is the total kinetic energy flux across magnetic surfaces, including convection, and in the case of a turbulent plasma, energy flows related with the presence of fluctuating particle motions in the wave fields ("sloshing"). The physical mechanism responsible for the toroidal contributions to energy transport is closely related to that inducing toroidal particle transport: the energy fluxes within magnetic surfaces, driven by radial gradients, anisotropy, and external electric fields, and in particular the diamagnetic fluxes, do result in not being divergence-free in toroidal geometry. This would lead to a net energy transport in vertical direction, if there were no forces along the magnetic field lines driving energy fluxes parallel to $B$ which restore the steady-state energy balance. As discussed before (see Sect. 3), these forces have to be such that also the steady-state particle balances are fulfilled. The energy fluxes connected with the drift motions perpendicular to the magnetic field induced by these forces and with anisotropy yield the toroidal energy transport effects found.

5. ENERGY TRANSFER

Information on the energy transfer between different particle species can be obtained from the energy balance which is contained in the balance of stresses (12). In fact, the energy balance is just the trace of Eq. (12), which upon introducing the (total) kinetic energy density

$$
\varepsilon_j = \frac{1}{2} \text{Tr} P_j
$$

(35)
of the species $j$ reads

$$
\frac{\partial \varepsilon_j}{\partial t} + \nabla \cdot \dot{Q}_j = \varepsilon_j \Gamma_{\varepsilon} \cdot \mathbb{E} + \frac{1}{2} \text{Tr} \left[ \frac{\partial P_j}{\partial t} \right]_{mf}
$$

(36)

From this form it is evident that the energy transferred to the species $j$ by microscopic (collisional or anomalous) interactions can be obtained by calculating

$$
\left[ \frac{\partial \varepsilon_j}{\partial t} \right]_{mf} \equiv \frac{1}{2} \text{Tr} \left[ \frac{\partial P_j}{\partial t} \right]_{mf}
$$

(37)
It must, however, be noted that in the case of anomalous effects prevailing, the result can be interpreted as an energy transfer between particle species only if there is no energy transferred to and possibly carried away by fluctuating electromagnetic fields. Moreover, contributions due to the fluctuating particle motions are again considered part of the energy densities of the particles rather than that of the waves.

6. COLLISIONAL TRANSPORT

In order to show how the previous scheme can be used to derive transport effects explicitly, let us consider the case in which binary collisions are the dominant transport mechanism. This is the only case for which a complete theory is available. For the sake of simplicity, only the "collision-dominated" regime, defined by

\[ \lambda_j \equiv v_{th,j} / u_j \ll qR \quad (38) \]

with \( \lambda_j, v_{th,j} \) and \( u_j \) the mean free path, the thermal velocity, and the collision frequency of particles of the species \( j \), respectively, and \( QR \) the "connection length" (\( R \) being the major radius of the torus and \( q \) the safety margin), will be considered. Here anisotropy effects are small, so that a macroscopic approach leads simply to explicit results. Nevertheless, the anisotropy contributions will be kept to display how they enter the scheme. Moreover, a quasi-neutral hydrogen plasma, consisting of electrons (e) and protons (i) with

\[ e_i = - e_e = e \quad \text{and} \quad n_i = n_e = n \quad (39) \]

will be treated. Further assumptions, consistent with the approximation of large Larmor frequency \( |\omega_{cj}| \) already introduced above, are that

\[ |\omega_{cj}| \gg u_j \quad (40) \]

and

\[ \rho_{Lj} \equiv v_{th,j} / |\omega_{cj}| \ll a \quad (41) \]

where \( \rho_{Lj} \) is the Larmor radius of species \( j \) and \( a \) the minor radius of the torus. No restriction will, however, be introduced as far as the form of the cross-sections of the magnetic surfaces is concerned.
Going now to calculate the collisional particle flux, one has for the collision terms in Eq. (11) (Ref. 11)

\[
\mathbf{m}_i \left( \frac{\partial \mathbf{J}}{\partial t} \right)_{\text{mf}} = -\mathbf{m}_e \left( \frac{\partial \mathbf{E}}{\partial t} \right)_{\text{mf}} = -e n \mathbf{J}_e + \frac{3}{2} \frac{e}{\omega_c T_e} n \mathbf{V}_\mathbf{e} - e n \mathbf{V}_\mathbf{e} + 0.71 n \mathbf{V}_\mathbf{e} \cdot \mathbf{E} \quad (42)
\]

where

\[
j = e(\mathbf{I}_i - \mathbf{I}_e) \quad (43)
\]

is the current density, \( T_e \) the electron temperature,

\[
m = \eta_{\parallel} (\mathbf{e}_X^e \mathbf{e}_X^e + \mathbf{e}_X^e \mathbf{e}_X^e) + \eta_{\parallel} \mathbf{e}_X^e \mathbf{e}_X^e \approx \mathbf{J} (\psi) \quad (44)
\]

the resistivity tensor with

\[
\eta_{\parallel} = 1.96 n \beta = \frac{m_e}{e^2 n T_e} \quad (45)
\]

and

\[
\mathbf{T}_e = \mathbf{v}^e = \frac{3 \sqrt{m_e T_e}}{4 \sqrt{2 \pi} e^2 n T_e} \quad (46)
\]

the electron collision time, \( \mathbf{v} \) denoting the Coulomb logarithm. Adding Eqs (11) for ions and electrons yields the total momentum balance

\[
\frac{1}{c} \mathbf{j} \times \mathbf{B} = \frac{3}{\partial t} \cdot (P_e + P_\parallel) \quad (47)
\]

with

\[
P = \frac{1}{3} T_r (P_e + P_\parallel) \approx P (\psi) \quad (48)
\]

the total scalar plasma pressure, the contribution due to anisotropy stemming from Eq. (47) being generally unimportant. Hence, one has

\[
\mathbf{j}_\psi \approx \frac{c}{B} (\nabla P)_\psi \quad (49)
\]

\[
\mathbf{j} \approx 0 \quad (50)
\]

Using now the continuity equation for the current density

\[
\frac{3}{\partial t} \cdot \mathbf{j} = 0 \quad (51)
\]

which immediately follows from Eqs (10) and (43) upon neglecting the small time derivative and which, for axisymmetric geometry, implies

\[
\frac{3}{\partial \Phi} \left[ \mathbf{j}_\psi (|\nabla \psi|, |\nabla \psi|) \right] = 0
\]
one finds furthermore

\[ j_\theta = |\nabla \psi| |\nabla \phi| f_1(\psi) = B_\theta f_1(\psi) \]

with \( f_1(\psi) \) an arbitrary function of \( \psi \), the latter form requiring also that \( \psi \) be identified with the poloidal flux per unit angle in toroidal direction. This allows us to express \( j_\parallel \) as

\[ j_\parallel = B f_1(\psi) \quad B_\theta \frac{c B_\phi}{B_\theta} (\nabla \cdot \psi) \]

Thus the freedom contained in the appearance of \( f_1(\psi) \) is just the one necessary to be able to fulfill the constraint (28). Taking into account that \( B_\phi \) is (practically) a vacuum field (cf. Eq. (50)) so that in axisymmetric geometry

\[ B_\phi = |\nabla \phi| f_2(\phi) \]

holds, one finds, in fact, from (28) after some algebra and with \( E^{(\text{ext})} = E^{(\text{ext})}_\phi \), corresponding to the case occurring in practice, that

\[ f_1(\psi) = \frac{c B_\phi}{B_\theta} (\nabla \cdot \psi) \quad \frac{\langle\langle E^{(\text{ext})}_\phi B_\phi + E^{(1)}_{1,\parallel} B \rangle\rangle}{\eta_\parallel \langle\langle B^2 \rangle\rangle} \]

with

\[ \langle\langle A \rangle\rangle \equiv \int_\phi \frac{d\theta}{|\nabla \phi| B_\theta} A \]

(55)

has to be valid. Note that the constraint (28) also implies \( \langle\langle E^{(1)}_{1,\parallel} B \rangle\rangle = \langle\langle E^{(1)}_{1,\parallel} B \rangle\rangle \), so that the result (54) does not depend on which particle species is considered for making anisotropy effects enter, as it should be. Hence, one has for the current density flowing along the magnetic field

\[ j_\parallel = \frac{B}{\eta_\parallel \langle\langle B^2 \rangle\rangle} \langle\langle E^{(\text{ext})}_\phi B_\phi + E^{(1)}_{1,\parallel} B \rangle\rangle - \frac{c B_\phi}{B_\theta} \left( \frac{1 - B^2}{\langle\langle B^2 \rangle\rangle} \right) (\nabla \cdot \psi) \]

(56)

which yields with Eqs (26), (27) and (42), apart from the term containing \( C^{(1)}_j \),
\[ \Gamma_i, \psi = \Gamma_e, \psi = -\frac{cB}{B^2} n E_{\phi}^{(\text{ext})} - \frac{c^2}{B^2} n \eta_{\perp} \left( (\nabla P)_\psi - \frac{3}{2} n (\nabla T_e)_\psi \right) \]

\[ - \frac{c B}{B^2 B_\theta} n \left[ E_{\phi}^{(\text{ext})} B_\phi - \frac{B^2}{<B^2>} \ll E_{\phi}^{(\text{ext})} B_\phi \gg \right] \]

\[ + \frac{c B}{B_\theta} \ll E_{i, \parallel}^{(1)} B \gg \]

\[ - \frac{c^2 B_\phi^2}{B^2 B_\theta^2} n \eta_{\parallel} \left[ 1 - \frac{B^2}{<B^2>} \right] (\nabla P)_\psi - 0.71 \frac{c B_\phi}{B^2} \frac{n}{e} (\nabla T_e)_\theta \] (57)

In Eq. (56), the contribution containing \( E_{\phi}^{(\text{ext})} \) is the current driven by the external electric field; the one containing \( E_{i, \parallel}^{(1)} \) is the bootstrap current which, as said above, is unimportant in the case under consideration, and the one proportional to \( (\nabla P)_\psi \) is the Pfirsch-Schlüter current. The relative weight of the latter is crudely

\[ \frac{(c/B_\rho)(a/R) |(\nabla P)_\psi|}{J_{\parallel}} \approx \left( \frac{a}{R} \right) B_\rho \] (58)

with \( B_\rho = \frac{\Theta_{\nabla P}}{B_\rho^2} \) the poloidal \( B \) of the plasma (note that \( 1-B^2/<B^2> > \) is of order \( a/R \)). For negative \( (\nabla P)_\psi \), that is for pressure decreasing outwards, it is flowing in the same direction as the current driven by the external electric field in the area farther away from the main magnetic axis than about the major radius \( R \) where the magnetic field is small, \( B < \sqrt{<B^2>} \), and in the opposite direction otherwise.

In the expression (57) for the particle flux, the first line contains the "normal" collisional cross-field transport (the pinch effect due to the external field, cross-field diffusion and Nernst effect, respectively), while the remainder are the collisional toroidal transport terms. The first of them, related to the external electric field, is generally small. The second, due to anisotropy, is unimportant in the regime considered. The important toroidal transport effects are connected with the last two terms. In fact, the typical order of these terms, with respect to the "normal" cross-field diffusion, is

\[ (B_\phi/B_\theta)^2(a/R) = q^2(R/a) >> 1 \] (59)

as explicitly seen for the term proportional to \( (\nabla P)_\psi \). The other one can be written in terms of radial gradients only upon determining \( (\nabla T_e)_\theta \) from
the scheme of Sect. 4 and the energy balances, but then takes a similar form\textsuperscript{12,13}, contributing both a further term proportional to the pressure gradient and a toroidal Nernst term proportional to $(\Omega T_e)_\psi$. The direction of these toroidal fluxes, for pressure and temperature profiles decreasing outwards, is outwards for the area away from the major axis where $B < \sqrt{\langle B^2 \rangle}$ and inwards in the opposite case.

In the particle flux averaged over the magnetic surface

$$
\bar{T}_{j,\psi} = \frac{\dot{\phi}}{\dot{\phi}} \frac{d\theta}{\sqrt{\theta^2 + \psi^2}}
$$

the toroidal contributions have only a relative weight $q^2$ because of a compensation of the outward and inward fluxes to lowest significant order in the inverse aspect ratio, here taken to be small. As long as the energy transfer between ions and electrons is not important, that is in the usual Pfirsch-Schlüter regime for which

$$
\frac{v_{\perp}^i}{v_{\perp}^e, (QR)} < 1.3 \left(\frac{m_i}{m_e}\right)\frac{1}{4}
$$

and furthermore

$$
\frac{v_{\perp}^i}{\omega_{ci}} < \left(\frac{B_0}{B}\right)^2
$$

must also be valid, and specifically for circular concentric cross-sections of the magnetic surfaces, one finds\textsuperscript{12,13}

$$
\bar{T}_{i,\psi} = \bar{T}_{e,\psi} = - \frac{c B_0}{B^2} n E_{\phi \text{ext}} - \frac{c^2}{B^2} n \eta_\perp \left(\frac{dP}{dr} - \frac{1}{2} \frac{dT_e}{dr}ight)
$$

$$
- 2q^2 \frac{c^2}{B^2} n \left[\left(\eta_\parallel + 0.50 \frac{T_e}{e^2\chi_{e,\parallel}}\right) \frac{dP}{dr} - 1.8 \frac{T_e}{e^2\chi_{e,\parallel}} \frac{dT_e}{dr}\right]
$$

upon introducing the radial coordinate $r'$ instead of $\psi$ and suppressing the unimportant terms of the toroidal contribution. In this relation the terms containing the parallel electron heat conductivity

$$
\chi_{e,\parallel} = 3.16 \frac{n T_e}{m_e^2 e^2}
$$
stem from the contribution to Eq. (57) containing \((\nabla T_e )_\theta\). Using the relations between \(x_{e,||}\), \(\eta_{||}\), and \(\eta_{\perp}\) following from Eqs (45) and (63), the result (62) can also be written

\[
\bar{T}_{i,\psi} = \bar{T}_{e,\psi} = \frac{c B e}{B^2} n E^{(\text{ext})} - \frac{e^2}{B^2} n \eta_{\perp} \left( \frac{dP}{dT} - \frac{3}{2} n \frac{dT_e}{dT} \right) - q^2 \frac{e^2}{B^2} n \eta_{\perp} \left( 1.31 \frac{dP}{dT} - 1.1 n \frac{dT_e}{dT} \right)
\]

(64)

Toroidal transport terms here can be recognized by the appearance of the factor \(q^2\).

The derivation of explicit relations for the energy fluxes along the magnetic field as well as for \((\nabla T_e )_\theta\) and \((\nabla T_i )_\theta\) by means of the scheme of Sect. 4 and the steady-state energy balances implies a procedure which, although somewhat more involved, is completely analogous to the steps taken to obtain \(j_{||}\) and \(A_{j,||}\). Somewhat more simply, one can obtain these relations also by using directly Braginskii's expressions for the collisional heat fluxes in the energy balance, taking account of the periodicity in the poloidal angle \(\phi\). The steady-state energy balances (cf. Eq. (36)) can be cast in the form

\[
\frac{3}{2} Q_e - \frac{T_e}{n} \nabla P_e = - \frac{3}{2} Q_i = \dot{c}_{e i}
\]

(65)

with

\[
\dot{c}_{e i} \equiv \frac{m_e}{m_i} \frac{n}{T_e} (T_e - T_i)
\]

(66)

describing the heat transfer between electrons and ions. To obtain this form, the steady-state momentum balances (cf. Eq. (11)) and the explicit expressions for \(\text{Tr}(\partial P_e/\partial t)_{mf}\) valid for collisional energy transfer have to be used.

As said above, upon solving consistently for the particle and energy fluxes along the magnetic field as well as for the corresponding driving forces, one can readily calculate the cross-field energy fluxes \(Q_j,\psi\) from Eqs (30) and (31). For their averages over a magnetic surface \(\overline{Q}_{j,\psi}\), one obtains in the limit of small inverse aspect ratio and circular concentric cross-sections for the usual Pfirsch-Schlüter regime as delimited by the conditions (38) and (61) the following results: the average ion energy
The flux is found to be

$$\vec{Q}_{i,\psi} = \frac{5}{2} T_i \tau_i - \chi_{i,\perp} \frac{dT_i}{dr} - \frac{25}{2} q^2 \frac{c^2 (nT_i)^2}{B^2 e^2 \chi_{i,\parallel}} \frac{dT_i}{dr}$$

(67a)

$$= \frac{5}{2} T_i \tau_i - \chi_{i,\perp} (1 + 1.6 q^2) \frac{dT_i}{dr}$$

(67b)

with

$$\chi_{i,\parallel} = 3.9 \frac{n T_i \tau_i}{m_i}$$

(68a)

$$\chi_{i,\perp} = 2 \frac{n T_i}{m_i \omega_i c \tau_i}$$

(68b)

the parallel and perpendicular ion heat conductivity and

$$\tau_i = \frac{3 \sqrt{m_i} T_i^{3/2}}{4 \sqrt{n} e^2 n \tau_i}$$

(69)

the ion collision time, while the average electron heat flux is

$$\vec{Q}_{e,\psi} = \frac{5}{2} T_e \tau_e - \left( \chi_{e,\perp} \frac{dT_e}{dr} - \frac{3}{2} n_1 \frac{c^2 n T_e}{B^2 e^2 \chi_{e,\parallel}} \frac{dT_e}{dr} \right)$$

$$- \frac{25}{2} q^2 \frac{c^2 (nT_e)^2}{B^2 e^2 \chi_{e,\parallel}} \left( \frac{dT_e}{dr} - \frac{0.28}{n} \frac{dP}{dr} \right)$$

(70a)

$$= \frac{5}{2} T_i \tau_i - \chi_{e,\perp} \left[ (1 + 0.85 q^2) \frac{dT_e}{dr} - (0.32 + 0.24 q^2) \frac{1}{n} \frac{dP}{dr} \right]$$

(70b)

with $\chi_{e,\parallel}$ as given by Eq. (63) and

$$\chi_{e,\perp} = 4.66 \frac{n T_e}{m_e \omega_e c \tau_e}$$

(71)

the parallel and perpendicular electron heat conductivity. In Eqs (67) and (70) the terms containing $\vec{P}_{i,\psi}$ describe convective energy transport, whereas the remaining terms are the heat fluxes of ions and electrons, respectively. Toroidal effects are again proportional to $q^2$. The forms (67b) and (70b) for the energy fluxes are obtained from Eqs (67a) and (70a), respectively, using the relations between $\chi_{i,\parallel}$, $\eta_1$, and $\chi_{i,\perp}$ as well as between $\chi_{e,\parallel}$, $\eta_1$, and $\chi_{e,\perp}$, following from the definitions of these transport coefficients. So as in the particle flux (see Eqs (62) and (63))
both a normal and a toroidal Nernst term proportional to $dT_e/dr$ appears, there is in the electron heat flux both a normal and a toroidal contribution proportional to $dP/dr$, sometimes called Dufour terms. The Nernst and Dufour effects are complementary cross effects. In fact, from Eqs (62) and (70a) it is readily seen that the coefficients of the corresponding terms fulfill Onsager's reciprocity relations. Note, furthermore, that order-of-magnitude-wise one has

$$|Q_{i,\psi}| \approx (m_i/m_e)^{1/2} |Q_{e,\psi}|$$  \hspace{1cm} (72)

which is a consequence of

$$|(\nabla T_i)_\theta| \approx (m_i/m_e)^{1/2} |(\nabla T_e)_\theta|$$  \hspace{1cm} (73)

which in turn is due to the fact that

$$x_{i,\parallel} \approx (m_e/m_i)^{1/2} x_{e,\parallel}$$  \hspace{1cm} (74)

whereas the diamagnetic heat fluxes in vertical direction (cf. Sect. 4), apart from the sign, are equal.

When the collisionality of the plasma is larger than allowed for by condition (61a), heat transfer between electrons and ions becomes important and affects the determination of the heat fluxes and forces along the magnetic field$^{12,13}$. The strongest change appears in the ion parallel heat flux and consequently in $(\nabla T_i)_\theta$. The physical reason for this is that the ion energy imbalance, caused by diamagnetic effects, can now more easily be compensated by heat transfer to the electrons and electron heat conduction along the magnetic field than by ion heat conduction (cf. Eq. (74)). For

$$\frac{v_i}{v_{th,i}} \propto (m_i/m_e)^{1/2}$$  \hspace{1cm} (75)

the latter channel of energy transport becomes negligible. As a result, one obtains complete temperature equilibration between ions and electrons on a magnetic surface,

$$T_i(\theta) = T_e(\theta)$$  \hspace{1cm} (76)

and $|Q_{i,\psi}|$ decreases by typically a factor $(m_e/m_i)^{1/2}$. Specifically for the case of a hydrogen plasma as considered here, the toroidal contributions to the ion and electron heat fluxes compensate each other.
On the other hand, when condition (61b) is violated, the relative variation of \( T_1 \) on a magnetic surface becomes of the order of the inverse aspect ratio, that is comparable with that of the magnetic field\(^{13}\). In this case one reaches a transport regime in which the fluxes are nonlinear in the gradients.

Finally, it is to be noted that the theory discussed here can readily be extended to describe a multi-species plasma\(^{14,15,16,17,18}\). This more general case is of practical importance as it yields a transport theory for toroidal plasmas containing impurities.

7. ANOMALOUS TRANSPORT

When applying the general transport theory as formulated in Sects 3 to 5 to anomalous transport problems, the first step is to calculate the relevant "collision" terms in the macroscopic equations (11) to (13). This can readily be performed using the form (3) of \((\partial f_j / \partial t)_{mf}\). Taking Eq. (2) also into account, one thus obtains\(^{1,2,3,19}\)

\[
\frac{m_j}{e_j} \left( \frac{\partial f_j}{\partial t} \right)_{mf} = <\tilde{n}_j \tilde{E}> + \frac{1}{c} <\tilde{r}_j \times \tilde{B}> 
\]

(77)

\[
\frac{m_j}{e_j} \left( \frac{\partial Q_{ij}}{\partial t} \right)_{mf} = \frac{1}{2} <\text{Tr} \tilde{\pi}_j \tilde{E}> + <\tilde{\pi}_j \cdot \tilde{E}> + \frac{1}{c} <\tilde{q}_j \times \tilde{B}> 
\]

(78)

\[
\frac{1}{2} \text{Tr} \left( \frac{\partial \pi_{ij}}{\partial t} \right)_{mf} = e_j <\tilde{r}_j \cdot \tilde{E}> 
\]

(79)

where the fluctuating macroscopic quantities, defined by replacing \( f_j \) by \( \tilde{f}_j \) in Eqs (6) to (9), as well as the fluctuating electric and magnetic fields are again denoted by a wiggle. These expressions have to be used in Eqs (27), (28), (30), (31) and (36). However, to come to an explicit transport theory, one needs to express the effects of the microfields in terms of macroscopic equilibrium quantities rather than in terms of fluctuations. This requires, at least in principle, the knowledge of the nonlinear dynamics of the fluctuations for the inhomogeneous equilibrium under consideration. In practice, one has largely to rely on a quasi-linear approximation to get some insight\(^{20}\).

Here, the difficult question of how to obtain the appropriate macroscopic expressions for the collision terms (77) to (79) for a given
turbulent equilibrium will not be tackled; we rather discuss how different kinds of effects enter into a general scheme of anomalous transport. Suppose, e.g., that for a two-species plasma, \( (m_j/e_j n_j)(\partial \Gamma_j/\partial t)_{mf} \) can be reduced to

\[
\frac{m_j}{e_j n_j} \left( \frac{\partial \Gamma_j}{\partial t} \right)_{mf} = E_j^{(3)} - \eta^* \cdot \mathbf{j}
\]

(80)

where \( E_j^{(3)} \) contains all driving forces, that is, all terms depending explicitly on gradients, and \( \eta^* \) is an anomalous resistivity tensor, which may depend on the poloidal angle \( \theta \) if the level of excitation of turbulence does. Assuming also that the collective interactions conserve the total particle momentum, the anomalous particle transport can be treated along the same lines as in the collisional case (see Sect. 6), due to the analogous structure of Eqs (80) and (42). As a result, the parallel current turns out to be

\[
J_\parallel = \frac{B \langle \eta^* B \rangle}{\langle n^* B^2 \rangle} - \frac{c B \phi}{B_\theta \langle \eta^* B^2 \rangle} \left( 1 - \frac{B^2 \langle \eta^* B \rangle}{\langle n^* B^2 \rangle} \right) (\nabla \psi)
\]

(81)

where

\[
E_\parallel^* = E_{\parallel, \text{ext}} + E_{\parallel, \text{J}} + E_{\parallel, \psi}
\]

(82)

\( E_{\parallel, \text{J}} \) being due to anisotropy (see Eq. (29)), while for the particle flux across the magnetic surfaces

\[
\Gamma_{j, \psi} = \frac{c B}{B_\theta} \eta \left[ E_{\text{ext}} + E_{(3)} - \frac{c B \phi}{B_\theta} \langle \eta^* \rangle (\nabla \psi) \right]
\]

(83)

is obtained. In Eq. (83), the first line corresponds to the "normal" transport effects, whereas the second is the toroidal contribution. As expected, the appearance of an anomalous resistivity just reduces the current driven by the external electric field and modifies somewhat the Pfirsch-Schlüter current, provided it is dependent on \( \theta \) (see the second term in Eq. (81)). The largest effect that might appear is that the Pfirsch-Schlüter current is increased by a factor of the order of the aspect ratio, requiring a poloidal variation of \( \eta^* \) of order 1. In the particle flux, the main effect of an anomalous resistivity is an increase
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of the "normal" cross-field diffusion, related with \( n_\perp^* \), and of the Pfirsch-Schlüter flux, related to \( n_\parallel^* \). On the other hand, the appearance of additional driving forces via \( E_j^{(1)} \) and \( E_j^{(3)} \) leads to new contributions to the parallel current and new transport terms, both in the "normal" and in the toroidal contribution to the particle flux. Note, however, that there is a difference between the way \( E_j^{(1)} \) and \( E_j^{(3)} \) enter into the particle flux: whereas \( E_j^{(3)} \) contributes both explicitly to the normal and toroidal transport term and to \( j_\parallel \), the latter fact implying a partial compensation of the contributions related to \( E_j^{(3)} \) in the toroidal term, the anisotropy effect \( E_j^{(1)} \) appears only via \( j_\parallel \) and, hence, is not subject to compensation.

An analogous analysis can be made for the energy fluxes if it is assumed that \( (\partial Q_j/\partial t)_{mf} \) can be written as a sum of terms related to forces and fluxes (cf. Ref. 1)).

An explicit treatment of anisotropy effects requires determining \( P_{j,\parallel} - P_{j,\perp} \) for the particle flux, and \( R_{j,\parallel} - R_{j,\perp} \) for the energy flux (cf. Eqs (29) and (34)). Within a macroscopic framework, the second and fourth moment of the Boltzmann equation yield a starting point for that, but a microscopic approach might be more convenient. It is important to note that even weak collisional effects are in general expected to be relevant here because they usually constitute the most important mechanism of isotropization. The macroscopic equations determining, e.g., \( P_{j,\parallel} - P_{j,\perp} \) read explicitly (cf. Eqs (2), (3), (12) and (36))\(^1, 19, 21\)

\[
\frac{\partial}{\partial t} \left( P_{j,\perp} + \frac{1}{2} P_{j,\parallel} \right) + \frac{\partial}{\partial r} \cdot \left( E_j - e_j \Gamma_j \cdot E \right) = e_j < \hat{r}_j \cdot \hat{E} > + c_1 \tag{84}
\]

\[
\frac{\partial}{\partial t} P_{j,\parallel} + \left[ \frac{\partial}{\partial r} \cdot \left\{ \int d^3 v m_j v v f_j \right\} \right] = e_{\parallel} \Gamma_{\parallel} - 2e_j \hat{r}_{j,\parallel} E_{\parallel} = 2e_j < \hat{r}_{j,\parallel} E_{\parallel} > + c_2 \tag{85}
\]

where \( c_1 \) and \( c_2 \) describe the collisional effects. Note that both these equations are exact as the approximation (22), for \( P_{j,\parallel} \) has not been used to derive them. Therefore, taking their difference to determine the small anisotropy \( P_{j,\parallel} - P_{j,\perp} \) is significant. Note also that Eq. (84) is just the energy balance. Turbulence-induced effects on the anisotropy of the pressure enter in Eqs (84) and (85) both via the terms on the right-hand
sides depending explicitly on fluctuations and via fluctuating contributions to the third moment. On the other hand, the non-fluctuating contributions to the third moment contain a basic source of anisotropy, present in all confinement topologies where the magnetic field strength varies along a field line. In toroidal geometry this enters specifically through the Pfirsch-Schlüter fluxes contained in the third moment. One can therefore try to derive a condition for anomalous effects being unimportant for the pressure anisotropy by requiring that the fluctuating contributions be smaller than the one related to the Pfirsch-Schlüter effect. This typically yields

\[ \frac{2|\gamma_j|}{\omega_{cj}} \frac{aR}{\rho_{L,j}^2} \frac{\varepsilon_{f_1}}{n_{f_1}} < 1 \]  

(86)

upon observing that

\[ \frac{1}{2} \text{Tr} \left( \frac{\partial^2}{\partial \mathbf{t}^2} \right)_{\text{mf}} = e_j < \mathbf{\Gamma}_j \cdot \mathbf{\dot{E}} > \equiv -2\gamma_j \varepsilon_{f_1} \]  

(87)

where \( \varepsilon_{f_1} \) is the energy density of the fluctuating wave fields and \( \gamma_j \) is the contribution of the species \( j \) to the (nonlinear) growth rate of the excited waves, is just the power density transferred from the fluctuating fields to the particle species \( j \). Condition (86) can be expected to ensure also that fluctuation-induced effects in the energy fluxes are small. Furthermore, it must be noted (cf. Eqs (28), (29), (33) and (34)) that only a \( \theta \)-dependent anisotropy being asymmetric with respect to the equatorial plane of the torus can induce a bootstrap current and transport.

As far as the anomalous energy transfer to the species \( j \), given by Eq. (87), is concerned it should be recalled that this corresponds to a simple transfer of energy between different species only if the turbulent state is steady and no fluctuating energy is convected out of the system.

Finally, it is to be observed that none of the arguments or formal steps used in this general discussion of anomalous transport has required relying on the linear approximation to the wave dynamics. All results obtained are therefore also applicable to nonlinearly saturated turbulent states. Any further step, and especially finding macroscopic expressions for the collision terms explicitly for a particular kind of wave excitation, however, is practically only possible by introducing such a linearization, i.e. adopting a quasi-linear framework. Applications on drift waves whereby, however, toroidal effects on the wave dynamics were disregarded, have been considered in Refs 1) and 3) (cf. also Ref. 22)).
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A FOKKER-PLANCK/TRANSPORT MODEL FOR NEUTRAL-BEAM-DRIVEN TOKAMAKS*

J. KILLEEN, A.A. MIRIN, M.G. McCoy
Magnetic Fusion Energy Computer Center,
Lawrence Livermore National Laboratory,
Livermore, California,
United States of America

Abstract

A FOKKER-PLANCK/TRANSPORT MODEL FOR NEUTRAL-BEAM-DRIVEN TOKAMAKS.

The application of non-linear Fokker-Planck models to the study of beam-driven plasmas is briefly reviewed. This evolution of models has led to a Fokker-Planck/Transport (FPT) model for neutral-beam-driven tokamaks, which is described in detail. The FPT code has been applied to the PLT, PDX and TFTR tokamaks, and some representative results are presented.

1. INTRODUCTION

The need for a Fokker-Planck [1] description of a beam-driven plasma was recognized in the 1950s. An early proposal [2] for a two-energy component DT reactor discussed the non-Maxwellian character of such a plasma, and in particular described the "depletion" of low-energy electrons caused by the presence of the hot ion component. This conjecture [3, 4] was studied using a two-species Fokker-Planck code [5], developed to calculate energy transfer from hot ions to cold electrons in a plasma [6].

Energy transfer between ions and electrons is usually calculated by using the Spitzer [7] formulae. These transfer rates are based on a quasi-equilibrium theory assuming that the electrons have Maxwellian velocity distributions. It was conjectured [3, 4] that the transfer rate would be less than the stated value in those cases where the ions are considerably hotter than the electrons. The ions exchange energy primarily with electrons whose velocities are lower than the mean ion velocity. Estimates were made [3, 4] that the slow electrons would be scattered by ions to higher velocities faster than they would diffuse downward in velocity to fill this hole in the distribution. This depletion of the small velocity end of the electron distribution was observed in the Fokker-Planck calculations [6]. The consequence in these cases is that the transfer rates are less than the Spitzer values.

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Fokker-Planck models are also needed for the study of beam-driven mirror-confined plasmas [8], because of the presence of the loss cone in velocity space and the ambipolar potential. Killeen and Futch [9, 10] and Fowler and Rankin [11, 12] solved the Fokker-Planck equations for both ions and electrons, assuming that the evolution of the distribution functions could be described by the equations for isotropic distributions, with certain factors included to take the presence of the loss cone and ambipolar potential into account. The Fowler and Rankin code was for a steady-state model, whereas the Killeen and Futch code was time-dependent and included the effects of charge exchange and time-dependent build-up of a plasma formed by neutral injection.

A multispecies model [13, 14] was developed in order to study beam-driven DT and D³He mirror reactors, including the effects of reaction products. The principal assumptions of this model are that the "Rosenbluth potentials" [1] are isotropic and that the distribution functions can be represented by their lowest angular eigenfunction. An extensive parameter study [14] was conducted, yielding values of the confinement parameter nτ and the figure of merit Q (the ratio of thermonuclear power to injected power) as a function of mirror ratio and injection energy.

The first injection of neutral beams into tokamak plasmas took place at the Culham, Princeton and Oak Ridge Laboratories in 1972–73. The injected ions were studied with linearized Fokker-Planck models [15–17] and the expected plasma heating was observed experimentally.

With the advent of much more powerful neutral beams, it is now possible to consider neutral-beam-driven tokamak fusion reactors. For such devices, three operating regimes [18] can be considered: (a) the beam-driven thermonuclear reactor; (b) the two-energy component torus (TCT); and (c) the energetic-ion reactor, e.g. the counterstreaming ion torus (CIT). To study reactors in regimes (b) or (c), a non-linear Fokker-Planck model must be used because most of the fusion energy is produced by beam-beam or beam-plasma reactions. Furthermore, when co- and counter-injection are used, or major radius compression is employed, a two-velocity space-dimensional Fokker-Planck operator is required.

Fortunately, a non-linear, two-dimensional, multi-species Fokker-Planck model [13, 19] had been developed for the mirror programme in 1973. This model was applied successfully to several scenarios of TCT operation [20–22].

An important element of these simulations is the calculation of the energy multiplication factor Q [19] for the various operating scenarios. This involves an accurate calculation of ⟨σν⟩ for each pair of reacting species. The methods developed for computing these multi-dimensional integrals are reported elsewhere [23, 24] and are briefly reviewed in Section 2.

The successful application of the two-dimensional Fokker-Planck model to the energy multiplication studies of TCT led to the formulation of a more complete model of beam-driven tokamak behaviour [25]. The Fokker-Planck/
Transport (FPT) code in its present form is described in Section 2. The FPT code has been evolving since 1975 [22], and it has been applied to a CIT reactor study [26] and to the large Princeton tokamaks [27–30]. Some representative results are presented in Section 3.

2. FOKKER-PLANCK/TRANSPORT MODEL

Neutral-beam-heated tokamaks are characterized by the presence of one or more energetic ion species which are quite non-Maxwellian, along with a warm Maxwellian bulk plasma. This background plasma may be described by a set of fluid equations. However, for scenarios in which there is a large energetic ion population, it is very important to represent the energetic species by means of velocity space distribution functions and to follow their evolution in time by integrating the Fokker-Planck equations. It is essential to utilize the full nonlinear Fokker-Planck operator to ensure that the slowing down and scattering of these energetic species are computed accurately and realistically.

The model presented here, in addition to solving one-dimensional radial transport equations for the bulk plasma densities and temperatures, solves nonlinear Fokker-Planck equations in two-dimensional velocity space for the energetic ion distribution functions. Moreover, neutral beam deposition and neutral transport are modelled using appended Monte Carlo codes developed elsewhere [31, 32].

2.1. Energetic ions

We consider an arbitrary number of energetic ion species whose presence derives from the ionization and charge exchange of injected fast neutrals. These species are described by distribution functions $f_b(v, \theta, r, t)$ in three-dimensional phase space, where $b$ denotes the particle species, $v$ is the velocity magnitude, $\theta$ is the pitch angle with respect to the magnetic field, and $r$ is the distance from the magnetic axis. We assume that the flux surfaces are concentric circular tori.

2.1.1. Fokker-Planck equations

The kinetic equation for the distribution function of energetic species $b$ is

$$\frac{\partial f_b}{\partial t} = \left( \frac{\partial f_b}{\partial t} \right)_c + H_b - S_{bc} + S_{bcx} + \left( \frac{\partial f_b}{\partial t} \right)_E + \left( \frac{\partial f_b}{\partial t} \right)_r - L_b - L_{orb}$$ (2.1)
The collision term \((\partial f_b/\partial t)_c\) is given by the complete non-linear Fokker-Planck operator as derived by Rosenbluth et al. [1]. It may be expressed in the form

\[
\frac{\partial f_b}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} \left( A_b f_b + B_b \frac{\partial f_b}{\partial v} + C_b \frac{\partial f_b}{\partial \theta} \right) + \frac{1}{v^2 \sin \theta} \frac{\partial}{\partial \theta} \left( D_b f_b + E_b \frac{\partial f_b}{\partial v} + F_b \frac{\partial f_b}{\partial \theta} \right)
\]  

where the coefficients \(A_b\) through \(F_b\) are sums of moments of the distribution functions of all charged species present [13, 19]. The quantity \(H_b\) is the source resulting from the injection of neutral species \(b\); \(S_{bc}\) represents the deceleration of energetic ions into the bulk plasma; \(S_{bcx}\) represents charge exchange between ion species \(b\) and the various neutral species. The quantity \((\partial f_b/\partial t)_E\) models the effect of the toroidal electric field. The term \((\partial f_b/\partial t)_r\) represents radial diffusion of the energetic ions. \(L^p_b\) is a fusion depletion term, and \(L^{orb}_b\) represents orbit losses. These terms are thoroughly described in the next two sections.

The numerical solution of this type of equation has been extensively studied [13, 19]. In the FPT code a new, fast, vectorized programme package [33] is utilized. We employ either implicit operator splitting or the Peaceman-Rachford Alternating Direction Implicit (ADI) method [34].

It is not actually necessary to solve for distribution functions \(f_b\) on every flux surface where the bulk plasma ions are defined. Treating the energetic ions in detail on every fifth flux surface combined with cubic splines of velocity-space-integrated quantities yields accurate answers in a good deal less computer time.

2.1.2. Neutral beam deposition

The energetic ion source term \(H_b\) is calculated using the FREYA neutral beam deposition code [31]. This is a Monte Carlo code which takes into account the geometry of the tokamak and the precise locations and optical properties of the neutral beam injectors. A pseudo-collision technique is employed, i.e. particle penetration is based on the minimum mean free path throughout the plasma, and resulting collisions are analysed, a posteriori, to see if they were genuine or false. This pseudo-collision technique enables one to compute potential collision points without calculating the intersection of the neutral path with each flux surface.

For use in the Fokker-Planck/Transport code, several improvements have been made to FREYA:

(a) A multispecies background is allowed, i.e. the neutral mean free path is based on charge exchange and impact ionization with an arbitrary number of
ion species (in addition to electron impact ionization). The ionization and
charge-exchange cross-sections are taken from Ref. [35].

(b) The reaction rate \( \langle \sigma v \rangle \) for charge exchange and ion impact ionization is
computed by averaging the product of the cross-section \( \sigma \) and the relative velocity
over the ion distribution function. A 2-D table look-up procedure is used.

(c) All collisions with multiply charged ions are treated as ionizations, and
only one charge state of any given impurity is considered. The total reaction rate
between a neutral and an impurity ion of charge \( Z \) is taken as the equivalent
proton rate times \( Z^{1.35} \) [36].

(d) When a neutral-beam atom undergoes a charge-exchange, its location
and energy are stored for later use in the neutral transport module, permitting
the modelling of multiple charge exchanges and/or re-ionization.

(e) The initial orbit of each deposited ion is analysed. If that orbit strikes
the limiter, the ion is discarded. This calculation assumes conservation of the
toroidal component of the canonical angular momentum [37].

It is not necessary to call FREYA each timestep, as the neutral-beam
deposition term is usually slowly changing.

2.1.3. Other source and loss terms

Energetic ion transfer: Each energetic ion species \( b \) has a corresponding
background plasma component. As an energetic ion decelerates, if it is not
lost it will eventually join the bulk plasma. This process is simulated by
transferring all "hot" ions below a specified energy from the energetic ion
distribution function to the corresponding bulk plasma component. This loss
term, denoted \( S_{bc} \), satisfies

\[
\frac{m}{2} \int \frac{d}{d\mathbf{v}} S_{bc}(v, \theta, r) \mathbf{v}^2 \int S_{bc}(v, \theta, r) \mathbf{d}v = \frac{3}{2} T_e(r)
\]

(2.3)

where \( T_e \) is the electron temperature.

Charge-exchange: The charge-exchange term is of the form

\[
S_{bcx} = -f_b \sum_c \tilde{n}_c \langle \sigma v \rangle_{cx}^{cb}
\]

(2.4)

where \( c \) runs over all neutral species (including neutral-beam atoms) and \( \tilde{n}_c \)
is the corresponding neutral density. The charge-exchange rate is taken from
Ref. [36]. As can be seen, the charge-exchange probability is assumed to be
independent of ion energy.
Toroidal electric field: The acceleration by the electric field in the toroidal direction is given by

$$\left( \frac{\partial f_b}{\partial t} \right)_E = -a_{\theta \mu} \frac{\partial f_b}{\partial v_{\theta \mu}} = -\frac{Z_b e E_{\theta \mu}}{m_b} \left( \cos \theta \frac{\partial f_b}{\partial v} - \frac{\sin \theta}{v} \frac{\partial f_b}{\partial \theta} \right)$$

(2.5)

Radial diffusion: In a neutral-beam-heated plasma, the fast ions will have a velocity only two to three times greater than that of the bulk ions. Thus it is reasonable to expect that the fast ions are subject to a certain amount of radial diffusion. We approximate this by the term

$$\left( \frac{\partial f_b}{\partial t} \right)_r = -\frac{f_b}{n_b} \cdot \frac{1}{r} \frac{\partial}{\partial r} \left( r D_b \frac{\partial n_b}{\partial r} \right)$$

(2.6)

where \( n_b \) is the hot ion density and \( D_b \) is a diffusion coefficient. This operator diffuses density but preserves velocity-space shape.

Fusion depletion: For D-T plasmas a fusion loss term is included:

$$L^D_T = \hat{n}_T \langle \sigma v \rangle_D f_D \quad , \quad L^T_D = \hat{n}_D \langle \sigma v \rangle_T f_T$$

(2.7)

Here, \( \hat{n}_D \) and \( \hat{n}_T \) represent the total (bulk + hot) deuteron and triton densities, and the fusion rate, which is based on a cross-section given in detail in Ref. [14], is taken to be independent of energy.

Orbit losses: Orbits through the various meshpoints (\( v, \theta, r \)) are analysed. This is complicated by the fact that whether or not an orbit intersects the limiter depends on the poloidal angle. We assume that the energetic ions are distributed uniformly with respect to poloidal angle, and we throw out an appropriate number, based on the fraction of orbits which do intersect the limiter.

2.2. Bulk plasma ions and electrons

We consider an arbitrary number of bulk plasma ion species which are assumed to be Maxwellian in velocity space. These species are described by densities \( n_a(r, t) \) and by a common temperature profile \( T_i(r, t) \). The electrons have a separately computed temperature profile \( T_e(r, t) \), and their density is determined by quasineutrality, i.e.

$$n_e = \sum_{\text{bulk}} Z_a n_a + \sum_{\text{energetic}} Z_b n_b$$

(2.8)
2.2.1. Transport equations

The ion densities and the ion and electron temperatures are described by the following set of equations:

\[
\frac{\partial n_a}{\partial t} = - \frac{1}{r} \frac{\partial}{\partial r} \left( r \Gamma_a \right) + \int S_{bc} \frac{d\upsilon}{d\upsilon} \delta_{ab} + S_{ai} + S_{acx} - L_a^\alpha \tag{2.9}
\]

\[
\frac{3}{2} \frac{1}{r} \left( \frac{\partial}{\partial r} \right) \left( r \sum_a n_a T_i \right) = - \frac{1}{r} \frac{3}{2} \left( \frac{\partial}{\partial r} \right) \left( r \sum_a Q_a \right) + \frac{3}{2} \frac{1}{r} \sum_a S_{ab} E_{bc} \frac{d\upsilon}{d\upsilon} \delta_{ab} + \sum_a S_{ai} \tilde{E}_a + W_{cx} - \frac{3}{2} \sum_a \frac{L_a^\alpha}{r} + \sum_a \alpha_{ab} Q_{ab} + Q_\Delta + \sum_a Q_{a\alpha} \tag{2.10}
\]

\[
\frac{3}{2} \frac{1}{r} \frac{Q_e T_e}{r} = - \frac{1}{r} \frac{3}{2} \frac{1}{r} \left( \frac{Q_e}{r} \right) + Q_{eb} - Q_\Delta + Q_{e\alpha} - \frac{3}{2} \frac{1}{r} \frac{Q_e T_e}{r^2} + j_\phi \phi \tag{2.11}
\]

The quantities \(\Gamma_a\), \(Q_a\), and \(Q_e\) are particle and energy fluxes; \(E_{bc}\) is the mean energy of decelerated energetic ions; \(S_{ai}\) is the ionization source and \(\tilde{E}_a\) is the energy of neutral species \(a\); \(S_{acx}\) and \(W_{cx}\) describe charge exchange; \(L_a^\alpha\) represents fusion depletion; \(Q_{ab}\) models heating by the energetic species; \(Q_\Delta\) is energy exchange between bulk ions and electrons; \(Q_{a\alpha}\) is \(\alpha\)-particle heating; \(\tau_r\) is the radiation loss time; and \(j_\phi \phi\) represents Ohmic heating.

2.2.2. Transport models

The particle and energy fluxes are written as linear combinations of the density and temperature gradients and of the toroidal electric field. This makes possible the representation of a full multispecies neoclassical transport model, as described in Ref. [25]. However, present-day tokamaks do not seem to obey neoclassical scaling laws [38]; hence, the following transport model is employed.

We write the particle flux \(\Gamma_a\) as

\[
\Gamma_a = D_a \frac{\partial n_a}{\partial r} - R_a E_\phi \tag{2.12}
\]

where

\[
D_a = D_{0a} + D_{1a} r^2 + D_{2a} n_e + D_{3a} n_e^2 \tag{2.13}
\]
and

\[ R_a = 2.48c(r/R)^{1/2} \frac{n_a}{B_0} \tag{2.14} \]

The first term represents anomalous transport and the second term the effects of the Ware pinch.

The energy fluxes are written in terms of their convective and conductive components:

\[ Q_a = \frac{5}{2} r_a T_i + k_{ia} n_a \frac{\partial T_i}{\partial r} \tag{2.15} \]

\[ Q_e = \frac{5}{2} r_e T_e + k_{ie} n_e \frac{\partial T_e}{\partial r} \tag{2.16} \]

where

\[ T_e = \sum Z_a \left( D_a \frac{\partial n_a}{\partial r} - 0.8 R_a E_\phi \right) \tag{2.17} \]

For the ion thermal conductivity we employ the neoclassical formula of Connor [39]:

\[ k_{ia} = \frac{1.48c^2(r/R)^{1/2}T_i}{e^2B_0^2} \frac{m_a}{Z_a^2} \left\langle \frac{x_a^2v_a^2}{<v_a^2>} - \frac{<x_a^2v_a^2>}{<v_a^2>} \right\rangle \tag{2.18} \]

where the quantity in brackets is defined in Ref. [39]. For the electron thermal conductivity we use an empirical formula:

\[ k_e = k_e0/n_e + k_e1/n_eT_e \tag{2.19} \]

The current density \( j_\phi \) is specified (usually parabolic to the three halves power), and the toroidal electric field \( E_\phi \) is related to the current density through

\[ E_\phi = n_s j_\phi \tag{2.20} \]

where \( n_s \) is the Spitzer resistivity [7].
2.2.3. Charge exchange

The charge-exchange source for species \(a \) is expressed as

\[
S_{acx} = \tilde{n}_a \sum \Sigma n_d \langle \sigma v \rangle_{cx}^{ad} - n_a \sum \tilde{n}_c \langle \sigma v \rangle_{cx}^{ca}
\]  

(2.21)

Here, the first sum runs over all charged species (including energetic ones) and the second sum runs over all neutral species (including neutral beam atoms). The term \(\tilde{n}_c\) represents the density of neutral species \(c\), and \(\langle \sigma v \rangle_{cx}^{ca}\) is the charge-exchange rate between neutral species \(c\) and ion species \(a\).

The energy gained by the bulk ions due to charge exchange is

\[
W_{cx} = \sum \Sigma \tilde{n}_a \langle \sigma v \rangle_{cx}^{ad} E_a - \sum \tilde{n}_c \langle \sigma v \rangle_{cx}^{ca} \cdot \frac{3}{2} T_i
\]  

(2.22)

where \(a\) runs over all singly-charged bulk plasma ions, \(c\) runs over all neutral species (including beam neutrals), and \(d\) runs over all ions (including energetic ions). Recall that any charge exchange between a neutral and a multiply charged ion is treated as an ionization.

2.2.4. Ionization

The ionization source for species \(a\) is

\[
S_{ai} = \tilde{n}_a \left( \langle \sigma v \rangle_{ie} \epsilon_v + \langle \sigma v \rangle_{ib} \epsilon_b \right)
\]  

(2.23)

where electron and ion impact ionization are taken into account. As just noted, charge-exchanges with multiply charged ions are included in the second term. The ionization rate formulas are based on Ref. [35].

The ionization energy source is merely equal to

\[
\sum \tilde{E}_a S_{ai}
\]

where \(\tilde{E}_a\) is the energy of the neutral species \(a\). There is a drawback in the model in that energetic neutrals, upon ionization, become part of the bulk plasma. Energy is conserved but momentum is not. The fact that this energetic tail is assumed to thermalize instantly no doubt distorts the energy transfer with electrons.
2.2.5. Radiation

We consider only impurity radiation. The radiation loss time is written as

$$\tau_r = \frac{3}{2} \frac{T_e}{n} \sum Z n_z L_z$$

(2.24)

where the sum is over all impurity species. The cooling rate $L_z$ is expressed implicitly as

$$\log_{10} L_z = \frac{5}{2} \sum A_i \left( \frac{1}{\log_{10} T_e} \right)^i$$

(2.25)

where the coefficients $A_i$ are enumerated in Ref. [40]. An arbitrary number of impurity species may be considered.

2.2.6. Energy transfer

The energy transfer rate between bulk ions and electrons is

$$Q_\Delta = \sum_a \frac{3}{2} n_a \frac{T_e}{T_i} \tau_{ea}$$

(2.26)

where $\tau_{ea}$ is the Spitzer energy-exchange time [7]. The above sum runs over all bulk plasma ions.

The heating of plasma ions and electrons by energetic ions is obtained by integrating the appropriate part of the Fokker-Planck collision operator. This results in the formula

$$Q_{ab} = \int_0^\infty f_a(v) v^2 dv \left[ \int_v^\infty f_b(x)x dx - \frac{m_a}{m_b} \frac{1}{v} \int_b^v f_b(x)x^2 dx \right]$$

(2.27)

where $a$ represents the plasma species, $b$ is the energetic species, and $f_{a,b}$ are the respective distribution functions.

Alpha-particle heating is computed in a similar manner.

2.2.7. Miscellaneous terms

Fusion depletion: For D-T plasmas a fusion loss term is included:

$$L_D^\alpha = n_D \hat{n}_T <\sigma v>_{DT} \quad , \quad L_T^\alpha = n_T \hat{n}_D <\sigma v>_{DT}$$

(2.28)
Here the symbols $n_D$ and $n_T$ stand for the densities of the bulk deuterons and tritons, whereas the "hatted" symbols $\hat{n}_D$ and $\hat{n}_T$ include both bulk plasma and energetic ion contributions. The fusion rate is taken to be independent of energy.

Energetic ion transfer: The $\delta_{ab}$ appearing in Eq. (2.9) is a symbolic way of stating that plasma species $a$ and energetic species $b$ must really be the same species (e.g. both deuterons) for the transfer term to take effect. The quantity $E_{bc}$ in Eq. (2.10) is the energy at which particles are transferred; in most cases, $E_{bc} = \frac{3}{2}T_e$.

2.2.8. Discretization of the transport equations

Equations (2.9)–(2.11) may be cast in the form

$$\frac{\partial \vec{u}}{\partial t} = \mathcal{L}(\vec{u})$$

(2.29)

where the vector $\vec{u}$ consists of the bulk ion densities and the ion and electron energy densities. An implicit, iterative difference scheme is employed, i.e. we approximate Eq. (2.29) by

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\Delta t} = \rho \mathcal{L}_d(\vec{u}^{n+1}) + (1-\rho) \mathcal{L}_d(\vec{u}^n)$$

(2.30)

where $\Delta t$ is the time increment, $\vec{u}^n = \vec{u}(t = n\Delta t)$, $0 \leq \rho \leq 1$, and the spatially discretized quantity $\mathcal{L}_d(\vec{u}^{n+1})$, which approximates $\mathcal{L}(\vec{u}^{n+1})$, is linearized with coefficients depending on the latest iterate. In particular, products of derivatives are written as

$$\left(\frac{\partial f}{\partial r} \frac{\partial g}{\partial r}\right)^{n+1} \approx \frac{1}{2} \left[\left(\frac{\partial f}{\partial r}\right)^{n+1} \left(\frac{\partial g}{\partial r}\right)^* + \left(\frac{\partial f}{\partial r}\right)^* \left(\frac{\partial g}{\partial r}\right)^{n+1}\right]$$

(2.31)

where $^*$ refers to the latest iterate and $^\wedge$ denotes a central difference approximation. Products of a function and a derivative are written as

$$\left(f \frac{\partial g}{\partial r}\right)^{n+1} \approx f^* \left(\frac{\partial g}{\partial r}\right)^{n+1}$$

(2.32)
and products of functions are written as

\[(fg)^{n+1} = \frac{1}{2} \left[ f^{n+1} g + f^* g^{n+1} \right] \tag{2.33} \]

Second derivatives are approximated as

\[ \left[ \frac{\partial}{\partial r} \left( D \frac{\partial h}{\partial r} \right) \right]_j = \left[ \left( D \frac{\partial h}{\partial r} \right)_{j+1/2} - \left( D \frac{\partial h}{\partial r} \right)_{j-1/2} \right] / \Delta r \tag{2.34} \]

with

\[ \left( D \frac{\partial h}{\partial r} \right)_{j+1/2} = \left( D_j + D_{j+1} \right) \left( \frac{h_{j+1} - h_j}{\Delta r} \right) \tag{2.35} \]

where the subscript \( j \) indexes the radial variable.

**An exception:** The ion heat convection term \( \frac{1}{2} \Gamma_\alpha T_i \) uses the latest iterate for \( \Gamma_\alpha \) and treats \( T_i \) implicitly, even though \( \Gamma_\alpha \) contains derivatives. That is (dropping subscripts), for \( \Gamma < 0 \),

\[ \frac{\partial}{\partial r} (\Gamma T) \approx \frac{\Gamma_{j+1/2} T_j - \Gamma_{j-1/2} T_j}{\Delta r} \tag{2.36} \]

This linearization is appropriate for present-day transport models, in which ion heat convection dominates ion heat conduction — a fact that necessitates both implicit treatment of \( T_i \) and upwind differencing (as opposed to central differencing) of the heat convection term [41].

The boundary conditions are rather straightforward. At the limiter we impose small values of \( n_a, T_e \) and \( T_i \). At \( r = 0 \) we employ conservation boundary conditions, i.e. Eqs (2.9)—(2.11) are used but with flux derivatives

\[-\frac{1}{r} \frac{\partial}{\partial r} (rF) \]

replaced by \(-2F/r\) evaluated one-half meshpoint in from the centre. With the proper numerical integration scheme, the total number of ions and the total ion and electron energies are properly conserved (modulo known source and loss terms). The resulting system of difference equations is block tridiagonal, and it is solved using standard methodology [34].
2.3. Neutrals

We consider an arbitrary number of monatomic neutral species, described by densities $n_a(r, t)$ and mean energies $E_a(r, t)$. These neutrals result from: (a) charge-exchange of injected beam neutrals; (b) gas puffing; and (c) recycling from the limiter and wall. Neutral transport is computed using the AURORA code of Hughes and Post [32]. Although AURORA is a 3-D Monte Carlo code, it does not take toroidal effects into account, but instead assumes a long straight cylinder. This, of course, results in some inaccuracies in the treatment of energetic neutrals. AURORA does not use a pseudo-collision technique. The local mean free path and distance travelled per zone must be computed for each particle. It is the time-consuming nature of this latter computation which necessitates the assumption of a cylindrical geometry rather than a toroidal one.

As is the case with FREYA, several improvements have been made in AURORA. It is now a multispecies neutrals transport code. An arbitrary number of charge exchanges involving an arbitrary number of species may be considered. The reaction rates $\langle \sigma v \rangle$ are computed as in FREYA, and all collisions with multiply charged ions are treated as ionizations. In addition, neutrals can be launched from any radius, thereby enabling consideration of neutrals arising from charge exchange of injected beam neutrals. The neutral density profiles computed by AURORA are scaled to yield the correct integrated ionization rate. Also, neutral transport need not be computed every timestep, as that procedure would be too time-consuming.

2.4. Fusion

There are three contributions to the fusion reaction rate: (a) thermonuclear reactions, denoted $R_{11}$; (b) "beam-target" reactions, denoted $R_{12}$; and (c) reactions among the energetic ions, denoted $R_{22}$. At each plasma radius, the fusion reactivities $\langle \sigma v \rangle_{11}$, $\langle \sigma v \rangle_{12}$ and $\langle \sigma v \rangle_{22}$ are evaluated numerically via a five-fold velocity-space integral [23, 24]:

$$R_{ij} = \int f_i(v_i) f_j(v_j) \sigma (v_i, v_j) |v_i-v_j| dv_i dv_j$$

(2.37)

The $R_{ij}$ are then integrated over the plasma volume, to give the total reaction rate.

2.4.1. Deuteron plasmas

In deuteron plasmas two types of fusion reactions occur:

$$D + D = T + p + 4.04 \text{ MeV}$$

$$D + D = ^3\text{He} + n + 3.27 \text{ MeV}$$

(2.38)
Each reaction probability is computed separately, based on cross-sections found in Ref. [14]. We are thus able to monitor both the neutron production rate and the total fusion power. Because these reactions occur at such a slow rate, it is not necessary to include fusion depletion terms, nor is it necessary to consider the effects of reaction products.

2.4.2. Deuteron-triton plasmas

Here it is necessary to consider only the reaction

$$D + T = \alpha + n + 17.58 \text{ MeV} \quad (2.39)$$

as the number of D-D reactions will be orders of magnitude smaller. (The fusion cross-section may be found in Ref. [14].) Unlike the D-D case, the effects of the resulting fusion products (namely $\alpha$-particles) must be considered.

The $\alpha$-particle velocity distribution is taken to be the angle-averaged distribution given in Ref. [18]. Alpha heating is computed through integration of the Fokker-Planck collision operator. For computational convenience, all heat destined to be transferred from the alphas to the energetic ions is added to the bulk plasma ions instead. The $\alpha$-particle density is reduced in order to take into account the fact that some of the $\alpha$-particles will be lost on their first bounce. For this purpose we employ the subroutine of Shumaker [37]. Depletion of deuterons and tritons as a result of fusion is also modelled. This treatment of $f_\alpha(y)$ is reasonable only when plasma temperatures are changing slowly.

3. APPLICATIONS

The Fokker-Planck/Transport code has been applied to several neutral-beam-injected tokamaks, including the Princeton Large Torus (PLT), the Poloidal Divertor Experiment (PDX), the Tokamak Fusion Test Reactor (TFTR) and the Divertor Injection Tokamak Experiment (DITE) [27–30, 42]. We briefly highlight principal results of our applications to PLT and TFTR.

3.1. Applications to PLT

The Princeton Large Torus (PLT) has achieved record-setting temperatures. At high beam powers (e.g. 2.3 MW) and low plasma densities (e.g. $n_e(0) = 5.5 \times 10^{13} \text{ cm}^{-3}$) ion temperatures as high as 5.5 keV are reported [43]. Moreover, the fractional hot ion density on axis is measured to be up to 30%, and theoretical analyses indicate that, at low density, the majority of the fusion neutrons result from either beam-beam or beam-target reactions [30].
FOKKER-PLANCK/TRANSPORT MODEL

TABLE I. REFERENCE PLT PARAMETERS

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Major radius</td>
<td>1.40 m</td>
</tr>
<tr>
<td>Minor radius</td>
<td>0.40 m</td>
</tr>
<tr>
<td>B\textsubscript{toroidal}</td>
<td>3.2 T</td>
</tr>
<tr>
<td>Plasma current</td>
<td>(~0.5) MA</td>
</tr>
<tr>
<td>Neutral beam energy</td>
<td>(~35) keV</td>
</tr>
<tr>
<td>Neutral beam power</td>
<td>Up to 3 MW</td>
</tr>
<tr>
<td></td>
<td>85% full energy</td>
</tr>
<tr>
<td></td>
<td>15% half energy</td>
</tr>
<tr>
<td>Injection angles</td>
<td>(0^o, 180^o)</td>
</tr>
</tbody>
</table>

Realizing that the energetic ions play a very important role and that it is crucial to model their time-evolution as realistically as possible, we proceed to analyse PLT with the FPT code. A brief summary of PLT parameters is given in Table I.

3.1.1. Steady-state calculations

The first set of calculations is designed to evaluate the treatment of the energetic ions. Using experimentally measured profiles of the electron density \(n_e\), the electron temperature \(T_e\), the toroidal electric field and the impurity content \(Z\text{-effective}\), where

\[
Z\text{-effective} = \frac{1}{n_e} \sum_{\text{ions}} Z_a^2 n_a
\]

and is taken to be independent of radius and due only to carbon; also, using estimates of the ion temperature and neutral density profiles, the Fokker-Planck equations for the energetic ions are iterated to steady state. (The bulk plasma ion density is continually adjusted to maintain the prescribed electron density and impurity content.) The computed neutron fluxes are then compared with the experimentally measured values. This comparison is carried out at peak neutron production, when the experiment itself has reached a quasi-steady state. Results are shown in Fig. 1.
We see that there is excellent agreement between code and experiment over a wide range of beam power (0.3—2.3 MW) and plasma density. (The line-averaged electron density \( \bar{n}_e \), which is defined as \( \frac{1}{a} \int_0^a n_e(r) \, dr \), where \( a \) is the plasma minor radius, ranges from 1.5 to \( 6.2 \times 10^{13} \) cm\(^{-3} \).) The principal uncertainties include: (a) the ion temperature and neutral density profiles; (b) the fact that the experiment is not at an absolute steady state; (c) identification of the impurity; (d) calibration; and (e) the fact that only a limited number of experimental results are used.

### 3.1.2. Time-dependent calculations

Having established that the FPT code realistically models the energetic ions, we now consider time-dependent modelling of the beam-injection phase of the experiment. For this purpose we make direct comparisons with August 1978 PLT shots.

At time 0 (the time when the beams are turned on), the density profile is assumed to vary as \( (1-r^2/a^2)^3 \). Carbon and iron impurities are chosen with \( n_{\text{iron}} = 0.1 \cdot n_{\text{carbon}} \) and Z-effective constant in radius. The line-averaged electron density \( \bar{n}_e \) is matched to the experimentally measured value. The electron temperature and bulk ion temperature profiles also vary as parabolic cubed, and the initial temperature values on axis \( T_e(0) \) and \( T_i(0) \) are estimated. The energetic ion density is, of course, assumed to be initially zero.

The code is then run for 150 ms, which is the approximate duration of beam injection. The amount of gas puffing is dynamically determined to match the
FIG. 2. Ratio of computed to experimental neutron flux versus injection power for transient PLT calculations.

FIG. 3. Ratio of computed to experimental neutron flux versus electron line density for transient PLT calculations.
experimentally measured electron line density $\bar{n}_e$. The impurity density profiles are adjusted to maintain a constant Z-effective, and a recycling coefficient $R_c$ of 0.9 is prescribed. (The recycling coefficient $R_c$ is defined as the neutral influx at the limiter divided by the ion outflux.)

We take as our primary transport model $D = 5\times10^{16}/n_e$, $K_e = 2.4\times10^{17}/n_e T_e$, $K_i$ neoclassical, and include the effects of the Ware pinch, where the electron density $n_e$ is in units of cm$^{-3}$ and the electron temperature $T_e$ is in keV. The code results at $t = 150$ ms are then compared with the experimental measurements.

Figures 2 and 3 compare the computed neutron fluxes with the experimentally measured values over a wide range of injection power and electron line density, respectively. Agreement to within a factor of 1.5 is obtained. The computed electron temperatures on axis are significantly lower than the experimental values, as can be seen in Fig. 4. This suggests that errors in the electron transport are being balanced by errors elsewhere in the model, to yield the correct neutron flux.

We next investigate the effect of varying the transport model. Detailed comparisons of the electron density and temperature profiles for PLT run number 88214 are shown in Figs 5 and 6. The computed density tends to be higher close to the axis and lower away from the axis, almost independent of transport model. Remember that the area under each density curve is the same, since the gas puffing rate has been dynamically chosen to match the experimentally measured $\bar{n}_e$. For all of the transport models considered, the computed $T_e$ is lower than the experimentally measured value, and the experimental profile is more peaked on axis than any of the computed profiles. The shape and magnitude of the electron temperature varies considerably as the electron thermal
conductivity $K_e$ is varied. Comparisons with other experimental shots indicate that this trend is not uncommon. Thus it is difficult to cite a particular transport model as being truly optimal.

3.2. Applications to TFTR

The Tokamak Fusion Test Reactor (TFTR), which is currently under construction, is expected to come on line some time in the early 1980s. It is
hoped that this machine will achieve breakeven, i.e. that the power produced from D-T fusion reactions will exceed the power of the injected neutral beams. In fact, proposals are under consideration to enhance the injection capability so that the ratio of output to input power (denoted as Q) exceeds 2.

Several aspects of TFTR operation have been examined using the FPT code. Since this work has been presented in detail elsewhere [29], we will briefly summarize our investigation and its principal results.

We first compare the perpendicular injection of 120-keV D° beams into a tritium bulk plasma fuelled by recycling and gas puffing with the co- and counter-injection of 100-keV D° and 150-keV T° beams, respectively, into a plasma with a very low recycling coefficient. At small beam powers, the injection of D° and T° neutral beams is advantageous because the higher temperature and increased energetic ion fusion reaction rate outweigh the lower density. However, at large beam powers, the larger density with D° on T combined with a reduced electron thermal conductivity and an increased α-particle heating rate counteract the energy sink introduced by the cold puffed gas and recycled neutrals, resulting in a substantially higher fusion rate.

We also examine the effects of varying the bulk ion diffusion coefficient, the energetic ion diffusion coefficient, the ion thermal conductivity and the recycling coefficient. We find that Q is highly dependent on the particle transport, thereby making it essential that definitive information from present experiments on the magnitude of the particle diffusion coefficient at high temperature be obtained.

4. CONCLUSIONS

We have described in detail a Fokker-Planck/Transport code which is applicable to tokamaks in which there is intense neutral beam injection. For such scenarios, where there is a large energetic ion population it is essential to represent these energetic species by velocity space distribution functions and to follow their evolution in time by integrating non-linear Fokker-Planck equations.

We have performed simulations of two large tokamaks — the Princeton Large Torus (PLT) and the Tokamak Fusion Test Reactor. Since the PLT is an active experiment, we have had the opportunity to make direct comparisons with the experimental results. We find that the computed neutron fluxes and the experimentally measured values agree to within 50% over a wide range of beam power and plasma density. For the TFTR we have compared two modes of operation — the injection of D° beams coupled with tritium gas puffing, and the injection of both D° and T° neutral beams. We see that TFTR performance depends strongly on injection power, plasma density (which is a function of the recycling coefficient and gas puffing rate), mode of injection, and the assumed transport model.
REFERENCES

PHYSICS OF WAVE-PARTICLE INTERACTIONS*

K. NISHIKAWA
Institute for Fusion Theory,
Hiroshima University,
Hiroshima,
Japan

Abstract

PHYSICS OF WAVE-PARTICLE INTERACTIONS.

The paper consists of sections on the following topics: Landau damping; trapped-particle instability; collisionless drift-wave instability; momentum exchange between wave and particle; mode-coupling effects; effects of mode coupling on drift waves; diffusion due to mode coupling; effects of mode coupling on impurity diffusion. There is an appendix on the physical derivation of the Landau-damping rate.

1. LANDAU DAMPING

Let me start with the physics of the well-known Landau damping in a uniform unmagnetized plasma. We consider a wave represented by an electrostatic potential $\phi(x)$ in the wave frame as shown in Fig.1. We assume that the wave amplitude is small enough for the linear approximation to be applicable to all particles in the wave field. In real situations, the wave potential always has a finite lifetime which we denote by $\tau$. It may be limited by the Landau damping itself or by some other mechanisms. Then the particles in the wave field are roughly divided into two groups: the resonant and the non-resonant particles. The non-resonant particles are those which move rapidly in the wave frame and feel both acceleration and deceleration by the wave field within the lifetime $\tau$, while the resonant particles are those which are almost at rest in the wave frame and remain either accelerated or decelerated during the entire interval $\tau$. Obviously, the average kinetic energy averaged over the time $\tau$ of a non-resonant particle is unchanged because of the cancellation of the acceleration and deceleration effects, whereas that of a resonant particle either increases or decreases owing to acceleration or deceleration by the wave field. Energy exchange between the wave and the particle therefore takes place only through the resonant particles. The condition that a particle of initial velocity $v$ in the laboratory frame be a resonant particle is roughly given by

$$|v - \omega/k|\tau < \lambda/2 = \pi/k$$  \hspace{1cm} (1)

* Dedicated to the sixtieth anniversary of Professor Hajime Narumi, Faculty of Science, Hiroshima University.
where $\omega$, $k$, and $\lambda$ are the frequency, wavenumber, and wavelength of the wave in question. A resonant particle can move either slightly faster or slightly slower than the phase of the wave, and can stay either in an accelerating or a decelerating phase of the wave. However, when $v < \omega / k$, the particle can remain in resonance with the wave for longer periods by staying in an accelerating phase than in a decelerating phase. The situation is reversed when $v > \omega / k$. Those resonant particles which have velocities slightly less than the phase velocity, i.e. $v < \omega / k$, will therefore be in an accelerating phase than in a decelerating phase, while those which have velocities slightly greater than $\omega / k$, i.e. $v > \omega / k$, will more likely be in a decelerating phase. Since for a Maxwellian plasma there are more particles with kinetic energy $mv^2/2$ less than $m(\omega / k)^2/2$ than those with $mv^2/2$ greater than $m(\omega / k)^2/2$, we find that for a Maxwellian plasma there are more resonant particles which are accelerated by the wave field than those which are decelerated by it. In other words, a net energy flow takes place from the wave to the particle in a Maxwellian plasma. This causes a damping of the wave energy. A more elaborate calculation shows that this physical picture gives the correct expression for the Landau damping (see Appendix).

We note that the Landau damping is a damping of the wave energy and not of the wave amplitude. Indeed, if the wave has a negative energy, the Landau damping causes an amplification of the wave amplitude. We also note that the Landau damping is a statistical phenomenon and hence is intrinsically reversible; this is demonstrated by the phenomenon of the plasma wave echo [1].

Finally, we note that in the presence of magnetic field the wave propagation perpendicular to the magnetic field has no resonant particles and hence no Landau damping, since the particles cannot move freely across the magnetic field. Landau damping in a magnetic field can take place only through the particle motion along the magnetic field and hence needs a wavenumber component along the magnetic field. The resonance condition is given by

$$\omega = k_n v_n$$

where $k_n$ and $v_n$ are the components of the wavenumber and the particle velocity along the magnetic field.

FIG. 1. Wave-frame potential energy of a particle of charge $q$. 

$q\varphi(x)$

accelerating phase

decelerating phase
2. TRAPPED PARTICLE INSTABILITY

When the wave amplitude increases, acceleration and deceleration by the wave field become significant and none of the particles can remain any longer in either an accelerating or a decelerating phase over the entire period \( \tau \). Even the particles which are at rest in the wave frame at \( t = 0 \) will acquire a finite velocity in this frame and move from one phase to the other within the time \( \tau \). Then the concept of the linear Landau resonance breaks down. Since in this case all the particles experience acceleration and deceleration, there is no net energy exchange between the wave and the particle; the wave becomes undamped and stationary. This is called the BGK wave [2]. In this case, the particles are divided into trapped and untrapped particles. A trapped particle oscillates back and forth in a potential energy trough in the wave frame, whereas an untrapped particle moves across the potential energy crests. For a periodic wave train, the deeply trapped particles (those which are trapped near the bottom of the potential energy) undergo harmonic oscillation with the bounce frequency \( \omega_b = k\sqrt{\phi/m} \), where \( \phi \) is the amplitude of the potential and the other notation is obvious.

The question arises whether this oscillation can resonate with another wave. If so, this resonance can cause an energy exchange between the trapped particles and another wave. Let us call the BGK wave the carrier wave and another wave, which is resonant with the trapped particle oscillation, the sideband wave. Let \( k, \omega \) be the wavenumber and frequency of the carrier wave and \( k', \omega' \) be those of the sideband wave. The resonance condition is then given by

\[
\omega' - k' \frac{\omega}{k} = \pm \omega_b
\]

(2)

The left-hand side is the oscillation frequency of the sideband wave as seen in the carrier wave frame or in the frame moving with the average speed of the trapped particles.

\[ \text{FIG.2. Two simplest forms of trapped-particle distributions in the wave frame.} \]
We now determine whether this resonance causes damping or amplification of the sideband wave. To this end, we first note that the trapped-particle velocity distribution, \( F_t(u) \), in the carrier wave frame \( (u = v - \omega/k) \) is an even function of \( u \). Two simplest forms of \( F_t(u) \) are shown in Fig.2. For the case of Fig.2(a), there are more deeply trapped particles than shallowly trapped ones, whereas for the case of Fig.2(b) the situation is reversed. In the wave frame, a relatively deeply trapped particle has less kinetic energy than a relatively shallowly trapped particle. Then, owing to the wave-particle resonance given by Eq. (2), a relatively deeply trapped particle will on average acquire energy from the sideband wave, while a relatively shallowly trapped particle will remain resonant by losing its energy and hence will on average lose energy to the sideband wave. In other words, for the case of Fig.2(a), the trapped particles will on average acquire energy from the sideband wave, while for the case of Fig.2(b) they will on average lose energy to the sideband wave. Now, a sideband wave can either have positive energy or negative energy in the carrier wave frame, depending on whether the phase velocity of the sideband wave is greater than or less than that of the carrier wave. To be more specific, let us consider the electron plasma wave (Fig.3). There are two sidebands corresponding to the \( \pm \) sign in Eq. (2), i.e. the upper sideband for the case \( \omega' - k' \omega/k = -\omega_b \) and the lower sideband for the case \( \omega' - k' \omega/k = \omega_b \). Obviously, the upper sideband has lower phase velocity than the carrier wave while the lower sideband has greater phase velocity than the carrier wave. Therefore, the upper sideband has negative energy while the lower sideband has positive energy. Then for the case of Fig.2(a) the upper sideband is amplified and the lower sideband is damped, while for the case of Fig.2(b) the situation is just reversed.

Finally, we note that in the presence of trapped particles there also exists non-resonant sideband instability similar to the oscillating two-stream instability. A more detailed analysis can be found in Ref. [3].
3. COLLISIONLESS DRIFT-WAVE INSTABILITY

We now consider a non-uniform plasma immersed in a uniform magnetic field $\vec{B}$, and discuss the physics of the collisionless drift-wave instability. We take the direction of the magnetic field in the z-direction and the density gradient in the negative x-direction, as shown in Fig. 4. Drift waves propagate primarily in the positive y-direction with a small-wavenumber component in the z-direction. It can also have a wavenumber component in the x-direction, but we shall not consider it since it is not important in the present argument. Let us consider a potential as shown in Fig. 5. This potential is primarily produced by an ion density perturbation which is shielded by the electron motion along the magnetic field (Debye shielding). The electron and the ion charge density perturbations are therefore almost cancelled (charge neutrality) and the density perturbations are nearly in phase with the potential perturbation. Now, owing to the $E \times B$ drift motion, the ions move either in the positive or in the negative x-direction.

![Figure 4](image1.png)

**FIG. 4.** Geometry of the model; vector $\vec{k}$ denotes drift-wave propagation direction.

![Figure 5](image2.png)

**FIG. 5.** Potential $\phi$, density perturbation $\tilde{n}$ and electric field $\vec{E}$ of a drift wave and the direction of the associated $\vec{E} \times \vec{B}$ drift of the particles.
depending on the potential phase. Note that the electrons also undergo \(E \times B\) drift motion, but it is masked by their motion along the magnetic field. In the phase where the ions drift from the high-density side to the low-density side (i.e. to the positive \(x\)-direction), the density perturbation, hence the potential perturbation as well, increases, while in the phase where the ions drift from the low-density side to the high-density side (i.e. to the negative \(x\)-direction), the density and the potential perturbation decrease. This causes a propagation of the perturbation in the positive \(y\)-direction. The phase velocity \(\omega/k_y\) of the perturbation can easily be estimated from the linearized ion continuity equation:

\[
\frac{\partial \tilde{n}_i}{\partial t} = - \nabla \cdot (n_{i0} \tilde{v}_i) = - \frac{E_y}{B} \frac{\partial}{\partial x} n_{i0}
\]

where the tilde denotes the perturbation and \(n_{i0}\) the unperturbed ion density. Note that there is an unperturbed ion velocity due to the ion diamagnetic flow, but this does not produce any material flow and hence makes no contribution to the continuity equation. In the last equation, we have retained only the \(E \times B\) drift motion. Using the Fourier representation and noting the charge neutrality, \(\tilde{n}_e = Z \tilde{n}_i\), and the linear Boltzmann law for the electron Debye shielding, \(\tilde{n}_e = n_{e0} e \phi / T_e\) (\(T_e\) is the electron temperature), we obtain from Eq. (3)

\[
\omega n_{i0} \frac{e \phi}{T_e} = - k_y B \frac{\partial n_{i0}}{\partial x} \phi \quad \text{or} \quad \frac{\omega}{k_y} = \frac{\kappa T_e}{eB}
\]

where \(\kappa = - n_{i0}^{-1} \partial n_{i0}/\partial x\) is the inverse of the density gradient scale length.

Now, since the perturbation has a wavenumber component \(k_z\) along the magnetic field, there exist resonant particles, i.e. those particles whose parallel velocities \(v_z\) are close to the parallel phase velocity \(\omega/k_z\). If there is no density gradient, this Landau resonance causes a damping of the wave, since there are more particles which are accelerated by the wave field than decelerated by it. We assume that the wave frequency satisfies the following inequality:

\[
v_e \gg \frac{\omega}{k_z} \gg v_i
\]

where \(v_e\) and \(v_i\) are the electron and the ion thermal speed, respectively. Because of this inequality, there are practically no resonant ions, whereas there are a sufficient number of resonant electrons. Therefore, the Landau damping is mainly caused by the electrons. The time rate of change of the number of resonant electrons can be estimated as follows:
\[
\left[ \frac{dv_z}{dt} \frac{\partial F_e}{\partial v_z} \right] v_z = \omega/k_z \\
\sim -\frac{e}{m} E_z \left( -\frac{1}{v_e^2} \frac{\omega}{k_z} \right) F_e \\
\sim \frac{e}{m} i k_z \phi \left( -\frac{m}{T_e} \frac{\omega}{k_z} \right) F_e \\
\sim -i\omega \frac{\bar{n}_e}{n_0} F_e 
\]  

(6)

where we assumed that the unperturbed electron distribution is Maxwellian and used the relations \( E_z = -ik_z \phi \) and \( e\phi/T_e \sim \bar{n}_e/n_0 \).

---

**FIG. 6. Components of the drift-wave electric field, \( E_y \) and \( E_z \), and the direction of the associated \( E \times B \) drift motion. The straight lines denote the equiphase surfaces where the electric field vanishes.**

Let us next examine the effect of the density gradient on the resonant particles. By the same mechanism as explained in the beginning of this section, the \( E \times B \) drift motion along the density gradient causes a change of number of the resonant particles. Note that the resonant particles do not move along the magnetic field in the wave frame, so that the \( E \times B \) drift makes an important contribution to the continuity equation for the resonant electrons as well as that for the ions. The time rate of change of the resonant particles due to the \( E \times B \) drift is given by

\[
\frac{\bar{E}_y}{B} \frac{\partial}{\partial x} F \left( v_z = \frac{\omega}{k_z} \right) = i \omega_\ast \frac{\bar{n}_e}{n} F \left( v_z = \frac{\omega}{k_z} \right) 
\]

(7)

where \( \omega_\ast \left( = k_y \kappa T_e/eB \right) \) is the drift frequency. As seen from Fig. 6, the particle drifts from the high-density side to the low-density side in the electron decelerating phase or in the ion-accelerating phase, and from low to high densities.
in the ion-decelerating or electron-accelerating phase. This means that the $E \times B$ drift increases (or decreases) the number of decelerating (or accelerating) electrons and that of accelerating (or decelerating) ions. Therefore, the electron (or ion) $E \times B$ drift tends to amplify (or damp) the wave energy. Neglecting the resonant ions and comparing Eqs (6) and (7), we find that the energy of the drift wave amplifies, and hence it becomes unstable, if $\omega < \omega_u$.

The frequency of the drift wave is determined by the ion continuity equation. In practice, since the ion $E \times B$ drift is impeded by the finite Larmor radius effect, the ion inertia effect, etc., the actual wave frequency always tends to be reduced as compared with the value estimated by Eq. (2), so that the collisionless drift wave is always unstable provided that the magnetic field is uniform in space.

4. MOMENTUM EXCHANGE BETWEEN WAVE AND PARTICLE

Every wave has a momentum as well as an energy. Let $W_k^+$ be the energy of the wave of wavenumber $k$ and frequency $\omega$. Then the wave momentum $P_k$ is given by

$$P_k = \frac{k}{\omega} W_k$$

(8)

Its time rate of change is therefore given by the same equation as the wave energy. In the frame moving with the group velocity of the wave it can be written as follows:

$$\frac{dP_k}{dt} = \frac{k}{\omega} \frac{dW_k}{dt} = .2 \gamma_k \frac{k}{\omega} W_k$$

(9)

where $\gamma_k$ is the growth rate of the wave.

In the presence of a magnetic field, the momentum change across the magnetic field due to wave-particle interaction causes a particle drift across the magnetic field. The drift velocity $\vec{V}_D$ is given by

$$\vec{V}_D = -\left(\frac{dP_k}{dt} \times \vec{B}\right)/qB^2n_{res}$$

(10)

where $q$ is the particle charge and $n_{res}$ the number of resonant particles which take part in the wave-particle interaction. Since the direction of $dP_k/dt$ is random in
general, the direction of $\vec{V}_D$ is also random, so that the resultant particle flow becomes a diffusion process. This is the physical mechanism of the anomalous diffusion due to the linear wave-particle resonance [4].

For the case of the drift wave, the resonant electrons (or ions) contribute to a net emission (or absorption) of the wave and hence a net momentum transfer to (or from) the wave in the positive $y$-direction. This results in a net drift of both the resonant electrons and ions to the positive $x$-direction or a diffusion density downward.

For quantitative estimate of the diffusion constant we make use of Eq. (9). The wave energy of the drift wave is given by

$$W_k = \frac{\epsilon_0 kD}{2} |\phi|^2 = \frac{n_0 T_e}{2} \left(\frac{\epsilon \phi}{T_e}\right)^2$$  \hspace{1cm} (11)

where $kD (=\sqrt{n_0 e^2/\epsilon_0 T_e})$ is the Debye wavenumber. Substituting (11) into (9) and then into (10), we obtain the electron diffusion flux as follows:

$$\Gamma_{ex} = n_e \text{res} \cdot V_{Dx} = \frac{1}{eB} \sum_k \left[ \frac{d\vec{P}_{ke}}{dt} \right]_{ey}$$

$$\Gamma_{ex} = \frac{1}{eB} \sum_k \frac{k_y}{\omega} \gamma_{ke} n T_e \left(\frac{\epsilon \phi}{T_e}\right)^2$$  \hspace{1cm} (12)

where the suffix $e$ stands for the electron contribution. Setting $\Gamma_{ex} = D_e \frac{\partial n_e}{\partial x}$, we obtain the electron diffusion constant $D_e$ as

$$D_e = \frac{1}{k^2} \sum_k \gamma_{ke} \left(\frac{\epsilon \phi}{T_e}\right)^2$$  \hspace{1cm} (13)

where we approximated $\omega$ by the drift frequency $\omega_\phi$. A similar expression is obtained for the ion diffusion constant:

$$D_i = -\frac{1}{k^2} \sum_k \gamma_{ki} \left(\frac{\epsilon \phi}{T_e}\right)^2$$  \hspace{1cm} (14)

where $\gamma_{ki} < 0$ since the resonant ion contributes to damping the wave. Since there are only few resonant ions, $|\gamma_{ki}|$ is small and hence the resulting ion diffusion is also small.
We note that the above expression for the diffusion constant can be obtained by a simpler argument as follows. As is well known, the diffusion process is described in terms of a random walk process and the diffusion constant is given by the formula

\[ D = \frac{\langle \Delta x^2 \rangle}{\tau} \]  

(15)

where \( \Delta x \) is the step length of the random walk and \( \tau \) the correlation time. We estimate \( \Delta x \) by \( \Delta v/\omega \) and \( \Delta v \) by the \( E \times B \) drift, \( \Delta v = E_y/B \sim k_y \phi/B \).

Substituting this expression into (15), we obtain

\[ D \sim \sum_k \frac{k_y^2}{\omega^2 \tau} \frac{\langle |\phi|^2 \rangle}{B^2} = \sum_k \left\langle \left| \frac{e\phi}{T_e} \right|^2 \right\rangle \left( \frac{\omega_e}{\omega} \right)^2 \frac{1}{\kappa^2 \tau} \]

(16)

Setting \( \tau \sim |\gamma_{ks}^{-1}| \), we obtain Eqs (13) and (14).

The above analysis can be generalized to the case of a diffusion due to electromagnetic fluctuations. When the plasma \( \beta \)-value exceeds the electron-to-ion mass ratio, the drift wave is accompanied by magnetic perturbations. In this case, the drift velocity \( \Delta v \) is modified by the magnetic Lorentz force as

\[ \Delta v \sim [(\hat{E}_y + v_z \hat{B}_x)/B]v_z = \omega/k_z \]

from which we obtain [5]

\[ D = \frac{1}{\omega^2 \tau} \frac{1}{B^2} \left\langle \left| \hat{E}_y + \frac{\omega}{k_z} \hat{B}_x \right|^2 \right\rangle \]

(17)

Since the magnetic correction to the drift wave arises owing to its coupling to the Alfvén wave, for which \( \hat{E}_y + (\omega/k_z) \hat{B}_x = 0 \), the diffusion constant tends to be reduced by the electromagnetic effect or by the finite-\( \beta \) effect. A more elaborate analysis is given in Ref. [5].

Let me conclude this section by referring to the concept of radio-frequency flux control (RFFC) which was proposed by Itoh and Inoue [6]. As shown above, the wave-particle interaction is accompanied by both energy and momentum exchange between the wave and the resonant particles. In the usual concept of
radio-frequency heating, only the energy exchange between the wave and the particles is taken into account. In RFFC, we consider the effect of momentum exchange as well, which determines the diffusion flux. As seen from the above argument, the wave-induced electron diffusion flux is proportional to $\gamma_{fe}$, which in turn is proportional to $(\omega_* - \omega)$. Therefore, if we can externally excite a wave of frequency greater than the drift frequency, the resulting electron diffusion flux becomes negative or density-upward. By this method we can in principle suppress the anomalous-diffusion loss of electrons and hence the resulting energy loss of electrons. The concept of RFFC is then to optimize the RF-heating of the plasma by taking into account both the heating rate and the energy flux due to wave-induced diffusion.

5. MODE-COUPLING EFFECTS

When an instability evolves to a turbulent state, we need to take into account the mode-coupling effect. Within the framework of the weak turbulence theory, the lowest-order mode coupling due to wave-particle interactions occurs under the resonance condition

$$\omega - \mathbf{k} \cdot \mathbf{v} = \omega' - \mathbf{k}' \cdot \mathbf{v}$$

in an unmagnetized plasma and

$$\omega - k_{||} v_{||} = \omega' - k_{||}' v_{||}$$

in a magnetized plasma, where the two wave modes $(\mathbf{k}, \omega)$ and $(\mathbf{k}', \omega')$ couple to each other via particles of velocity $\mathbf{v}$. The process is called non-linear Landau damping since it can be regarded as Landau damping of the beat mode $(\mathbf{k} - \mathbf{k}', \omega - \omega')$.

The physical picture of this process can be explained as follows (Fig.7). Suppose a wave specified by $(\mathbf{k}, \omega)$ is incident on a charge cloud which is moving with velocity $\mathbf{v}$. The wave electric field produces an oscillating current, denoted by the current density $\mathbf{j}$, due to oscillation of this charge cloud. In the frame moving with the charge cloud, the oscillation frequency is given by $(\omega - k \cdot v)$. This oscillating current in turn emits another wave $(\mathbf{k}', \omega')$ which has the same frequency in the moving frame as the oscillating current, i.e. $\omega' - k' \cdot v = \omega - k \cdot v$. Alternatively, we can look at this process as an absorption of a quantum of the wave $(\mathbf{k}, \omega)$ accompanied by an emission of a quantum of the wave $(\mathbf{k}', \omega')$.

In other words, it is an induced scattering of a wave from $(\mathbf{k}, \omega)$ to $(\mathbf{k}', \omega')$ on the particle of velocity $\mathbf{v}$. Of course, the roles of $(\mathbf{k}, \omega)$ and $(\mathbf{k}', \omega')$ can be reversed.
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FIG. 7. Diagram showing an incident wave of \((\mathbf{k}, \omega)\) producing an oscillating current density \(\mathbf{j}\) which consists of a charge cloud moving with velocity \(\mathbf{v}\) and emits a wave of \((\mathbf{k}', \omega')\).

Since the charge cloud is always accompanied by a shielding charge, the oscillating current consists in general of two parts: the current due to the test charge cloud and that due to the shielding charge. Since the shielding charge has a sign opposite to the test charge, the two currents tend to cancel each other. If the cancellation is complete, the present mode-coupling process does not occur. Thus for the present mode-coupling process to occur, the existence of resonant particles (those which satisfy the resonance condition (18) or (19)) is not sufficient; there should, in addition, be no complete cancellation of the oscillating current.

To be more specific, let us consider the case of electron plasma waves in an unmagnetized plasma. In this case, the shielding charge consists mainly of electrons, as they are more mobile than ions. The wavelengths of the electron plasma waves are much greater than the Debye shielding length \(\lambda_D\), i.e. \(k\lambda_D\), \(k'\lambda_D \ll 1\). The velocity of the test charge which satisfies the resonance condition (18) is mostly less than the electron thermal speed \(v_e\), since \(|(\omega - \omega')(\mathbf{k} - \mathbf{k}')| \sim |\mathbf{k} + \mathbf{k}'|\lambda_D v_e\). Then the shielding of the test charge is almost complete within a distance much less than the wavelength. Suppose the test charge consists of electrons, then the oscillation speed induced by the wave field is about the same for both the test charge and the shielding charge. As a result, the current cancellation becomes almost complete. In other words, the present mode coupling does not occur significantly when the test charge consists of electrons. The mode coupling does occur if the test charge consists of ions. In this case, the oscillation speed of the test charge is much less than that of the shielding charge because of the mass difference, and hence the current cancellation never occurs. The oscillating current here mainly consists of the shielding electrons. Thus the dominant non-linear wave-particle interaction for the electron plasma wave is the induced scattering on ions.

6. EFFECTS OF MODE COUPLING ON DRIFT WAVES

Let us consider the effect of the above mode-coupling process on the drift wave. Consider first the case of an electron test charge. In this case, it is
efficiently shielded by the electron motion along the magnetic field, since 
\( k_B \lambda_D \ll 1 \), so that the oscillating current is almost completely cancelled for the same reason as in the case of the electron plasma wave. For the case of ion test charge, however, the electrons cannot shield it efficiently because of the rapid gyrating motion of the test ion. In other words, since the ion Larmor radius is much greater than the electron Larmor radius, the electron can shield the test charge only by the motion along the magnetic field, but since \( \omega_{ci} > k_B v_e \), even this motion cannot follow the ion motion. On the other hand, the surrounding ions can contribute to shielding the test ion by their Larmor motion. The guiding centres of the shielding ions are distributed over the distance \( \rho_i \) away from the test charge, where \( \rho_i \) is the ion Larmor radius. Now, if the wavelength perpendicular to the magnetic field \( \lambda_p \sim 2\pi/k_L \) is much greater than the ion Larmor radius, i.e. if \( k_L \rho_i \ll 1 \), then both the test ion and the shielding ion feel the same phase of the wave field and hence their oscillation speeds are about the same and therefore the oscillating current is almost cancelled. In other words, there is practically no mode coupling for the mode \( k_L \rho_i \ll 1 \). For the mode \( k_L \rho_i \gg 1 \), however, the test charge and the shielding charge feel different phases of the wave field, so that the current cancellation becomes incomplete. This is the case in which the present mode coupling plays an important role. We note that it is also the region where the collisionless drift wave has the largest growth rate. Thus we conclude that the induced scattering on ions becomes an important non-linear process for the collisionless drift mode of largest growth rate.

The mode-coupling effect on the drift wave can be looked upon, alternatively, as follows. We consider a test wave specified by \((k, \omega)\). The other drift waves constitute a background for this mode. We denote the electric field due to this background fluctuation by \( \vec{E} \). Then the fluid undergoes a \( \vec{E} \times \vec{B} \) drift motion which causes a Doppler shift of the test wave frequency:

\[
\omega \rightarrow \omega - k \cdot (\vec{E} \times \vec{B}/B^2) = \omega + \Delta \omega
\]  

(20)

Since \( \vec{E} \) is random, \( \Delta \omega \) can be considered a random-frequency modulation of the test wave. Now, effects of a random-frequency modulation have been extensively studied in the field of magnetic resonance. According to this theory, a weak random-frequency modulation (weak in the sense that \( |\Delta \omega| \ll \omega \)) produces an effective damping of the wave in the long time limit [7]:

\[
\gamma_{\text{eff}} = \langle \Delta \omega^2 \rangle/\omega
\]  

(21)

This is the effective damping due to the present mode-coupling process. We evaluate \( \gamma_{\text{eff}} \) using the perpendicular wavenumber \( k_L \) of the background fluctuation as follows:
\[ \gamma_{\text{eff}} \sim \sum_{\vec{k}_\perp} \left( \frac{\vec{k} \times \vec{k}_\perp}{\omega} \right)^2 \frac{\langle |\phi|^2 \rangle}{B^2} \sim \sum_{\vec{k}_\perp} \left( \frac{\vec{k} \times \vec{k}_\perp}{\omega} \right)^2 \frac{\langle |\phi|^2 \rangle}{k^2 \kappa^2} \omega \left\langle \frac{|\phi|}{T_e} \right\rangle^2 \] (22)

where we approximated \( \omega \) by \( \omega_\ast \).

This formula allows us to evaluate the saturation level of the drift wave, i.e. balancing this effective damping with the linear growth rate \( \gamma_L \), we obtain for the typical value of \( \kappa_\perp \)

\[ \left\langle \frac{|\phi|}{T_e} \right\rangle^2 \sim \frac{k^2}{\kappa^2} \frac{\gamma_L}{\omega} \] (23)

This is the formula obtained by Kadomtsev et al. [8] by a more elaborate mode-coupling calculation. Substitution of this saturation level into Eq. (13) yields the following explicit expression for the electron diffusion constant:

\[ D_e \sim \sum_k \frac{\gamma_L}{k^2} \left\langle \frac{|\phi|}{T_e} \right\rangle^2 \sim \sum_k \frac{\gamma_L}{k^2} \frac{\gamma_L}{\omega} \] (24)

which is the Kadomtsev formula for the wave-induced diffusion constant [8]. We note here that although we used the mode-coupling theory for evaluation of the saturation level, the expression for the diffusion constant is based on the linear wave-particle interaction process.

7. DIFFUSION DUE TO MODE COUPLING

In the process of induced scattering of drift waves on ions, the net absorption or emission of wave quanta does not occur. Since in most cases \( \omega \gg \kappa_\parallel v_\parallel \), the energy exchange between the wave and the ion is negligibly small. However, a significant momentum transfer occurs between the wave and the ion. The direction of the momentum transfer is random, so that the resulting drift of the ion takes place in a random direction. Because of the density gradient, this random drift causes a diffusion density downward.

As mentioned before, each ion can contribute either to a test charge or to a shielding charge of another test charge. When an ion contributes to a test charge, its diffusion is in the direction of low plasma density. However, a shielding charge consisting of ions for test ion charge is a hole surrounding the test charge. Therefore, diffusion of a shielding charge is nothing but a diffusion of an ion density hole. Obviously the hole density is high in the high plasma density region,
so that a diffusion density downward of the hole implies a diffusion density of the particle upward. In other words, any flow of ions (test charge) is always accompanied by an opposite flow of other ions (shielding charge) to maintain the local charge neutrality. The net diffusion flux of the ion then consists of the sum of the ion flow flux as test charge, $\Gamma_{xt}$, and the counter-ion flow flux as shielding charge, $\Gamma_{xs}$:

$$\Gamma_x = \Gamma_{xt} + \Gamma_{xs}$$  \hspace{1cm} (25)

with a large cancellation of the two fluxes:

$$\Gamma_{xt} \sim -\Gamma_{xs} \gg \Gamma_x$$  \hspace{1cm} (26)

The actual ion diffusion flux, of course, consists of the sum of $\Gamma_x$ due to the induced scattering process and the diffusion flux due to the linear wave-particle interaction process. The relative importance of the two fluxes depends on the fluctuation spectrum; if it is localized to the region where $\omega \gg k_{||} v_i$, the induced scattering process is dominant, while if the spectrum spreads over the region where $\omega \ll k_{||} v_i$, the linear process becomes more important. In any case, when the plasma is in a strictly stationary state (stationary turbulence), the diffusion becomes completely ambipolar [4]:

$$\Gamma_{ix} = \Gamma_{ex}$$  \hspace{1cm} (27)

since otherwise a local charge accumulation takes place.

We now examine the energy flux of ions due to the induced scattering process under consideration. As mentioned before, the guiding centre of a shielding ion is located at a distance $\rho_i$ away from the test charge, where $\rho_i$ is the Larmor radius of the shielding ion. Therefore, an ion of large Larmor radius can shield another ion far away from it (Fig.8). In other words, an ion of large Larmor radius can contribute to the shielding of other ions more efficiently than an ion of small Larmor radius. Since the test charge and the shielding charge are almost equal in magnitude, we can conclude from this observation that an ion of large Larmor radius (or an ion of large perpendicular energy) is more likely to act as a shielding charge than as a test charge, while that of small Larmor radius (or that of small perpendicular energy) is more likely to act as a test charge than as a shielding charge. Since the test charge diffuses toward the low plasma density side while the shielding charge diffuses in the opposite direction, we conclude that ions of small perpendicular energy tend to diffuse density downward, but those of large perpendicular energy tend to diffuse density upward. We thus expect that the perpendicular ion energy flows from the low-density to the high-density side.
This argument is valid when there is no temperature gradient and no net particle diffusion. For more precise argument, we have to examine the effects of both the temperature gradient and the energy flow due to the net particle diffusion. Let us first consider the latter effect, neglecting the temperature gradient. Because of the inequality (26), the energy carried by the net diffusion is usually small as compared with the energy carried by \( \Gamma_{xt} \) or \( \Gamma_{xs} \). Therefore the particle diffusion will not significantly modify the above argument. Let us next introduce the temperature gradient, \( \partial T_\perp / \partial x \), where \( T_\perp \) is the temperature of the perpendicular ion energy. We assume that the ratio

\[
\eta \equiv \frac{\partial \log T_\perp / \partial x}{\partial \log n_i / \partial x}
\]

(28)

is positive as in the usual low-\( \beta \) tokamak. If \( \eta \) is non-zero, there are more particles of high energy in the high-density side than in the low-density side, so that an additional energy diffusion arises towards the low-density side. Since the relative importance of this additional energy flow increases as \( \eta \) increases, we find that the net perpendicular ion energy flow becomes negative (temperature upward) when \( \eta \) is less than a certain critical value \( \eta_c \) and positive when \( \eta > \eta_c \). A more elaborate calculation described in Ref. [9] indeed shows this property with \( \eta_c \) ranging between 1 and 2. The analysis given in Ref. [9] also shows that the magnitude of the 'negative' heat flux for \( \eta = 0 \) is of the order of the neoclassical value at the plateau regime and that the parallel energy flow is positive but is small compared with the perpendicular energy flow.

8. EFFECTS OF MODE COUPLING ON IMPURITY DIFFUSION

Essentially the same argument as the ion energy flow can be applied to the impurity diffusion due to the induced scattering of drift waves on ions. We assume that the impurity concentration is sufficiently small for the linear
dispersion characteristics of the drift wave to be little affected by the presence of impurities. If the thermal speed of heavy impurity ions is small compared with the parallel phase velocity of the wave, we can neglect the linear interaction between the wave and the impurity ion. Then the dominant effect of the drift wave on the impurity diffusion arises from the induced scattering process. Since the ratio \((Z_e/m)\) of the impurity ion is different from that of the host ion, the wave-induced current oscillation for a test impurity charge does not cancel that for the shielding charge, which consists mainly of host ions. We can also expect that the cancellation for the case where the test charge consists of host ions will become incomplete when impurity ions contribute to the shielding. We can therefore expect a significant mode-coupling effect on the impurity diffusion.

For the same reason as mentioned in the previous section, those impurities whose Larmor radii are larger than the Larmor radius of the host ion will diffuse toward the high-density side (negative diffusion), while those with smaller Larmor radii than the host ion will diffuse toward the low-density side (positive diffusion).

A similar argument can be applied to the mutual diffusion of a D-T mixture. If the temperatures of deuterium and tritium are the same, tritium will diffuse inward and deuterium will diffuse outward.

A more elaborate argument for the impurity diffusion can be found in Ref. [4].

Appendix

PHYSICAL DERIVATION OF LANDAU DAMPING RATE

Consider a wave represented by the electric field

\[ E(x, t) = E_0 \cos(kx - \omega t) \]

where \(E_0\) is taken to be constant. Let \(v_1\) and \(v_2\) be the two resonance velocities \((v_1, v_2 = \omega/k)\) with \(v_1 < v_2\). Consider the particles which are accelerated from \(v_1\) to \(v_2\) (or decelerated from \(v_2\) to \(v_1\)) during the time interval \(\tau\) and let their number be \(N(v_1 \rightarrow v_2)\) (or \(N(v_2 \rightarrow v_1)\)). The change of energy, \(\delta[mv^2/2]\), of the resonant particles owing to these processes is given by

\[
\delta [mv^2/2] = \frac{1}{2} m (v_2^2 - v_1^2) \{N(v_1 \rightarrow v_2) - N(v_2 \rightarrow v_1)\} = m \delta v_1^2 \left\{ N(v_1 \rightarrow v_2) - N(v_2 \rightarrow v_1) \right\} \quad (A-1)
\]
where $\delta v = (v_2 - v_1)$, and we approximated $(v_1 + v_2)$ by $2 \omega/k$. The quantity $N (v_1 \rightarrow v_2)$ can be written as the product of the transition probability $W(v_1 \rightarrow v_2)$ of a particle to be accelerated from $v_1$ to $v_2$ in time $\tau$ and the average number of particles at initial velocity $v_1$:

$$N (v_1 \rightarrow v_2) = W(v_1 \rightarrow v_2) F(v_1) \tag{A-2}$$

Similarly, we have

$$N (v_2 \rightarrow v_1) = W(v_2 \rightarrow v_1) F(v_2) \tag{A-3}$$

We calculate $W(v_1 \rightarrow v_2)$ using the solution of the equation of motion:

$$\delta v = \frac{q}{m} E_0 \int_0^\tau dt \cos \left[ kx (t) - \omega t \right]$$

$$\approx \frac{q}{m} \frac{E_0}{k\Delta v} \left\{ \sin (kx_0 + k\Delta v \tau) - \sin kx_0 \right\}$$

$$\approx \frac{q}{m} E_0 \tau \cos kx_0 \tag{A-4}$$

where $\Delta v = (v_1 - \omega/k)$, and we approximated $x(t)$ by $x_0 + v_1 t$, $x_0$ being the initial position of the particle. In the last line, we assumed that $|k\Delta v\tau| < 1$, which is the condition for resonant particles. We show the right-hand side of (A-4) in Fig.9, which shows that, for given $\tau$, $\delta v$ determines $x_0$ and vice versa. The probability $W(v_1 \rightarrow v_2)$ is then the probability for a particle at $x = x_0(\delta v)$ at $t = 0$. For the case of $W(v_2 \rightarrow v_1)$, we simply replace $\delta v$ by $-\delta v$. From the symmetry of Fig.9 we see that

$$W (v_1 \rightarrow v_2) = W (v_2 \rightarrow v_1) \equiv W (v_1, v_2) \tag{A-5}$$
Using this relation as well as (A-2) and (A-3) in (A-1), we obtain

\[
\delta \left[ \frac{1}{2} m v^2 \right] = m \delta v \frac{\omega}{k} \ W(v_1, v_2) \left\{ F(v_1) - F(v_2) \right\}
\]

\[
= -m \delta v^2 W(v_1, v_2) \left[ \frac{dF}{dv} \right]_{v=\omega/k}
\]

(A-6)

The time rate of change of the kinetic energy of the resonant particles can then be given by summing (A-6) over all the resonant particles and dividing it by \(\tau\). The summation over the resonant particles can be replaced by the integration with respect to \(v_1\) and \(v_2\) or, equivalently, with respect to \((v_1 + v_2)/2\) and \(\delta v\), over the resonance region. Integration with respect to \((v_1 + v_2)/2\) simply gives the resonance width \(2\pi/k\tau\), so that we get

\[
\sum_{\text{res. particles}} \delta \left[ \frac{1}{2} m v^2 \right]/\tau = \sum_s \int_{\delta v > 0} d\delta v \ m \delta v^2 W(v_1, v_2) \left[ \frac{dF}{dv} \right]_{v=\omega/k} \frac{2\pi}{k\tau^2}
\]

\[
= \frac{2\pi}{k} E_0^2 \sum_s \frac{q_s}{m_s} \langle \cos^2 kx_0 \rangle_s \left[ \frac{dF_s}{dv} \right]_{v=\omega/k}
\]

(A-7)

where we used (A-4) and

\[
\langle \cos^2 kx_0 \rangle_s = \int_{\delta v > 0} d\delta v \cos^2 kx_0 \ W(v_1, v_2)
\]

(A-8)

the suffix \(s\) denoting the particle species. As seen from Fig.9, we can replace the above integration by an appropriate integration with respect to \(x_0\). The integration region is from \(kx_0 = 0 \sim \pi/2\) and \(3\pi/2 \sim 2\pi\), so that we have \(\langle \cos^2 kx_0 \rangle_s = 1/4\). Equating (A-7) to \(-dW_k/dt\), we finally get

\[
\frac{dW_k}{dt} = -\frac{\pi}{2k} E_0^2 \sum_s \frac{q_s}{m_s} \left[ \frac{dF_s}{dv} \right]_{v=\omega/k}
\]

which is the well-known result of Landau damping.

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HIGH-INTENSITY CO₂ LASER-PLASMA INTERACTION EXPERIMENTS

A.A. OFFENBERGER
University of Alberta,
Edmonton, Alberta,
Canada

Abstract

HIGH-INTENSITY CO₂ LASER-PLASMA INTERACTION EXPERIMENTS.

High-intensity irradiation of solid and gas targets have shown clear evidence for a variety of laser-induced parametric processes. These include stimulated Brillouin and Compton scattering, filamentation, two-plasmon decay, harmonic generation, ponderomotive profile modification with accompanying resonance absorption, and the emission of fast particles and high-energy X-rays related to these various processes. Underdense laser-plasma interaction phenomena are discussed in an attempt to correlate theory and experiment.

INTRODUCTION

Interesting and important regimes of nonlinear phenomena result when electromagnetic radiation incident on a plasma is intense enough to couple linear plasma modes to the external pump permitting growth of the waves [1-14].

The primary motivation in these lectures is laser-driven fusion. Induced plasma density fluctuations can strongly influence energy coupling and transport in the laser-target interaction. Both scattering and absorptive processes can be modified by the presence of a strong pump wave as well as energy transport (through wave-particle interactions leading to anomalous collisional processes) [15,16].

Laser-induced fusion requires high-intensity irradiation of targets. Recall that for intensity I, the associated field strength $E_0$ is given by

$$\frac{E_0^2}{8\pi} c = I$$

(1)
and the oscillating velocity of an electron in a field \( E_0 \cos \omega_0 t \) is given by

\[
\nu_0 = \frac{eE_0}{m\omega_0} \approx \lambda_0 \sqrt{I} \\
= 25.7 \sqrt{I} \lambda^2 \text{ cm/sec}
\]  

(2)

where \( I = \text{watt/cm}^2 \), \( \lambda = \text{wavelength in \( \mu \text{m} \).} \) Thus for CO\(_2\) laser radiation, \( \lambda = 10.6 \mu \text{m} \)

\[
\nu_0 = 272 \sqrt{I} \text{ cm/sec}
\]  

(3)

For experiments to be discussed here, \( I \leq 10^{13} \text{ watt/cm}^2 \) whereby \( \nu_0 \leq 8.6 \times 10^8 \text{ cm/sec} \) and the oscillation energy \( \frac{1}{2} m\nu_0^2 = 210 \text{ eV} \). Irradiated material is instantly ionized, producing a hot plasma corona with which the incident radiation can interact.

Plasma scale lengths \( L \) may be determined by either focal dimensions or hydrodynamic flow. If the focused spot size is less than ion expansion \((c_s t)\), then \( L \) is set by the focusing optics. On the other hand, when hydrodynamics dominate, ion inertia limits the outward plasma flow to speeds of \( \approx 10^7-10^8 \text{ cm/sec} \) for temperatures of 100 eV \( \rightarrow 10 \text{ keV} \) and ion mass \( M = M_\text{H} \). This in turn determines the plasma scale length \( L \) which for \( c_s = 10^7 \text{ cm/sec} \) and laser pulse length \( \tau = 1-10 \text{ nsec} \) is 0.1-1 mm.

Consider then an electromagnetic wave with a field

\[
E_0 = E_\infty \ e^{-k_0 \cdot x} \cos(k_0 \cdot r - \omega_0 t)
\]  

(4)

incident on a plane target:
Waves for which
\[ \omega > \omega_p \] can propagate

where
\[ \omega < \omega_p \] are evanescent

\[ \omega^2_p = \frac{4\pi ne^2}{m}, \quad \omega^2_p/\omega^2_o = n/n_c \] (5)

Different density regimes permit different parametric couplings
(3-wave, 4-wave). An electrostatic fluctuation \( \delta n(k, \omega) \) with associated
field \( E(k, \omega) \) can be coupled by \( E, (k, \omega) \) to sidebands \( E, (k \pm k_o, \omega \mp \omega_o) \)
which may be electrostatic or electromagnetic in nature. These side­
bands, in turn, couple with \( E, \) through the ponderomotive force pro­
portional to \( \nabla (E, \cdot E, \) to accentuate the fluctuation field \( E(k, \omega) \).
We will not discuss higher order modes \( E, (k \pm 2k_o, 2 \omega \mp \omega_o) \) here.

Dispersion relations for the important natural modes in a
plasma are:

- **Transverse**
  \[ \omega^2_o = k_o^2 c_s^2 + \omega^2_p \] (t)

- **Electrostatic, longitudinal**
  \[ \omega^2_e = \omega^2_p + 3k^2 v_e^2 \] (l)

- **Ion, longitudinal**
  \[ \omega^2_s = k^2 c_s^2/[1 + k^2 \lambda_D^2] \] (s)

where
\[ v_e^2 = T_e/m, \quad c_s^2 = (2T_e + 3T_i)/M \]
\[ \lambda_D^2 = T_e/4\pi ne^2 \]

are electron thermal speed, ion acoustic speed, Debye length. We
note
\[ \omega_p \lambda_D = v_e \] (8)

Important parametric decays include:

- **Absorption**
  \{ t \rightarrow \ell + s \} electron-ion decay (EID)
  \{ t \rightarrow \ell + \ell' \} two-plasmon decay (2\( \omega_p \))

- **Scattering**
  \{ t \rightarrow t' + s \} stimulated Brillouin (SBS)
  \{ t \rightarrow t' + \ell \} stimulated Raman (SRS)

Conservation of energy and momentum impose phase-matching con­
ditions for all such processes:

\[ \omega_o = \omega_1 + \omega_2 \]
\[ k_o = k_1 + k_2 \] (9)

which restrict the density regions accessible to various decays.
Thus we find from frequency matching

\[ \begin{align*}
\text{EID} & \quad \omega_1 = \omega_k, \quad \omega_2 = \omega_s \ll \omega_1, \quad \omega_0 = \omega_k - \omega_p \\
2\omega_p & \quad \omega_1 = \omega_k - \omega_p = \omega_0/2, \quad \omega_0 = 2\omega_p \\
\text{SBS} & \quad \omega_1 = \omega_o - \omega_s, \quad \omega_2 = \omega_s \ll \omega_o, \quad \omega_0 > \omega_1 \\
\text{SRS} & \quad \omega_1 = \omega_o - \omega_k, \quad \omega_2 > \omega_p \quad \omega_0 \geq 2\omega_p
\end{align*} \]

the following density regimes of interest:

\[ \begin{align*}
\text{At } n_c & \quad - \text{ electron-ion decay } \quad t \rightarrow k+s \\
& \quad - \text{ oscillating two-stream instability} \\
& \quad - \text{ resonance absorption (linear process)} \\
\text{Below } n_c & \quad - \text{ stimulated Brillouin } \quad t \rightarrow t'+s \\
\text{At } n_c/4 & \quad - \text{ two-plasmon } \quad t \rightarrow k+l' \\
\text{Below } n_c/4 & \quad - \text{ stimulated Raman } \quad t \rightarrow t'+l
\end{align*} \]

In addition to the parametric decays discussed above, modulation \((k \rightarrow o)\) and filamentation \((k \cdot k_0)\) instabilities can result. Filamentation is a 4-wave process (both \(E_+\) and \(E_-\) enter) leading to beam-focusing effects which may severely modify target irradiation symmetry.

We note that decays giving rise to electrostatic waves also lead to heating as the wave energy is converted to thermal energy. In addition, strong ion fluctuations react back on the pump wave by influencing the electron motion. This can give rise to anomalous absorption, especially at \(n=n_c\), though it also holds true for short-wavelength fluctuations \((k \lambda_p \sim 1)\) even for \(n \geq n_c/4\) [16].

In these lectures we will concentrate on underdense \((n\lessgtr n_c)\) plasmas since:

(a) there is more experimental data and theory to compare;
(b) there is ready diagnostic access and plasma information to be obtained;
(c) the lower density regions are of considerable importance to efficient energy coupling.

With a few exceptions, critical layer \((n=n_c)\) phenomena have yet to be thoroughly studied, though effects are crucial to efficient absorption.
I am going to be using data from gas target experiments to illustrate what has been observed and compared with theoretical predictions in order to verify the existence and characteristics of: (i) stimulated Brillouin (Compton) scattering, (ii) filamentation, and (iii) two-plasmon decay. Since I will use data from my own laboratory extensively, it is appropriate to acknowledge the many individuals and laboratory programs which have contributed to the current state of knowledge on laser-plasma interaction [17-37].

GAS TARGETS

A variety of gas targets have been used to simulate solid targets for laser-plasma interaction studies. These include: (i) free jet expansion of high-pressure gas into vacuum through a small orifice, and (ii) supersonic laminar jets formed by expanding high-pressure gas through shaped orifices (see Figs 1 and 2). Virtues of the gas target include:

(a) variable gas density \( n \) from \( n < n_c \) to \( n > n_c \)
(b) non-destructive target which can be pulsed just prior to firing laser
(c) variable mass \( M \) and atomic number \( Z \)
(d) access of focused beam to gas-vacuum interface (no detonation waves)
(e) good diagnostic access for measurements
Since all densities are present in a solid target irradiation experiment, it is difficult to isolate and study individual parametric instabilities. The variable-density gas target is ideally suited for such experiments. Moreover, a variety of gases can be studied, though we have concentrated on $\text{H}_2$ and $\text{O}_2$ in the results to be discussed here.

Characteristics of the laser and plasma include:

- **Laser**
  - $\text{CO}_2$ gain-switched, $\lambda = 10.6 \, \mu\text{m}$
  - 40-60 Joules in 40 nsec pulse length
  - focused intensity for:
    - stable resonator, $I \leq 10^{12} \, \text{watt/cm}^2$
    - unstable resonator, $I \leq 10^{13} \, \text{watt/cm}^2$
LASER-PLASMA INTERACTION EXPERIMENTS

Plasma - density typically $0.1 \, n_c < n < n_c$
- temperature $50 \, \text{eV} < T_e < 160 \, \text{eV}$
- scale length $100 \, \mu\text{m} < L < 4 \, \text{mm}$
- variety of gases employed, though results here are chiefly for $H_2$ gas unless otherwise noted

THEORY

It may be shown by solving $\mathbf{\nabla} \cdot q (\mathbf{E}+v \times \mathbf{B})$ to second order that a ponderomotive force results, arising from $v \cdot \nabla v$ and $v \times \mathbf{B}$ accelerations

$$f_p = -\frac{e^2}{2m_w^2} \nabla (E_o + E_1 E_0 \cdot E_1) \quad (10)$$

The equivalent ponderomotive force on the ions ($\propto M^{-1}$) is neglected in the following equations. Solving

$$\begin{align*}
\frac{\partial f_e}{\partial t} + v \cdot \frac{\partial f_e}{\partial x} - \frac{eE}{m} \cdot \frac{\partial f}{\partial y} & = 0 \\
\frac{\partial f_i}{\partial t} + v \cdot \frac{\partial f_i}{\partial x} + \frac{Ze}{m} E \cdot \frac{\partial f_i}{\partial y} & = 0 \\
v \cdot E & = 4\pi e \left\{ Zf_i \, dv - f_e \, dv \right\}
\end{align*} \quad (11)$$

as a perturbation expansion, one obtains the dispersion relation [13] for the coupled linear modes in the presence of a pump wave $E = E_o e^{i(\mathbf{k}_o \cdot \mathbf{r} - \omega_o t)}$

$$\frac{1}{\chi_e(k,\omega)} + \frac{1}{1+\chi_i(k,\omega)} = \frac{k^2 v_o^2}{4} \left\{ \frac{|k-xe_o|^2}{k_D^2} - \frac{|k \cdot e_o|^2}{k^2 \omega^2} \right\}$$

$$+ \frac{|k_+xe_o|^2}{k_+D^2} - \frac{|k_+e_o|^2}{k_+^2 \omega^2} + 2 \frac{k \cdot k_0 c^2}{2 \omega_0^2 - \omega^2} \quad (12)$$

where $v_o = \frac{eE_o}{m \omega_o} e_o$

$$\begin{align*}
k_{\pm} & = k \pm k_o \\
\omega_{\pm} & = \omega \pm \omega_o
\end{align*} \quad (13)$$

Stokes (-) \quad Anti-Stokes (+)

$$D_\pm = k_{\pm} c^2 - \omega_{\pm}^2 = k^2 c^2 \pm 2k \cdot k_0 c^2 + 2 \omega_0 - \omega^2 \quad (14)$$
The equations are valid for low-frequency waves $\omega \ll \omega_o$, along with the requirement that either $\omega / \omega_o \ll 1$ or $kD \ll 1$ for all $\omega / \omega_o$. The plasma dielectric constant

$$\varepsilon = 1 + \chi_e(k, \omega) + \chi_i(k, \omega)$$  \hspace{1cm} (15)$$

where the electron and ion susceptibilities are given by

$$\chi_{e,i}(k, \omega) = \frac{\omega^2}{k^2} \int \frac{\delta f_{e,i}}{\varepsilon - \omega k \cdot \nu} d\nu$$  \hspace{1cm} (16)$$

We note that the $k \nu \omega_o$ terms give rise to electromagnetic instabilities and $k \cdot v_o$ terms give rise to electrostatic instabilities. The nature of the decays are determined by the denominator terms $\varepsilon_\pm D_\pm$. When $\varepsilon_\pm = 0$, $\omega = \omega_s$ we find electrostatic decay and absorption. When $D_\pm = 0$, $\omega = \omega_s$ we find electromagnetic decay and scattering.

Corresponding to the various decays, the associated density fluctuations can be written

$$\delta n_e = \int \delta f_e d\nu = - \frac{(1+\chi_e)}{4\pi e} \frac{i k \cdot E}{2}$$ \hspace{1cm} (17)$$

$$\delta n_i = \int \delta f_i d\nu = - \frac{\chi_i}{4\pi Z e} \frac{i k \cdot E}{2}$$ \hspace{1cm} (19)$$

$$\delta n_e = \int \delta f_e d\nu = - \frac{\chi_e i k \cdot E}{4\pi Z e}$$ \hspace{1cm} (20)$$

Consider, for example, only one high-frequency sideband $\omega_-, k_-$ (Stokes) and a low-frequency electrostatic mode $\omega \ll \omega_o$. Then for $D_- = 0$, $D_+ \neq 0$ (nonresonant)

$$D_- = (k^2 c^2 - 2k_o \cdot k c^2) = 0$$ \hspace{1cm} (21)$$

implies $k = 2k_o$ has the lowest threshold for instability. This corresponds to backscatter. The term $|k_+ \nu \omega_o|^2 / k_- D_-$ describes stimulated Brillouin and Raman scattering.

Keeping the $|k_+ \nu \omega_o|^2 / k_+ D_+$ term as well as $D_-$ for small $k$ leads to modulational instability and for $k \perp k_o$ leads to filamentation.
Hydrodynamic and Kinetic Regimes

If the phase velocity of the wave \( v \omega = \omega / k \) satisfies \( v \omega \gg v_{th} \), where \( v^2 = T_e / m, T_i / M \) the wave is weakly Landau damped. If, however, \( v \omega \sim v_{th} \), strong Landau damping leads to induced Compton scattering of the transverse wave by the particles [38-40]. This provides a heating mechanism through direct energy transfer from photon to particle in a scattering event.

For electrons, Landau damping is given by

\[
\frac{\gamma_e}{\omega} = \sqrt{\frac{\pi}{8}} \frac{1}{(k \lambda_D)^3} \exp \left[ - \frac{1}{2(k \lambda_D)^2} - \frac{3}{2} \right]
\]  

which is negligible for \( k \lambda_D \ll 1 \). As \( k \lambda_D \to 1 \), waves become heavily damped.

For ions, Landau damping is given by

\[
\frac{\gamma_i}{\omega} = \sqrt{\frac{\pi}{8}} \left\{ \left[ \frac{m}{M} + \frac{ZT_e}{T_i} \right] \text{exp} \left[ - \frac{ZT_e}{2T_i} - \frac{3}{2} \right] \right\}
\]  

which is negligible for \( ZT_e / T_i \gg 1 \). As \( ZT_e / T_i \to 1 \), ion sound waves become heavily damped.

Appropriate limits of the susceptibility must be used [see, for example, Refs 14, 40].

Hydro

\[
\chi_e = - \frac{\omega^2 e}{\omega^2 - 3k^2 v_e^2} ; \chi_i = - \frac{\omega^2 p_i}{\omega^2 - 3k^2 v_i^2}  
\]  

where \( \omega = \gamma_e, \omega = -i \gamma_i, s \)  

Kinetic

\[
\frac{\omega}{2\gamma_s} \to \frac{1}{(1 + \beta^2 \text{Re} F) + (\beta^2 \text{Im} F)^2} 
\]

\[
\tau = \beta^2 \text{Im} F  
\]

\[
\frac{\gamma_s^2}{2} \left[ \frac{ZT_e}{T_i} \frac{\Delta \omega}{kv_i} \right] \exp \left[ - \frac{1}{2} \left( \frac{\Delta \omega}{kv_i} \right)^2 \right] 
\]  

The definition for \( \beta^2 \) is given in Eqn. (49).
We shall concentrate on SBS, SCS and filamentation as important ion-related phenomena. These instabilities can dominate in the underdense region, leading to strong scattering and self-focusing of the incident radiation. In addition, we shall discuss the \(2\omega_p\) decay (though the dispersion eqn. (12) is not valid and must be modified).

**General Criteria for Instability**

A general requirement for instability can be described by the condition:

\[
\text{gain from pump wave} > \text{damping losses}
\]

Since the gain varies as \(v_o^2 \sim E_o^2/\omega_o^2\), the left-hand side may be expected to scale approximately as \(II^2\).

The effective damping losses indicated on the right-hand side will be determined by one of several possible mechanisms in the laser-plasma interaction region. These include, for plasmas which are:

- homogeneous — collisions, Landau damping \((\gamma_c, \gamma_L)\)
- finite length — convective losses \((k\nu)\)
- inhomogeneous — mismatch in energy and momentum

\[
\omega_o = \omega_1 + \omega_2, \quad k_o = k_1 + k_2
\]

Evolution of plasma conditions — density, temperature or scale length — during the interaction may also modify the instability threshold or growth, effectively turning off the decay process. In addition, pump depletion, particle trapping, particle heating, wave-breaking can limit fluctuation levels and scattering.

We may summarize the absolute or convective nature and conditions for instability for:

(a) infinite, homogeneous plasma, weak damping \([13,14]\)

\(\rightarrow\) absolute instability;

(b) finite, homogeneous, weak damping \([9]\)

\(\rightarrow \gamma_L/|v_1 v_2|^{1/2} > \pi/2\) absolute

\(< \pi/2\) convective;

(c) inhomogeneous, weak damping \([7]\)

\(\rightarrow \pi v_o^2 /k'|v_1 v_2| >> 1\)
(d) finite length, strong damping [14]

\[ \gamma_0^2 / \gamma_p v_1 \gg 1 \]  

where \( \gamma_0 \) = homogeneous growth rate  
\( \gamma_p \) = plasma wave damping rate \((\gamma_s, \omega_s)\)  
\( v_1, v_2 \) = group velocities of decay waves  
\( k' = d(k_0 - k_1 - k_2)/dx \) = wavenumber mismatch

For large damping and a strong pump wave, convective growth normally dominates with particle trapping and heating leading to wave-breaking and saturation.

**Random Phase vs. Coherent Waves**

If the interacting waves can be characterized as random phase, i.e.

\[ \Delta\Omega t >> 1 \]  

where \( \Delta\Omega \) = bandwidth of pump or sidebands  
\( t \) = characteristic time of process or experimental time

then \( |E|^2 \), the mean square value of the field strength, is the relevant parameter. Tsytovich [38] discusses this case thoroughly and gives probabilities for various decays using a quantum-particle description.

Interestingly, one can show that growth rates for SCS calculated from both random phase and coherent wave theory in the limit of strong ion Landau damping (i.e. kinetic regime) are the same. This is not a surprising result for waves with a broad frequency spectrum, i.e. \( \Delta\Omega \sim \gamma_s \omega_s t^{-1} \). This result again shows the equivalence of particle and wave descriptions.

**Linear Equations**

For slowly varying amplitudes, the coupled equations for the 3-waves can be written [14]

\[ \left[ \frac{\partial}{\partial t} + \left( \frac{c_p}{\partial x} + \gamma_p \right) \right] a_o(x,t) = \frac{\omega_p^2}{4\omega_o} \bar{a} a_o \]

\[ \left[ \frac{\partial}{\partial t} - \left( \frac{1}{\partial x} + \gamma \right) \right] a_-(x,t) = \frac{\omega_2}{4\omega_o} \bar{a} a_o \]

\[ \left[ \frac{\partial}{\partial t} + \left( -c_p \frac{\partial}{\partial x} + \gamma_p \right) \right] \bar{a}(x,t) = -\frac{1}{2} \frac{m}{M \omega} \frac{k^2 v^2}{a_o} a_+ a_0 \]
where the normalized amplitudes of the waves are defined by

\[
a_\circ = \frac{E_\circ(x,t)}{E_\circ(0,0)} = \text{incident wave} \quad a_-(x,t) = \frac{E_-(x,t)}{E_\circ(0,0)} = \text{scattered wave}
\]

\[
\eta = \frac{\delta n}{n_\circ} = \text{density wave} \quad \gamma, \gamma_p = \text{wave damping rates}
\]

\[(34)\]

\(c_p - v\) allows for plasma flow, where \(c_p\) = electron or ion group velocity.

In the limit of large plasma wave damping, convective growth will prevail in steady-state. Writing \(z = k_o x\), Eqns (33) reduce to

\[
\begin{align*}
\frac{\partial a_\circ}{\partial z} &= -\left(\frac{1}{4} \frac{\omega^2}{\omega_o^2} \frac{\nu^2}{v_e^2} \frac{\omega}{2\gamma_p} \right) a_\circ^2 a_0^4 \\
\frac{\partial a_-}{\partial z} &= -\left(\frac{1}{4} \frac{\omega^2}{\omega_o^2} \frac{\nu^2}{v_e^2} \frac{\omega}{2\gamma_p} \right) a_o^2 a_-^4 \\
\eta &= -\left(\frac{1}{2} \frac{\omega}{\omega_o^2} \frac{\nu^2}{v_e^2} \right) a_-^2 a_o^4
\end{align*}
\]

\[(35)\]

These equations can be integrated \([14,41]\) to give (on \(0 < x < L\)) an invariant

\[
|a_\circ|^2 - |a_-|^2 = \text{const.}
\]

\[(36)\]

and an implicit relation for the magnitude of the scattered wave:

\[
\frac{|a_-(x)|^2}{|a_-(0)|^2} = \left[|a_\circ(0)|^2 - |a_- (0)|^2 \right] e^{-2\eta k_o L} \left[|a_\circ(0)|^2 - |a_- (0)|^2 \right] \\
\frac{|a_- (0)|^2}{|a_\circ(0)|^2} = \left[|a_\circ(0)|^2 - |a_- (0)|^2 \right] e^{-2\eta k_o L} \left[|a_\circ(0)|^2 - |a_- (0)|^2 \right]
\]

\[(37)\]

where \(\eta = \frac{1}{4} \frac{\omega^2}{\omega_o^2} \frac{\nu^2}{v_e^2} \frac{\omega}{2\gamma_p}\)

Noting that \(|a_\circ(0)|^2 = 1\), identifying

\[
|a_- (0)|^2 = R = \text{reflectivity of scattered wave} \\
|a_- (L)|^2 = \varepsilon = \text{noise level of scattered wave referenced to the incident wave}
\]

\[(38)\]
and defining
\[ G = 2\eta k_o L \]  
the reflectivity equation may be written as
\[ R(1-R) = e^{G(1-R)} - R = e^{G(1-R)} \]  
\[ \text{(39)} \]

\[ \text{STIMULATED BRILLOUIN (COMPTON) SCATTERING} \]

Thresholds and growth rates for the different plasma conditions can be shown to be:

**Homogeneous:**
- **Threshold:**
  \[ \frac{v_o^2}{v_e^2} > 8 \frac{\gamma_s \omega_o}{\omega_s \omega_p} \]
  \[ > 4 \frac{\gamma_s \omega}{\omega_o} \]  
  \[ \text{(41)} \]
- **Growth:**
  \[ \gamma_o = \frac{1}{2} \frac{v_o}{c} \left( \frac{\omega_o}{\omega_s} \right)^{1/2} \omega_p \]
  \[ \text{(42)} \]

**Inhomogeneous:**
- **Threshold:**
  \[ \frac{v_o^2}{v_e^2} > \frac{8}{k_o L} \]
  \[ \text{(43)} \]
- **Growth:**
  \[ \frac{2\pi v_o^2}{|k'|\sqrt{v_1 v_2}|} \]
  \[ \text{(44)} \]

**Finite length, homogeneous, strong damping:**
- **Threshold:**
  \[ \frac{2\gamma_o^2 L}{\gamma_s^c} \geq 1 \]
  \[ \text{(45)} \]
- **Growth:**
  \[ G = \frac{1}{4} \frac{\omega_p^2}{\omega_s \omega_e} \frac{v_o^2}{v_e^2} \frac{\omega_s^2}{2\gamma_s} \frac{k_o L}{c_s} \]
  \[ \text{(46)} \]

The latter criterion dominates over inhomogeneity if [9]
\[ \frac{\gamma_s}{|k'|c_s} > \xi_{\text{plasma}} \]  
\[ \text{(47)} \]

For Brillouin scattering this condition would appear to be normally satisfied. For kinetic scattering SBS→SCS, we use Eqn. (26) to replace \( (\omega_s/2\gamma_s) \).

**Question:** When does linear Landau damping dominate nonlinear damping? Since the linear Landau damping decrement \( \gamma_s^L \) varies exponentially with \( (ZT_e/T_i^L) \) whereas the nonlinear damping \( \gamma_s^{NL} \)
FIG. 3. Ion acoustic dispersion characteristics for various ion wave-phase velocities \((\omega/k)\) showing the transition from weak ion Landau damping to strong kinetic damping.

\[ \beta = \sqrt{ZT_e/T_i} \]

\( a \) 1.00  
\( b \) 1.50  
\( c \) 1.75  
\( d \) 2.00  
\( e \) 2.25  
\( f \) 2.50  
\( g \) 2.75  
\( h \) 3.00

\( \frac{1}{\beta^2} \text{Im} \left( \frac{1 - \lambda_p}{1 + \lambda_p} \right) \)

\( x = \sqrt{\frac{m_1}{2kT_i}} \frac{(\omega - \omega_0)}{k^2 \omega_0} \)

We might expect that if strong ion heating occurs, i.e. \( T_i \rightarrow 2T_e \), then linear damping prevails. Now \( \beta \) in Eqn. (26) is given by

\[ \beta^2 = \frac{1}{1 + k^2 \lambda_L^2} \frac{ZT_e}{T_i} = \frac{ZT_e}{T_i} \]

for \( k \lambda_D \ll 1 \), which is normally satisfied.

We can approximately calculate the ion heating by balancing energy gain from Brillouin reflection with that free streaming away in the ions [27]

\[ n_i v_i T_i = I R \frac{\omega_e}{\omega_0} \]

which reduces to

\[ \left[ \frac{T_i}{ZT_e} \right]^{3/2} = \frac{1}{\beta^3} = R \frac{\omega_o}{\omega_p} \frac{v_o^2}{v_e} \]
We take \( \beta \leq 2 \) as the transition to strong linear damping (see Fig. 3) and thereby expect \( \gamma_s^L \) to dominate \( \gamma_s^{NL} \) if

\[
\frac{v_o^2}{v_e^2} > \frac{0.12}{R} \frac{\omega_p^2}{\omega_o^2}
\]

(52)

The parameter \( v_o^2/v_e^2 \) for 10.6 \( \mu \)m can be expressed in terms of \( \Lambda^2 \) and \( T_e \) as

\[
\frac{v_o^2}{v_e^2} = \frac{0.375}{T_e} \frac{\Lambda^2}{I}
\]

(53)

for \( \Lambda^2 \) in units of \( 10^{12} \) watt-\( \mu \)m\(^2\)/cm\(^2\), or for CO\(_2\) laser radiation

\[
\frac{v_o^2}{v_e^2} = 42 \frac{I}{T_e}
\]

(54)

with \( I \) in units of \( 10^{12} \) watt/cm\(^2\) and \( T_e = eV \). We now discuss experimental stimulated scattering results of 3 papers for under-dense hydrogen plasma and variable focused laser characteristics.

**Paper I:** J.Appl.Phys. 47, 1451(1976)

A schematic diagram of the laser-heated solenoid geometry used to investigate SBS is shown in Fig. 4 along with the measured spectral shape of the back-scattered radiation in Fig. 5 and growth rate in Fig. 6.

From the width of the spectrum, we would guess that kinetic scattering prevails (i.e. \( \nu_s \sim v_1 \)). Moreover, the bandwidth \( \Delta \nu = 2 \gamma_s \) and experimental time \( t \sim 10^{-8} \) sec suggest that the theory of Tsytovich should be applicable. The kinetic scattering rate is shown to be

\[
\gamma_k = \frac{(2\pi)^3}{(kT_i)^3} \frac{\pi R_b(x)}{k_0^3} \sec^{-1}
\]

(55)

which for \( R_o = e^2/mc^2 \), and identifying \( \omega_s/2\gamma_s \) of Eqn. (26) as \( \tau' \),

\[
\tau' = \frac{1 + \chi_i}{1 + \chi_i + \chi_e}
\]

\[
= \frac{\tau}{(1 + \beta^2 Re F_1)^2 + \tau^2} = \frac{\pi \beta^2 x \gamma_b(x)}{\tau}
\]

(56)
FIG. 4. Schematic diagram of laser-heated solenoid experiment used to experimentally investigate stimulated Brillouin backscatter from underdense hydrogen plasma.

FIG. 5. Spectrum of backscattered power from underdense hydrogen plasma at a focused CO$_2$ laser intensity of $\approx 10^{11}$ W cm$^{-2}$. 
FIG. 6. Backscattered power versus incident laser power from which the growth rate can be calculated.

where

$$\tau = \sqrt{\frac{\pi}{2}} \left( \frac{ZT}{T_1} \right) \left( \frac{\Delta \omega}{\nu_1} \right) \exp \left[ - \frac{1}{2} \left( \frac{\Delta \omega}{\nu_1} \right)^2 \right]$$

and

$$F_1(x) = 1 - 2xe^{-x^2} \int x^2 e^{t^2} dt + i\sqrt{\pi} x e^{-x^2}$$

(57)

It is then straightforward to show that the exponential growth

$$G = \gamma_k L/c$$

can be reduced to

$$G = \frac{1}{4} \frac{\omega_2}{\omega_0} \frac{\nu^2}{\nu_e} \tau' k_0 L$$

which agrees exactly with Eqn. (46) - the latter result derived from 2(Imk, )L by Cohen and Max. Thus in the kinetic limit, random phase and coherent pumps give the same induced Compton scattering rate.

Experimental parameters (measured or calculated from a hydrodynamic code) include:

- $$\varepsilon = 3.3 \times 10^{-7}$$
- $$R = 0.2\%$$
- $$I \leq 10^{11} \text{ watt/cm}^2$$
- $$L = 4.4 \text{ mm}$$
- $$T_e = 50 \text{ eV (calculated), } n_e = 4 \times 10^{17} \text{ cm}^{-3}$$
The scattered power is observed to vary exponentially with incident power, i.e. \( R = \epsilon \exp(G) \), for which the calculated \( G = 8.7 \). From the spectral shift and \( G \), we find \( \Gamma_i = 10 \text{eV} \), which agrees with the calculated value. The spectral width, however, is too large for the \( \beta^2 = 5 \) value.

**Question:** is linear Landau damping or collisional and nonlinear damping expected to prevail? From Eqn. (52), we calculate that ion heating requires \( \frac{\nu_o^2}{\nu_e^2} = 2.4 \gg \left( \frac{\nu_o^2}{\nu_e^2} \right)_{\text{expt.}} = 0.084 \). In fact, the maximum heating, Eqn. (51), is \( \approx 1 \text{ eV} \). We conclude linear damping does not limit wave growth or account for the full spectral width. Similarly, collisional damping \( \gamma_i = 10^{10} \text{sec}^{-1} \) is insufficient to account for an effective damping \( \Gamma_{\text{eff}} \approx \frac{\omega_s}{2} \sim 5 \times 10^{10} \text{sec}^{-1} \) required to explain the spectral width.

On the other hand, for nonlinear Landau damping to account for this width, ion density fluctuations of \( \sim 20\% \) are necessary. At first glance this seems unlikely, since the average \( \delta n/n \) calculated from Eqns (18), (20) is \( \sim 1.5\% \) for backscatter (note \( \delta n_e \approx \delta n_i \) for low-frequency fluctuations). Instantaneous values, however, are likely higher since the growth times are much less than detector response times. In fact, the experimentally derived growth rate \( \gamma_k = Gc/L = 5.9 \times 10^{11} \text{sec}^{-1} \) is larger than \( \omega_s = 1.1 \times 10^{11} \text{sec}^{-1} \) whereby strong ion turbulence would be expected. While not definitive, this evidence along with the inadequacy of linear Landau and collisional damping
would suggest that nonlinear Landau damping may account for the spectral width.

The experimental growth rate \( \gamma_k \) shows good agreement with the theoretical value of \( = 5 \times 10^{11} \text{sec}^{-1} \) derived from random phase (kinetic) Compton scattering.

**Paper II:** Can. J. Phys. 56, 381(1978)

A schematic diagram of the gas cell optical geometry is shown in Fig. 7 along with temporally resolved backscatter in Fig. 8, spectral profile in Fig. 9 and growth of backscatter in Fig. 10.

Growth and decay of SBS irregularly in time over a period equal to the incident laser pulse width is evident. Strong turbulence and random-phase characteristics seem likely responsible for the behaviour
FIG. 9. Spectrum of backscattered power from underdense plasma at a focused CO$_2$ laser intensity of $\approx 10^{12}$ W cm$^{-2}$. Correct wavelength error should be 6 Å.

FIG. 10. Backscattered power versus incident laser power for focused CO$_2$ laser intensity of $< 10^{12}$ W cm$^{-2}$.
as in the previous experiment.

Experimental parameters (again, measured or calculated) are:
\[ \epsilon = 2.2 \times 10^{-4} \]
\[ R = 20\% \]
\[ I \leq 10^{12} \text{ watt/cm}^2 \]
\[ L = 600 \mu\text{m} \]
\[ T_e = 110 \text{ eV}, T_i = 60 \text{ eV (calculated)}, n_e = 1.5 \times 10^{18} \text{ cm}^{-3} \]

The observed exponential variation of scattered power with incident power gives a value for \( G = 6.8 \). Note that the noise level \( \epsilon \) is substantially larger than in I. Again, from the spectral width we anticipate kinetic scattering and convective growth. The growth and shift are consistent with initial growth from plasma conditions given above, though the width implies broadening during scattering. In this experiment, however, conditions for appreciable ion heating exist since \( \left( \frac{v_i}{v_e} \right)^2 = 0.53 \) is greater than the required value of 0.09. Thus linear Landau damping should govern. We calculate from Eqn. (51) that the ions can be heated by \( 70 \text{ eV} \), which is more than adequate to account for \( \Gamma_{\text{eff}} \omega_s = 1.76 \times 10^{11} \text{ sec}^{-1} \). Also \( v_i \sim 10^9 \text{ sec} > 1/\tau_{\text{Laser}} \), which should permit adequate thermalization.

The experimental growth rate \( \gamma_k = 3.4 \times 10^{12} \text{ sec}^{-1} \) shows good agreement with the theoretical Compton value of \( 2.7 \times 10^{12} \text{ sec}^{-1} \) derived for initial \( T_i = 60 \text{ eV} \). Fluctuation levels \( \delta n/n = 22\% \) are calculated for maximum laser irradiance.


The experimental arrangement is the same as for II. Laser focusing was modified by changing the cavity to produce lower divergence, hence smaller spot size and higher intensity. The spectrum of scattered light is shown in Fig. 11 and growth of backscatter in Fig. 12.

In this case, clear saturation of backscatter is observed at a peak level of 60\%. Again, strong temporal modulation is seen.

Experimental parameters are:
\[ \epsilon = 6 \times 10^{-4} \]
\[ R = 60\% \]
\[ I \leq 10^{13} \text{ watt/cm}^2 \]
\[ L = 200 \mu\text{m} \]
\[ T_e = 100 \text{ eV}, T_i = 50 \text{ eV (calculated)}, n_e = 2 \times 10^{18} \text{ cm}^{-3} \]
FIG. 11. Spectrum of backscattered power from the gas cell target at a focused CO$_2$ laser intensity of $\approx 10^{13}$ W cm$^{-2}$.

The observed saturation in scattering implies, from $R(1-R) = \varepsilon \exp G(1-R)$, a value for $G = 15$. Again, from the spectral width we expect kinetic scattering and convective growth. From the width and $G$ value we find $T_i = T_e = 100$ eV, again implying ion heating during scattering. The ponderomotive term $(v_o^2/v_e^2)$$_{\text{expt}} = 4.2$ is much larger than the value required for effective ion heating ($0.04$).

Thus linear Landau damping should dominate, as is indeed observed. From Eqn. (51), ion heating of up to 500 eV is possible though the ion collisional rate of $v_i^{-1} \approx 10^8$ sec$^{-1}$ is slow enough to prevent equilibration of ion Compton heating during the pulse time.

The experimental growth rate $\gamma_k = 2.3 \times 10^{13}$ sec$^{-1}$ is in complete agreement with the theoretical Compton value for the plasma parameters.
LASER-PLASMA INTERACTION EXPERIMENTS

FIG. 12. Stimulated backscatter signal versus incident laser power, showing saturation of the instability for intensity < $10^{13}$ W·cm$^{-2}$.

Given above. Fluctuation levels $\delta n/n_0$ are calculated for the observed reflectivities.

Since $v_0$ $\sim$ $v_p$, particle trapping effects are expected to dominate. Furthermore, when damping is large the instability becomes convective, in which case, should ponderomotive pressure exceed thermal pressure, particle trapping and wave breaking will occur before pump depletion [14]. Filamentation could augment the ponderomotive effects. Thus wave breaking is thought to be the principal saturation mechanism.
Note in this experiment that, even for $T_i \sim T_e$ and short scale length, scattering is severe. It would therefore appear that ion heating as a saturation mechanism is limited. Importantly, the homogeneous growth rate $\gamma_o = \nu_o / 2c x (\omega_o / \omega_s)^{1/2} \omega_{pi} = 1.1 \times 10^{12} \, \text{sec}^{-1}$ for this set of plasma parameters is $\ll \gamma_k = 2.3 \times 10^{13} \, \text{sec}^{-1}$. Stimulated Compton scattering in the kinetic regime is far greater than stimulated Brillouin scattering in the hydrodynamic limit.

A schematic diagram of the optical geometry on the supersonic laminar jet (with O₂ gas) is shown in Fig. 13. Spectra of the backscattered radiation for various laser-target conditions are shown in Fig. 14, and the variation of backscatter with incident pump is shown in Fig. 15.

Markedly different results are seen in this experiment with targets having a critical layer (n=nₐ) in the axial direction. High-density plasma results in specular reflection of ω₀ radiation as well as down-shifting and decreasing the stimulated backscatter.
For typical parameters:  
\[ \varepsilon = 6 \times 10^{-4} \]
\[ R = 20\% \]
\[ I = 5 \times 10^{12} \text{ watt/cm}^2 \]
\[ L = 100 \mu \text{m} \]

with measured \( T_e = 160 \text{ eV} \), the broad spectrum again implies kinetic scattering. From the experimental gain \( G = 5.6 \), with heavy damping \( T_i \gg Z T_e \), we calculate that scattering is occurring near \( n/n_c = 0.24 \).

The ponderomotive term \( \left( \frac{v_o^2}{v_e^2} \right)_{\text{expt}} = 2.6 \) is greater than the value required for linear Landau damping (0.14) to dominate. The potential ion heating from Eqn. (51) is 1600 eV for \( Z = 6 \), and the ion collision rate \( v_i \approx 7 \times 10^9 \text{ sec}^{-1} \), which is large enough for thermalizing the ion Compton heating.

The growth rate is calculated to be \( \gamma_k = 1.7 \times 10^{13} \text{ sec}^{-1} \) and the fluctuation level \( \delta n/n \approx 0.7 \). Scattering is observed to be short-lived.
for the critical layer target as compared to the underdense target. Saturation of scattering at a lower value in this experiment seems likely due to a combination of short scale length, filamentation and perhaps also to the simultaneous occurrence of other effects. For example, as seen from the spectral shifts, a supersonic flow on the underdense side due to strong ponderomotive pressure would reduce the density significantly.

While this data has not been fully analysed, it is given to show that effects very different to those observed in strictly underdense plasma can result with a critical layer target; in particular, stimulated backscatter can decrease.

In summary:

(a) For finite $ZT_e/T_i$, SBS evolves to SCS. Induced scattering heats ions, providing positive feedback to maintain kinetic regime.

(b) Scattering rates can be far greater for SCS than for SBS.

(c) With strong ponderomotive pressure and large ion wave damping (SCS regime), particle trapping-heating-waved breaking is an important saturation mechanism.

(d) Even with strong damping, long scale length plasma will generate large scatter. Short $L$ and laser pulses seem desirable to limit backscatter.

(e) Broad ion turbulence is generated in SCS. This will enhance the anomalous collision frequency for electrons, leading to greater absorption of the incident laser radiation. It will also inhibit heat transport by the electrons.

**FILAMENTATION**

Filamentation is a four-wave nonresonant process leading to growth of striations across the beam [13,40,42]. It arises for $k \perp k_0$ from both $D_+$, $D_-$ being comparable.

\[
\text{Homogeneous: threshold } \frac{v_o}{c} > \sqrt{2} k_L \lambda_D \quad (58)
\]

\[
\text{growth (max) } \gamma_o = \frac{v_o}{c} \frac{\omega_{pi}}{\sqrt{2}} \quad (59)
\]
FIG. 16. X-ray signals detected through various aluminium foil thicknesses (a) - 3 μm; (b) - 12.5 μm; (c) - 18.5 μm) from underdense hydrogen plasma for focused CO₂ laser intensity \( \leq 10^{13} \text{W/cm}^2 \) on gas cell target.

In the hydrodynamic limit \( v_{ph} \gg v_e \), SBS dominates filamentation. In the kinetic regime \( v_{ph} \approx v_e \), however, growth of filamentation is comparable to growth of SCS. Since it is not a normal mode, damping is not a problem as for scattering.

Optimum growth occurs for

\[
\frac{k_e^2 c^2}{\omega_p^2} = \frac{1}{4} \frac{v_{ph}^2}{v_e^2} \frac{1}{(1+T_e/T_i)}
\]
Experimental Evidence for Filamentation

The experimental set-up using the gas cell target is described in paper III. Plasma and laser parameters are: $I \leq 10^{13}$ watt/cm$^2$, $n \sim 0.2 n_c$, $T_e = 100$ eV, hydrogen. The gas is important since bremsstrahlung for higher Z plasma would mask the non-thermal plasma emission associated with filamentation. For any quantitative calculation, it is necessary to observe hot X-ray signals superimposed on the cold thermal emission - measured using scintillator and photomultiplier with various absorbing foils.

The temporal behaviour of the X-ray emission is shown in Fig. 16. For the upper trace with 3μm Al foil, clear high-energy X-ray signals are seen superimposed on the cold X-ray background. The measured energy spectra of the hot and cold components are shown in Fig. 17,
from which we obtain $T_{cold} = 90$ eV and $T_{hot} = 1.2$ keV. The cold temperature was obtained at a time just prior to onset of the high-energy X-rays. It is known from our SCS results that $T_c = 100$ eV during the time of $T_{hot}$ emission.

With this information we may then make an approximate calculation of the emitting volume $V_h$ or characteristic linear dimension $L$ by equating $V_h = L^3$. From the ratio of the pulse heights $P_{h,c}$ and the known foil transmission factors for different temperatures one can solve

$$\frac{P_h}{P_c} = \frac{I_h}{I_c} \times \frac{n_n^2}{n_c^2} \times \frac{V_h}{V_c},$$

for $V_h$. Now the SCS results have shown $\delta n/n \sim 1$, which for $\nu_n^2/\nu_e^2 > 1$ implies wave-breaking as the dominant saturation and particle heating mechanism. Thus all local particles are expected to be heated, i.e. $n_h = n_c = n_i$. From pinhole photographs, we find $V_c = (400 \, \mu m)^3$. The calculated dimensions for many such shots ranged from 14 to 22 $\mu m$ with average $L = 18 \, \mu m$.

In order to confirm the existence of these small characteristic dimensions, photographic imaging of the backscatter was undertaken as well. Direct evidence was found in the form of intense filaments of 20 $\mu m$ size.

Additional evidence for correlation between SCS and the high-energy X-ray emission is shown in Fig. 18. The X-rays show a saturation characteristic with increased laser power identical to that found for backscatter. This similarity suggests that wave-breaking is responsible for both phenomena—decay of SCS and emission of hot electrons. Indeed, conversion of the ponderomotive energy to thermal by equating $E^2/8\pi = n_0 T_e$ gives $T_e = 1$ keV, a result very near the measured $T_{hot}$.

The threshold intensity for growth of filaments of 18 $\mu m$ size ($k = 3500 \, cm^{-1}$) is $\sim 10^{13}$ watt/cm, nominally the mean available intensity. Known hot spots in the beam, however, have sufficient intensity to permit growth of size $> \lambda_0$. We may calculate from Eqn. (61) for
FIG. 18. Plot of high-energy X-ray component showing saturation characteristic nearly identical to that found for stimulated backscatter.

$$\frac{v_0^2}{v_e^2} = 4, \frac{T_e}{T_i}, \frac{\omega_p^2}{\omega_0^2} = 0.2, \text{that } \lambda_0 \text{ optimum is } 1900 \text{ cm}^{-1} \text{ or } \lambda_i = 33 \mu m.$$  

Importantly, since thresholds and growth are comparable for SCS and filamentation in the kinetic regime, we expect them to occur simultaneously, as is observed in this experiment.

**TWO-PLASMON DECAY**

The modified dispersion relation keeping nonlinearities in the equations of motion and continuity give thresholds and growth rates of:

Homogeneous: 

$$\text{threshold} \quad \frac{v_0^2}{c^2} > \left( \frac{\gamma_p}{\omega_p} \right)^2$$  \hspace{1cm} (62)

$$\text{growth (max)} \quad \gamma_0 = \frac{k v_0}{2}$$  \hspace{1cm} (63)
Inhomogeneous: threshold

\[ \frac{2\pi \gamma_o^2}{k' |v_1 v_2|} \gg 1 \]  \hspace{1cm} (64)

growth

\[ G = \frac{2\pi \gamma_o^2}{k' |v_1 v_2|} \]

Absolute, inhomogeneous:

\[ \frac{\nu_o^2}{\nu_e^2} > \frac{3}{k_o L} \]  \hspace{1cm} (65)

Features of this decay include:

(a) Matching conditions \( \omega_o = \omega_k + \omega_\perp \), \( k_o = k_k + k_\perp \), localize the instability to near quarter-critical density.

(b) Maximum growth calculated from the \([ (k_o \cdot k_\perp) (E_o \cdot k_\perp) ] \) term \([12]\) is expected for decays at 45° w.r.t. \( E_o \) and \( k_o \).

(c) Inhomogeneity dominates boundedness \([9]\) if

\[ \frac{\gamma_o}{|k'| |v_1 v_2|^{1/2}} < \xi \quad \xi = \text{plasma length} \]

This is satisfied for most situations.

(d) In addition, growth is optimized for \( k_\perp \gg k_o \) but for weak Landau damping we require \( k_L \ll 1 \). The exponential gain \( G \) is then given by

\[ \frac{2\pi \gamma_o^2}{k' |v_1 v_2|} = \frac{\pi}{4} \frac{k_o^2 v_o^2}{\omega_p} \frac{1}{\omega_p} \frac{1}{3k_L v_e} \]

\[ = \frac{\pi}{6} \frac{k_o v_o^2}{k \nu_e^2} \frac{1}{\omega_p} \frac{1}{3k_L v_e} \]

where

\[ \alpha = \frac{|k_k^2 - k_\perp^2|}{k_k k_\perp k_o} (k_L \cdot e_o) = 0(1) \]  \hspace{1cm} (67)

\(^1\) w.r.t. = with regard to.
Using $k \lambda_D \leq 0.2$ as the Landau criterion and $k_o = \sqrt{3} \omega_p/c$ from matching, we obtain a threshold (growth) condition [35]:

$$I_{th} \geq \frac{0.73T^{1/2}}{a^2L}$$

with $T = eV$, $L = mm$, $I$ in units of $10^{10}$ watt/cm².

**Frequency Shift**

In a cold plasma, phase matching dictates that the decay will occur exactly at $n_c/4$. For a finite-temperature plasma, however, the Langmuir wave frequency is shifted slightly from $\omega_L = \omega_p$ to

$$\omega_L^2 = \omega_p^2 + 3k^2v_e^2$$

$$= \omega_p^2(1 + 3k^2\lambda_D^2)$$

Subsequent Thomson (Raman mode) scattering of either the incident or stimulated backscatter wave off the enhanced electron fluctuations at $\omega_L = \omega_p = \omega_o/2$ will lead to generation of $3\omega_o/2$ emission from the plasma.

In particular, if the incident radiation at $\omega_o$ is Thomson-scattered off the decay towards the input optics (direction of detection of $3\omega_o/2$ emission), then the scattered frequency

$$\omega' = \omega_o \pm \omega_L = \omega_o \pm (\omega_o - \omega_L)$$

$$= \frac{3}{2} \omega_o - \frac{3}{4} k\lambda_D^2 \omega_o, \quad k\lambda_D \ll 1$$

![Decay Diagram](attachment:image1.png)

![Thomson Scattering Diagram](attachment:image2.png)
If, however, the SBS radiation at \((\omega_o - \omega_s)\) is Thomson-scattered off \(\omega_s' = \omega_o/2\), then

\[
\omega' = \frac{3}{2} \omega_o - \omega_s - \frac{3}{4} k_x^2 \lambda_D^2 \omega_o
\]  

(69)

Consider the mean angle of acceptance w.r.t. normal incidence of the focusing mirror at \(7^\circ \pm 7^\circ\) for our experiment, it is easy to show that the latter scattering process is more likely for reasonable levels of SBS. The relative frequency shift

\[
\frac{\Delta \omega}{3\omega_o/2} = \frac{\omega' - 3\omega_o/2}{3\omega_o/2}
\]

is then given by

\[
\frac{\Delta \omega}{3\omega_o/2} = -\left(\frac{2}{3} \frac{\omega_s}{\omega_o} + \frac{1}{2} k_x^2 \lambda_D^2\right)
\]  

(70)

The alternative of Thomson scattering off the forward decay \(k_x\) is unlikely for underdense plasma because of the difficulty of detection in the backward direction. With a high-density region, it may be anticipated that Thomson scattering off both forward and backward decays is likely with both (+ve) and (-ve) \(\Delta \omega\) components. Indeed, solid target experiments have shown this [34].
Scattered Signal at $3\omega_o/2$

Liu and Rosenbluth [43] have calculated the expected scattered wave amplitude $E_s$ due to the incident-beam Raman scattering off the fluctuations $\omega_p = \omega_o/2$.

\[
(c^2 \gamma^2 - \frac{\omega^2}{\omega_t^2} - \omega_p^2) E_s = E_0 \omega_p^2 \frac{\delta n}{n} e_o \cdot \mathbf{e_p}
\]

where $\delta n/n$ is determined from theory of the $2\omega_p$ decay. With minor modifications, we use their theory to calculate the expected $3\omega_o/2$ emission for this experiment.

Integrating Eqn. (71), we find

\[
\frac{|E_s|^2}{|E_o|^2} = \left[ \frac{\omega_p^2 e_o \cdot \mathbf{e_p}}{2k_s k_{sx}^{1/2} c^2} \right]^2 \left( \frac{\delta n}{n} \right)^2
\]

where $k' = dk'_x/dx$ and the density fluctuation level is related to the electrostatic wave value $E$ by

\[
\left( \frac{\delta n}{n} \right)^2 = k_{sx}^2 \lambda_D^2 \frac{|E|^2}{4\pi nT}
\]

For incident intensity near threshold, pump limited growth is approximately given by

\[
\frac{|E|^2}{|E_o|^2} = \frac{1}{2} \left[ \frac{k' c \gamma \ln \left( \frac{I}{I_{th}} \right)}{2\pi \gamma_o^2} \right]^{1/2}
\]

\[
= \frac{1}{2} \left[ 5 \frac{V e}{c} \frac{1}{(k_o^2 \lambda_D^2)(k_x^2 \lambda_D^2)} \ln \left( \frac{I}{I_{th}} \right) \right]^{1/2}
\]

for the instability criterion given in Eqn. (68). We now turn to experimental results of two-plasmon decay.

**Paper V:** Phys. Rev. A 18, 746 (1978)

The experimental set-up is the same as for paper II, with $I < 10^{12}$ watt/cm$^2$, hydrogen gas, $T_e = 115$ eV, $n = n_c/4$, $L = 600$ $\mu$m.

$2\omega_p$ instability threshold. The "threshold" behaviour for $3\omega_o/2$ emission is shown in Fig. 19, where $I_{th}$ corresponding to unity on
FIG. 19. Dependence of $3\omega_0/2$ emission on incident CO$_2$ laser power for hydrogen gas cell target. Unity on the abscissa corresponds to $\approx 2 \times 10^{11}$ W cm$^{-2}$.

The inhomogeneous "threshold" as given by Eqn. (68) for $\lambda = 1$ is $> 1.3 \times 10^{11}$ watt/cm$^2$, showing reasonably good agreement with experiment. In comparison, the absolute threshold predicted by Eqn. (65) would be $= 2.6 \times 10^{10}$ watt/cm$^2$ for the same parameters, a value considerably smaller than the experimental one.
Spectrum. We now interpret the spectrum of $3\omega_o/2$ emission, as shown in Fig. 20, which was not attempted in the publication. The experimental $\Delta \omega = -4.2 \times 10^{11}$ sec$^{-1}$. From Eqn. (70), with $\omega_s = 1.76 \times 10^{11}$ sec$^{-1}$ and the average value for $k_s = 1.41 \omega_o/c$, determined from wave-matching conditions for $k_s$ at a 7° angle w.r.t. the backward direction, we calculate

$$\frac{\Delta \omega}{3\omega_o/2} = \frac{2}{3} \frac{\omega_s}{\omega_o} + \frac{1}{2} k_s^2 \lambda_D$$

or $\Delta \omega = -4.17 \times 10^{11}$ sec$^{-1}$. While the focusing mirror is capable of collecting scattering from a spectrum of $k_s$, fluctuations, this has the effect of broadening the $3\omega_o/2$ feature rather than materially changing the average $\Delta \omega$. The chief contribution to the spectral width, however, is from the broad SBS probe wave ($\Delta \omega = \omega_s$).

Agreement with experiment is very good, indicating that the combination of ion shift due to the backscattered wave serving as probe and Raman shift due to finite temperature seem adequate to account for the measured value.
3ω₀/2 emission. Finally, we calculate the expected 3ω₀/2 scattering collected by the input mirror. For our scattering geometry with k₀ = 0.56 ω₀/c, e₀ · e_p = sin 17.9; I/I_th = 2, we find

\[
\frac{|E|^2}{|E_o|^2} = 0.8 \frac{c}{v_e} \left[ \frac{k_n I}{I_{th}} \right]^{1/2}
\]

\[
\left( \frac{\delta n}{n} \right)^2 = \frac{v_e}{c} \left[ \frac{k_n I}{I_{th}} \right]^{1/2} \frac{E_o^2}{4\omega n} = 7.2 \times 10^{-3}
\]

\[
\frac{|E_S|^2}{|E_o|^2} = 2 \times 10^{-2} \left( \frac{\omega_L}{c} \right) \left( \frac{v_e}{c} \right)^2 \left( \frac{\delta n}{n} \right)^2 = 1.35 \times 10^{-4} \left( \frac{\delta n}{n} \right)^2
\]

FIG. 21. 3ω₀/2 emission from supersonic oxygen gas target versus incident laser intensity showing threshold and saturation.
With a density fluctuation level \( \delta n/n = 8.5 \times 10^{-2} \), and remembering that \( |E_o|^2 \) for scattering is 0.2\( |E_{inc}|^2 \) from the level of SBS, we find
\[
\frac{|E_o|^2}{|E_{inc}|^2} = 2 \times 10^{-7}
\]
This compares very well with the experimental value of \( = 3 \times 10^{-7} \).

In summary, the foregoing model would appear to correctly predict the essential features of two-plasmon decay-threshold, spectra and absolute level of \( 3\omega_o/2 \) emission.


Saturation in \( 3\omega_o/2 \) has been observed in our \( CO_2 \) laser irradiated supersonic \( O_2 \) target as shown in Fig. 21. The calculated \( I_{th} = 9.2 \times 10^{11} \) watt/cm\(^2\) for \( L = 100 \) \( \mu \)m and \( T_e = 160 \) eV, though ponderomotive forces at \( \omega_p = \omega_o/2 \) will likely reduce the localized value of \( L \) below 100 \( \mu \)m, increasing the threshold [44]. Calculated and experimentally measured levels of \( 3\omega_o/2 \) emission are comparable to those found in V.

**REFERENCES**


Stimulated Brillouin

Filamentation


Two-Plasmon Decay
STABILITY OF PLASMA VORTICES

R.N. SUDAN
Laboratory of Plasma Studies,
Cornell University,
Ithaca, New York,
United States of America

Abstract

STABILITY OF PLASMA VORTICES.

The stability of plasma vortices possessing internal fields and flow against incompressible perturbations is examined by a variational technique due to Woltjer. It is shown that there is a cusp-like equilibrium composed of oppositely oriented vortices which is stable.

This lecture is concerned with examining the stability of plasma vortices which possess both internal currents and mass flow. The presence of fluid velocity in equilibrium rules out stability analysis by the conventional mhd energy principle [1]. Instead we adopt the method formulated by Woltjer [2] and Chandrasekhar [3] in which a variational principle is established by minimizing the system energy while keeping a number of global integrals of motion constant. More recently, Taylor [4] has employed this technique to establish the minimum-energy state for a toroidal pinch bounded by a rigid, infinitely conducting toroidal vessel. Similarly, Rosenbluth and Bussac [5] have evaluated the stability of a zero-pressure, force-free, spheromak against internal perturbations. They have extended the analysis to also include deformations of the plasma surface.

The material of this lecture is a commentary with some extensions on the previous analysis contained in Ref. 6.

1. Plasma Model

We treat the plasma in the two-fluid approximation, neglecting both electron inertia $\omega_e << 1$ and collisions $\omega_e \tau_e > 1$, where $\tau_e$ is the electron collision time and $\omega_e$ is the electron gyro-frequency. Thus, the ion momentum balance is given by

$$\frac{2}{t} \gamma = -V_{2} + V \times V \times V - \frac{V_p}{\rho} + \frac{9}{m} (E + V \times B) \quad (1)$$

where p, y, and $\rho$ are the pressure, velocity, and mass density, respectively. For the electrons we have

$$E + u \times B = 0 \quad (2)$$
where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields and $\mathbf{u}$ is the electron velocity. Equations (1) and (2) are supplemented by the equations of continuity and adiabatic pressure laws, for both electrons and ions, together with Faraday's law expressed through the curl of Eq. (2).

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{u} \times \mathbf{B}$$  \hspace{1cm} (3)

Taking the curl of Eq. (1), assuming $V_p \times V_p = 0$, we obtain for the vorticity $\omega = \nabla \times \mathbf{v}$,

$$\frac{\partial \omega}{\partial t} = \nabla \times \mathbf{v} \times \omega = -\frac{q}{m} \left( \frac{\partial}{\partial t} \frac{\mathbf{B}}{\rho} - \nabla \times \nabla \times \mathbf{B} \right)$$  \hspace{1cm} (4)

In the limit $q/m \to 0$ we obtain from Eq. (4) the law of conservation of circulation for hydrodynamics and in the limit $m/q \to 0$ we obtain the flux conservation law of magnetohydrodynamics. With the help of the continuity equation, (4) may also be written as

$$\frac{d}{dt}[(\mathbf{B} + \frac{m}{q} \omega)/\rho] = [(\mathbf{B} + \frac{m}{q} \omega)/\rho] \cdot \nabla \mathbf{v}$$  \hspace{1cm} (5)

where $d/dt = (\partial/\partial t + \mathbf{v} \cdot \nabla)$. A fluid element of length $d\mathbf{r}$ in motion obeys $d(d\mathbf{r})/dt = d\mathbf{r} \cdot V\mathbf{v}$. By comparison we conclude that if $d\mathbf{r}$ is initially oriented along the direction of the vector $(\mathbf{B} + m/q \omega)/\rho$, then it continues to do so in its subsequent motion. We now take the initial conditions for the plasma-vacuum interface to be

$$a) \hat{n} \cdot \mathbf{B} = \hat{n} \cdot \mathbf{u} = 0; \hspace{0.5cm} b) \hat{n} \cdot \mathbf{j} = 0 \text{ or } \hat{n} \cdot \mathbf{u} = \hat{n} \cdot \mathbf{v}$$  \hspace{1cm} (6)

where $\hat{n}$ is the unit normal to the surface and $\mathbf{j}$ is the current density. Equations (3) and (5) ensure that the condition (6), if satisfied initially, will be satisfied for all time.

2. Constants of Motion

The set of equations (1)-(5) possess, in principle, an infinite number of integrals of motion, but the following are easily established

$$N = \int \rho \, d^3x \hspace{1.5cm} K = \int \hat{A} \cdot \mathbf{B} \, d^3x$$

$$G = \int \nu \cdot (\mathbf{B} - \frac{m}{2\rho} \cdot \mathbf{u})d^3x \hspace{1.5cm} L = \int \mathbf{r} \times \rho \mathbf{v} \, d^3x$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{r}$ is the position vector.
It is straightforward to show that $N$ and $L$ are constants. To derive $K$, we recognize $\mathbf{E} = -\mathbf{A} - \mathbf{V}\phi$ and $\mathbf{E} \cdot \mathbf{B} = 0$, and

\[
\frac{dK}{dt} = \int d^3x \left( \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \hat{\mathbf{n}} \mathbf{u} \right) + \int d^2x (\hat{\mathbf{n}} \cdot \mathbf{u}) (\mathbf{A} \cdot \mathbf{B})
\]

\[
= \int d^3x \left( -\mathbf{V} \cdot \mathbf{B} + \mathbf{V} \cdot (\mathbf{u} \mathbf{B}) \times \mathbf{A} \right) + \int d^2x \hat{\mathbf{n}} \cdot \mathbf{u} \mathbf{A} \cdot \mathbf{B}
\]

\[
= \int d^2x \hat{\mathbf{n}} \cdot \mathbf{B} (\mathbf{u} \cdot \mathbf{A} - \phi)
\]

\[
= 0
\]

since $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$. The proof for $G$ is more elaborate and is given in the Appendix.

3. Extremum States with Incompressible Flow

We limit further discussion to (i) incompressible flow, i.e. $\nabla \cdot \mathbf{V} = 0$ and (ii) nonrotating equilibria, i.e. $L = 0$. The system energy for incompressible flow is

\[
W = \int d^3x \left[ \frac{1}{2} \rho \mathbf{V}^2 + \frac{1}{2} \mathbf{B}^2 \right]
\]

If $W$ attains a minimum value consistent with $K$ and $G$ being constant at their initial values, we conclude that such a state is stable. We minimize $W$ by applying Lagrange's method; thus,

\[
\delta W - \lambda \delta K - \mu \delta G = 0
\]

for independent variations in $\delta A$ and $\delta V$; $\lambda$, $\mu$, and $\nu$ are the Lagrange multipliers. Note that the density $\rho$ is not varied because of the assumption of incompressibility. We obtain the following equations for the extremum states:

\[
\nabla \times \mathbf{B} = \lambda \mathbf{B} + \mu \omega (7)
\]

\[
\mu \omega = \frac{q}{m} (\rho \mathbf{V} - \mu \mathbf{B}) (8)
\]

Substituting (7) and (8) in (1) and (2) and setting $\partial / \partial t = 0$ we obtain

\[
\nabla \rho + \rho \nabla \mathbf{V}^2 / 2 = 0 (9)
\]
which is Bernoulli's law. In what follows we shall assume that the density \( \rho \) is constant everywhere initially and, therefore, is constant for all time since \( \nabla \cdot \mathbf{v} = 0 \). Equations (7) and (8) have the solution

\[
\nabla \times \mathbf{B} = k \mathbf{B} \tag{10}
\]
\[
\mathbf{v} = \beta \mathbf{B}/\sqrt{\rho} \tag{11}
\]

where

\[
k = \frac{\lambda}{1 - \mu \beta / \sqrt{\rho}} = \frac{q}{m} \left( \frac{\beta \sqrt{\rho} - \mu}{\mu \sqrt{\rho}} \right) \tag{12}
\]

and (9) integrates to

\[
p + \frac{1}{2} \rho \mathbf{v}^2 = p + \frac{\beta^2}{2} \mathbf{B}^2 = p \text{ (const.)} \tag{13}
\]

From Eqs. (7), (8), (10), (11), and (12) we obtain

\[
k^2 - \frac{q \sqrt{\rho}}{m} \left( \frac{m \lambda}{\rho \sqrt{\rho}} + \frac{\sqrt{\rho}}{\mu} - \frac{\mu}{\sqrt{\rho}} \right) k + \frac{q \rho \lambda}{m \mu \sqrt{\rho}} = 0 \tag{14}
\]
\[
\beta^2 + \beta \left( \frac{m \lambda}{\rho \sqrt{\rho}} - \frac{\sqrt{\rho}}{\mu} + \frac{1}{\sqrt{\rho}} \right) + 1 = 0 \tag{15}
\]

Thus, in general, there are two solutions \( k_+ \), \( k_- \) and correspondingly \( \beta_+ \) and \( \beta_- \). It can be easily established that

\[
(k_+ - k_-)^2 = (\beta_+ - \beta_-)^2 \tag{16}
\]

The solution of Eqs (7) and (8) is, therefore,

\[
\mathbf{B} = a_+ \mathbf{b}_+ \text{ or } a_- \mathbf{b}_- \tag{17}
\]
\[
\mathbf{v} = \beta_+ a_+ \mathbf{b}_+ / \sqrt{\rho} \text{ or } \beta_- a_- \mathbf{b}_- / \sqrt{\rho} \tag{18}
\]

where \( \nabla \times \mathbf{b}_+ = k_+ \mathbf{b}_+ \), \( \mathbf{b}_+ \) and \( \mathbf{b}_- \) are the eigenfunctions corresponding to eigenvalues \( k_+ \), \( k_- \), and \( \hat{n} \cdot \mathbf{b}_+ = 0 \) at the plasma vacuum boundary. From Eq. (10) we can establish that

\[
(k_+ - k_-) \int \mathbf{b}_+ \cdot \mathbf{b}_- \, d^3x = \int \nabla \cdot \mathbf{b}_+ \times \mathbf{b}_- \, d^3x = \int \hat{n} \cdot \mathbf{b}_+ \times \mathbf{b}_- \, d^2x \tag{19}
\]

For \( k_+ \neq k_- \) we must have \( \int \mathbf{b}_+ \cdot \mathbf{b}_- \, d^3x = 0 \) and, furthermore, the eigenfunctions are normalized as \( \int \mathbf{b}_+^2 \, d^3x = \int \mathbf{b}_-^2 \, d^3x = 1 \).
4. Minimum Energy States

From (17) and (18) the expression for the total energy may be expressed as \( J \frac{b^2}{T} \frac{d^3x}{d^3x} = 1 \),

\[
W = \frac{1}{2} (1 + \beta^2) a^2 + \frac{1}{2} (1 + \beta^-) a^-^2
\]

(20)

Furthermore, since \( A = \beta/k + \gamma x \) we may write

\[
K = a^2/k^+ \text{ or } a^2/k^-
\]

(21)

This representation of \( K \) is valid in a singly connected region of plasma (for instance, a sphere). In a multiply connected region, like a torus, \( x \) is a multi-valued function and an additional term \( \Delta x \int_s d^2x \hat{n} \beta \) appears [7]; \( s \) is the torus minor cross-section and \( \Delta x \) is the jump in \( x \) in going around the torus once. In what follows we shall limit ourselves to singly connected regions although the generalization to multiply connected regions can be undertaken. Substituting (21) in (20) we obtain

\[
W = \frac{1}{2} \frac{k^+}{k} K (1 + \beta^2) + \frac{1}{2} \frac{k^-}{k} K (1 + \beta^-^2)
\]

(22)

where \( k^- \) and \( \beta^- \) can be expressed in terms of \( k^+ \) and \( \beta^+ \) through (16) and \( \beta^+ \beta^- = 1 \). It is clear that as a function \( a^2/a^2_+ \), \( W \) has no minimum except when \( a^- = 0 \) or \( a^+ = 0 \). Thus, the total energy

\[
W = (1 + \beta^2) k^+ K \text{ or } (1 + \beta^-^2) k^- K
\]

(23)

depending on which is the smaller value. We now evaluate the expression for \( G \) as

\[
\frac{G}{K} = 2 \beta k/\sqrt{\rho} + m \beta^2 k^2/q \rho
\]

(24)

Solving for \( \beta k \) we obtain

\[
\beta^+ k^+ = \frac{q}{m} \sqrt{\rho} \left[ (1 + mG/qK)^{1/2} - 1 \right] \equiv \alpha^+_+\]

(25)

Note that \( m/q \rho = (m/m_e)^{1/2} \lambda_e \), where \( \lambda_e \) is the electromagnetic screening length. Also note that \( mG/qK > -1 \) because \( \beta k \) is a real quantity. In terms of \( \alpha \),

\[
W = (k + \alpha^2_+/k) K
\]

(26)
FIG. 1. Variation of $W_{\text{min}}$ and $\beta_{\text{min}}$ with $\alpha_+$ for a set of discrete eigenvalues $k_0, k_1, k_2, ...$.
Note that in between transitions to a new mode, $W_{\text{min}}$ varies as $\alpha_+^2$ and $\beta_{\text{min}}$ as $\alpha_+$.

The minimum value of $W$ determined by $\partial W/\partial k = 0$ is given by

$$k_{\text{min}}^2 = \alpha_+^2, \quad \beta_{\text{min}} = 1 \quad \text{and} \quad W_{\text{min}} = 2|\alpha_+|K$$

(27)

because we note that $|\alpha_+| < |\alpha|$. Since the eigenvalues $k$ form a discrete set, the expressions (27) are valid only for the higher mode numbers or large values of $\alpha$. The actual variation of $W_{\text{min}}$ and $\beta_{\text{min}}$ with $\alpha_+$ is given in Fig. 1. Thus, given $G, K$ and $\rho$, $\alpha_+$ can be computed and $W_{\text{min}}$ and $\beta_{\text{min}}$ are determined by the eigenvalue $k_{\text{min}}$ that falls closest to $\alpha_+$. Such an equilibrium state will be stable to any internal incompressible perturbation that leaves the plasma vacuum interface undisturbed.

5. Axisymmetric Stable Equilibria

To obtain an equilibrium solution the pressure balance condition

$$(p + \frac{1}{2} B_\perp^2)_{\text{II}} = B_\perp^2$$

(28)
at the plasma vacuum boundary must be observed and, furthermore, the plasma vacuum interface must coincide with a magnetic surface for both the vacuum and plasma fields. This results in the vacuum equation for $B$ to be overdetermined if the plasma boundary is given a priori. We are thus led to conclude that the shape of the plasma surface is unknown and has to be determined from the nature of the field at infinity and the plasma invariants. Recognizing the Bernoulli relation (13), we can rewrite (28) as

\[
P + \frac{1}{2} (1 - \beta^2) B_{\perp}^2 = B_{\parallel}^2
\]  

(29)

For $\alpha_+$ close to an eigenvalue $k_\parallel$, we see from Fig. 1 and (27) that $\beta \rightarrow 1$, which leads to

\[
B_{\parallel}^2 = \text{(constant)}
\]  

(30)

on the plasma surface. If the field at large distances from the plasma is uniform then the plasma surface will be a toroid. In the large aspect ratio $R/a \gg 1$ this torus will have circular cross-section, such that

\[
B_\theta = \left(\frac{I}{4\pi R}\right) \ln\left(\frac{16\pi R B_s/I}{\ln(4/a)} - \frac{1}{2}\right)
\]  

(31)

where $I = [(2P)^{1/2} - B_\parallel]$ is the azimuthal surface current, $B_s$ is the vacuum field on the surface, and $B_\theta$ is the field at infinity when $a/R$ is finite then the cross-section is D-shaped (Fig. 2) given by

\[
(1 - \frac{1}{4}[12 \ln (4/\sigma) - 15] \sigma^2 \cos 2\theta) \sigma = \text{const.}
\]  

(32)

where $\sigma = \exp(\Xi)$ and $\Xi, \theta, \phi$ are toroidal co-ordinates defined by $\Xi + i\theta = \ln(p + iz + R)/(p + iz - R)$. 

FIG. 2. Schematic of $D$ cross-section toroidal plasma given by Eq. (32). Only the poloidal field lines are shown.
A quadrupole vacuum field determines a cusp-shaped boundary. The internal fields form two vortex rings with magnetic fields, shown in Fig. 3, such that the toroidal fields are oppositely oriented. Near the ring cusp it can be shown that in terms of a local co-ordinate system $x, y, z$ and $r, \phi, z$ (see Fig. 3), the plasma boundary surface is given by $y = \pm (x)^{3/2}$ and the vacuum vector potential is

$$A_z = rB \sin \phi + \frac{15}{16} \frac{r^2}{B} \sin 2\phi + r^{3/2} \sin \frac{3}{2} \phi + \ldots$$

(33)

and $B^2$ is constant on the plasma surface. Similar expressions are obtained for the point cusp.

6. Stability to Surface Deformations

To establish the stability against surface deformations we follow Rosenbluth and Bussac and perturb the surface by a displacement $\xi \hat{n}$, allowing the interior to relax to a minimum energy state for the perturbed surface consistent with $K$ and $G$ being preserved. Stability is determined by whether the pressure imbalance across the surface reinforces or opposes the displacement, i.e. if

$$\delta E = \int_S d^2x \, \xi \left( B_{11}^2 - B_{12}^2(1-\beta^2) \right) + \xi \frac{3}{8} \left( B_{11}^2 - B_{12}^2(1-\beta^2) \right) \left( B_{11}^2 - B_{12}^2(1-\beta^2) \right) > 0$$

(34)
we have a stable system; $B_1^{(1)}$ and $B_1^{(2)}$ are the perturbed fields in plasma and vacuum, respectively. For minimum energy equilibria with $\beta > 1$ the influence of the internal fields is annihilated by the velocity perturbations and this expression simplifies to

$$\delta E = \int d^2x \xi (B_{II} \cdot B_1^{(1)} - \xi B_{II}^2/\rho) > 0 \tag{35}$$

where $\rho$ is the radius of curvature of the plasma boundary considered positive for a concave surface viewed from within the plasma. The first term of $\delta E$ is always positive. For equilibria where $\rho$ is everywhere negative, $\delta E$ is always positive. Thus, the cusp equilibria shown in Fig. 3 is always stable to both surface and internal incompressible perturbations for $\beta = 1$. When $\rho$ is positive as for the equilibria of Fig. 2, criteria (34) can establish stability by computing $B_{II}^{(1)} = \nabla \psi$, with $\nabla^2 \psi = 0$ and $\partial \psi/\partial n = \nabla \times \xi \times B_{II}$ as the plasma boundary condition.

An approximate estimate of the $B_{II} \cdot B_1^{(1)}$ term for a large aspect ratio ring can be made as follows. Let $B_1^{(1)} = \nabla \psi$; then with $\psi = \exp(i m \phi + n \theta)$, where $\phi$ and $\theta$ are the toroidal and poloidal angles respectively, we have

$$\nabla^2 \psi = \nabla^2 \psi - \left(\frac{m^2}{a^2} + \frac{n^2}{R^2}\right) \psi = 0$$

Thus, at the plasma surface $r = a$, where $\hat{n} \cdot B_{II} = 0$

$$\hat{n} \cdot B_1^{(1)} = \hat{n} \cdot \nabla \times \xi \times B_{II} = \imath m \nabla \psi / a = \nabla \psi = -(m^2/a^2 + n^2/R^2)^{1/2} \psi$$

and

$$\int d^2x \xi^* B_1^{(1)} \cdot B_{II} \bigg|_{r=a} = \int d^2x \xi^2 \frac{m^2}{a^2} B_{II}^2 / (m^2/a^2 + n^2/R^2)^{1/2}$$

and

$$\delta E = \int d^2x \frac{\xi^2 B_{II}^2}{a} \left(\frac{m^2}{(m^2 + n^2 a^2/R^2)^{1/2} - 1}\right)$$

i.e. the ring is unstable for

$$m^2 < \frac{1}{2} \left(1 + (1 + 4n^2 a^2/R^2)^{1/2}\right)$$

In particular, the ring is thus unstable to interchange modes with $m = 0$.

The ballooning mode stability analysis for a ring of D-shaped cross-section, Eq. (32), is not so straightforward because all the modes with quantum number $m$ are coupled in this geometry.
7. **Plasma Injection Parameters**

From the condition $\alpha e = k_{\min} \sim \kappa/R$, where $R$ is scale length of the plasma and $\kappa$ is the lowest eigenvalue (for a sphere $\kappa \sim 4.5$), we obtain for $mG/q\kappa \geq 1$

$$N_{r_i} = \kappa^2 R$$  \hspace{1cm} (36)

where $r_i = Z^2 e^2/\sqrt{m_e}$, the ion classical radius, and $N$ is the total number of plasma ions. This expression is remarkably similar to the number of particles required in an ion ring [9],

$$N_{r_i} = \zeta R$$  \hspace{1cm} (37)

where $\zeta$ is the field reversal factor. The 'injected' energy to form such a state is

$$W_{\text{inj}} \geq \frac{4\pi}{3} B^2 R^3$$

and the energy per ion is

$$\varepsilon = 4\pi B^2 R^2 r_i / 3 \kappa^2$$

For $R \sim 0.5$ m, $B \sim 5$ kG, $\kappa \sim 5$, one requires $W_{\text{inj}} \sim 100$ kJ, for $N \sim 5 \times 10^{18}$ and $\varepsilon \sim 125$ keV. The total injected charge is 800 mC and could be provided by an intense ion beam sources capable of delivering a total of 0.5 MA over 2 $\mu$s.

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The time rate of charge of G is written as
\[
\frac{dG}{dt} = \int d^3x \left( (\vec{v} \cdot \vec{B} + \vec{v} \cdot \vec{B}) + \frac{m}{2q} (\vec{v} \cdot \vec{\omega} + \vec{v} \cdot \vec{\omega}) \right) \\
+ \int d^2x \left( \vec{n} \cdot \vec{v} \right) \left[ \vec{v} \cdot (\vec{B} + \frac{m}{2q} \vec{\omega}) \right]
\] (A1)

where \( \frac{\partial v}{\partial t} = \vec{\omega} \), etc. From Eqs. (1), (2) and (4) we obtain
\[
\int d^3x \frac{\vec{B} \cdot \vec{\omega}}{\vec{B} \cdot \vec{v}} = \int d^3x \left( \vec{B} \cdot \vec{v} \times \vec{w} - \frac{\vec{B} \cdot \vec{v}}{\rho} \right) - \int d^2x \vec{n} \cdot \vec{B} \frac{v^2}{z}
\] (A2)
\[
\frac{m}{2q} \int d^3x \frac{\omega \cdot \vec{v}}{\omega \cdot \vec{v}} = \int d^3x \left( \frac{1}{2} \omega \cdot \vec{E} + \frac{1}{2} \omega \cdot \vec{v} \times \vec{B} - \frac{m}{2q} \frac{\omega \cdot \vec{v}}{\rho} \right)
\] (A3)
\[
\frac{m}{2q} \int d^3x \frac{\omega \cdot \vec{v}}{\omega \cdot \vec{v}} = \int d^3x \left( \frac{m}{2q} \omega \cdot \vec{v} \times \vec{v} \times \vec{w} - \frac{1}{2} \omega \cdot \vec{v} + \frac{1}{2} \omega \cdot \vec{v} \times \vec{v} \times \vec{B} \right)
\] (A4)

Furthermore,
\[
\int d^3x \frac{(\vec{B} + \frac{m}{2q} \omega) \cdot \vec{v} \rho}{\rho} = \int d^3x \frac{\vec{v} \rho}{\rho} \cdot \vec{v} \times \left( \vec{A} + \frac{m}{2q} \omega \right)
\] (A5)
\[
= \int d^3x \frac{\vec{v} \rho}{\rho} \times \left( \vec{A} + \frac{m}{2q} \omega \right)
\]
\[
= \int d^2x \frac{\vec{v} \rho}{\rho} \times \left( \vec{A} + \frac{m}{2q} \omega \right)
\]
\[
= 0
\] (A5)

since \( \hat{n} = \frac{\vec{v} \rho}{|\vec{v} \rho|} \) at the plasma boundary surface and \( \vec{v} \rho \times \vec{v} \rho = 0 \). We may also write
\[
\int d^3x \omega \cdot \vec{E} + \int \omega \cdot \vec{B} = \int d^2x \frac{\hat{n} \cdot \vec{v}}{\hat{n} \cdot \vec{v} \times \vec{E}} = \left[ \frac{d^2x}{\hat{n} \cdot \vec{v} \times \vec{v} (\vec{v} \times \vec{B})} \right]
\]
\[
= \left[ \frac{d^2x}{\hat{n} \cdot \vec{v} \times \vec{B}} - \left( \frac{\hat{n} \cdot \vec{v}}{\hat{n} \cdot \vec{v} \times \vec{E}} \right) \right]
\] (A6)
because at the boundary $\hat{n} \cdot u = \hat{n} \cdot v$:

$$
\int d^3x \, \nabla \times v \times B = \int \omega \times v \times B = \int d^2x \, \nabla \times \nabla \times v
$$

$$
= \int d^2x \, [\nabla \times (\n\times B)] - [\nabla \times (\n \times v)]
$$

(A7)

$$
\int d^3x \, \nabla \times v \times v = \int d^2x \, \nabla \times [\n \times v]
$$

(A7)

Introducing (A7)-(A8) in (A1) we find that the volume integral of (A1) reduces to the surface integral

$$
\int d^2x \, \{n \times B (\hat{\omega}) + \frac{m}{2} (\hat{\omega}) \frac{v^2}{2} - \hat{n} \times v \times (\hat{\omega} + \frac{m}{2} v \hat{\omega})\}
$$

Substituting this in (A1) we obtain

$$
\frac{dG}{dt} = \int d^2x \, \{n \times B (\hat{\omega}) + \frac{m}{2} (\hat{\omega}) \frac{v^2}{2}\}
$$

$$
= 0
$$

since according to Eqs (3), (4) and (5), if the conditions $\hat{n} \cdot B = 0 = \hat{n} \times \omega$ are satisfied initially they will always be satisfied.

REFERENCES

Part 3

REVIEWS
SMALL-SCALE EXPERIMENTS IN POLOIDAL FIELD SYSTEMS

B. LEHNERT
Royal Institute of Technology,
Stockholm,
Sweden

Abstract

SMALL-SCALE EXPERIMENTS IN POLOIDAL FIELD SYSTEMS.

In the field of plasma physics and controlled fusion, small-scale experiments fulfil a number of important needs. For example, they provide flexible and simple means for plasma research by (a) backing up large-scale experiments through model studies and detailed investigations of specific problems; (b) development of diagnostic methods; and (c) producing simple methods for testing new ideas and principles. These possibilities are illustrated by some experiments performed at the Laboratory of the Royal Institute of Technology, Stockholm, with rotating plasmas confined in poloidal magnetic field configurations. In particular, these experiments deal with plasma/neutral-gas interaction, the cold-mantle concept, and associated stability questions.

1. INTRODUCTION

Experimental research on magnetically confined plasmas has so far been performed in the frame of a great variety of confinement schemes and in devices of varying sizes. For some of these schemes it has become clear that the parameter range of a fusion reactor can only be reached for plasmas with large characteristic dimensions $L_{\perp}$ in the directions perpendicular to the confining field and at considerable magnetic field strengths $B$, as well as by means of large powers $P$ for plasma heating. Although this seems to be the case for the currently most interesting approaches to magnetic confinement, small-scale experiments should still fulfil a number of important functions in fusion research.

This paper makes a contribution to the discussion of possibilities and properties of small-scale experiments that can be performed at moderate values of $L_{\perp}$, $B$ and $P$ which are available at laboratories of small and medium size. A strictly defined upper borderline for the parameters of such experiments cannot, of course, be given, but here we shall refer mainly to the ranges $L_{\perp} \lesssim 0.1$ m, $B \lesssim 1$ T, and $P \lesssim 1$ MW.

In Section 2 some general points are made on small-scale experiments and their potentials. These are followed in Section 3 by a description of the effects of plasma/neutral-gas interaction which have lately become the subject of increasing interest in fusion research and whose features can often be easily investigated in small-scale experiments. Section 4 summarizes some characteristics of poloidal magnetic field systems and their use in small-scale experiments.
Finally, in Section 5 some concrete experimental illustrations are given on the behaviour of plasmas confined by such systems.

This review is mainly illustrated by results obtained at the Royal Institute of Technology in Stockholm [1]. It must, however, be stressed from the beginning that important contributions to the subject are also due to other authors and laboratories that are not explicitly mentioned.

2. POTENTIALITIES OF SMALL-SCALE EXPERIMENTS

Although fusion research has made considerable progress in large sectors of plasma physics, fusion technology and reactor system studies, a final solution of the magnetic confinement problem has not yet been achieved. Much work still remains in plasma physics before an optimal and practically feasible reactor scheme can be realized. In this context, plasma research by means of small-scale experiments becomes clearly justified for several reasons, some of which are as follows:

(a) Further basic investigations on plasma physics in general are needed.
(b) Small experiments are flexible and inexpensive with respect to parameter variations and thus provide efficient, cheap and time-saving studies of a number of relevant physical phenomena.
(c) Existing large-scale experiments can be backed up by detailed investigations and model studies on a small scale (e.g. in relation to start-up, equilibrium, stability and heating, defined by the problems of burn-out, the cold-mantle (gas blanket) concept, plasma transport and impurities, and certain wave and instability phenomena.
(d) Small experiments are convenient as a first step in development and training for experimental techniques, including diagnostics.
(e) Small experiments can be used for pioneering work, i.e. in tests of new ideas and principles during the first steps of their development.

We shall also raise the somewhat provocative question, whether plasma conditions similar to those in a future fusion reactor become realizable only in very large and expensive devices or whether such conditions can also be reached in devices "the size of a writing desk". The scaling laws so far obtained for tokamaks and similar schemes certainly indicate that large linear dimensions, strong magnetic fields and considerable heating powers are required if this type of device is to reach thermonuclear plasma conditions. However, it has not
been proved that other more efficient confinement schemes could not exist by which thermonuclear conditions can be realized on a modest scale and by means of smaller technical resources. To illustrate this, we consider for a moment the idealized situation which would arise in looking for a scheme that permits classical and stable confinement of a high-β plasma in a closed magnetic bottle [2]. From a steady one-fluid macroscopic model, the energy containment time $\tau_E$ of a plasma body of average radius $a = L_\perp$ in the direction
perpendicular to a confining magnetic field of average strength $B$ becomes determined by

$$n\tau_E = \frac{3f_V k_B^2 a^2 / T / C_p (\ell \ln \Lambda)}{1 + (C_r B^2 a^2 / C_p \ln \Lambda) (1 + k_{cb} B^2 / T / n)} \; (1)$$

Here SI units are adopted; $n$ and $T$ stand for spatial mean values of ion density and temperature:

$$C_p = f_p C_r k' \, , \, C_r = f_b C_b k_b \, , \, k_{cb} = f_c C_c k' / f_b C_b k_b$$

$f_V$, $f_p$, $f_b$, $f_c$ are dimensionless factors of order unity arising from mean-value formation across the plasma body; $k_b = 1.7 \times 10^{-46}$ and $k_c = 8 \times 10^{-24}$ represent bremsstrahlung and cyclotron radiation; and $k_\Lambda = 1.5 \times 10^{-42}$ represents heat losses by conduction across the magnetic field. Further, the dimensionless coefficient $C_p$ represents, in a semi-empirical way, the total energy losses due to plasma transport with $C_p \approx 1$ in the strictly classical case and $C_p > 1$ in other cases, whereas the dimensionless coefficient $C_b$ includes impurity effects on bremsstrahlung, and the coefficient $C_c < 1$ in cases where reabsorption of cyclotron radiation becomes important.

As an illustration of Eq.(1) we choose a pure plasma with $a = 0.05$ m; $n = 5 \times 10^{20}$ m$^{-3}$; a $\beta$-value $\beta < 0.5$ in the range $10^6 < T < 10^8$ K; $B = 2.5$ T; $f_p = f_b = f_V = f_c = 0.5$; $c_b = 1$; and $c_c = 0.1$. We then obtain the results given by the full ($C_p = 1$) and broken ($C_p > 1$) lines in Fig.1. In this figure some recent values of $n\tau_E$ for large tokamaks [2] have also been plotted, with corresponding equivalent values of $C_p$ given in brackets. The figure thus indicates that if a closed confinement scheme can be found which is stable and obeys the laws of classical transport, thermonuclear plasma conditions could become realizable even in small devices and by means of modest technical resources. Thus, there is still much to be gained in improving plasma confinement, even in cases where the idealized limit of classical transport cannot be fully attained. Needless to say, efforts to find more efficient confinement schemes than the present ones are clearly justified at this stage of fusion research.

3. EFFECTS OF PLASMA/NEUTRAL-GAS INTERACTION

During both creation and maintenance of a laboratory plasma, problems of neutral-gas interaction are inevitable. The simplified picture of a fully ionized plasma body surrounded by a perfect vacuum can seldom be realized, especially with limited technical facilities. Furthermore, in a full-scale fusion
reactor the plasma is likely to interact with a more or less pronounced flux of neutral particles arriving from surrounding vessel walls. There are also reactor schemes, such as those with a cold mantle, where a partially ionized boundary layer is purposely included. Before describing particular experiments, we shall therefore outline some of the general features and problems of plasma/neutral-gas interaction [3—9].

3.1. Neutral-gas penetration into a hot plasma

The interaction between a plasma body and a neutral gas depends on the way in which neutral particles can penetrate into the plasma, under current processes of ionization, elastic, charge-exchange, and other types of non-elastic collisions. In a first simplified approach, the neutral particle population can be divided into two components of slow (cold) and fast (hot) neutrals. The former consists of those neutrals which have interacted with the vessel walls, or other low-temperature regions surrounding the plasma body, and have adopted the temperature $T_{ns}$ of these regions. Before being ionized during their flow into the plasma, some of the slow neutrals become converted by collisions with the ions into fast neutrals with temperature $T_{nf}$ of the order of the corresponding local ion temperature $T_i$. Under certain conditions, and especially at low plasma densities, some of these fast neutrals are, in their turn, scattered back into the vessel walls, thereby creating a secondary neutral flux with a broad spectrum of energies.

To put it simply, the penetration lengths of slow and fast neutrals can be estimated from the joint ionization and diffusion processes taking place in a nearly static plasma of average ion density $n$ and temperature $T_i \approx T_e \approx T$ of ions and electrons. With an average temperature $T_{ns} \approx 2m_e^2eT/3m_n (\xi + \xi_{ins})$ of the slow neutrals, resulting from simultaneous heating by electron-neutral impacts during the penetration into the plasma [6], the corresponding penetration length becomes [3, 7]

$$L_{ns} = 1/n_{cs}, \quad 1/\sigma_{cs} = \left[\frac{2kT_{ns}}{m(\xi + \xi_{ins})} (2\xi + \xi_{ins})\right]^{1/2}$$  (2)

The penetration length of fast neutrals is further given by [3, 7—9]

$$L_{nf} \approx 1/n_{cf}, \quad 1/\sigma_{nf} = \left[\frac{2kT}{m_\xi(2\xi + \xi_{inf})}\right]^{1/2}$$  (3)

In Eqs (2) and (3), $m_i, m_e, m_n$ stand for the ion, electron and neutral particle masses,

$$m = m_i + m_e, \quad \xi = \langle \sigma \rangle_{en\wedge en} w_{en}$$

$$\xi_{\text{inv}} = \langle \sigma \rangle_{\text{in\wedge in\wedge v}}, \quad \xi_{en} = \langle \sigma \rangle_{en\wedge en}$$
TABLE I. SUBDIVISION OF DENSITY REGIMES FOR NEUTRAL GAS PENETRATION INTO A PLASMA OF AVERAGE ION DENSITY $\bar{n}$ AND CHARACTERISTIC MACROSCOPIC DIMENSION $L_b$.

<table>
<thead>
<tr>
<th>Parameters and properties</th>
<th>Plasma types</th>
<th>Permeable</th>
<th>Impermeable</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Dilute</td>
<td>Non-dilate</td>
</tr>
<tr>
<td>Ion density $\bar{n}$</td>
<td>$\bar{n} \ll n_{cs} \ll n_{cf}$</td>
<td>$n_{cs} \ll \bar{n} \ll n_{cf}$</td>
<td>$n_{cs} \ll n_{cf} \ll \bar{n}$</td>
</tr>
<tr>
<td>Slow neutrals:</td>
<td>$\ll 1$</td>
<td>$\gg 1$</td>
<td>$\gg 1$</td>
</tr>
<tr>
<td>$\bar{n}/n_{cs} = L_b/L_{ns}$</td>
<td>Free streaming</td>
<td>Diffusion</td>
<td>Diffusion</td>
</tr>
<tr>
<td>Fast neutrals:</td>
<td>$\ll 1$</td>
<td>$\leq 1$</td>
<td>$\gg 1$</td>
</tr>
<tr>
<td>$\bar{n}/n_{cf} = L_b/L_{nf}$</td>
<td>Free streaming</td>
<td>Free streaming</td>
<td>Diffusion</td>
</tr>
</tbody>
</table>

Neutral densities —
In case of moderately high $\beta$ values and a steady state:

<table>
<thead>
<tr>
<th>In main plasma body</th>
<th>Small</th>
<th>Negligible</th>
<th>Negligible</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_{ns}/\bar{n}$</td>
<td>Negligible</td>
<td>Small</td>
<td>Negligible</td>
</tr>
<tr>
<td>$n_{nf}/\bar{n}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>In near-wall regions</th>
<th>Small</th>
<th>Moderately large</th>
<th>Small</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_{ns}/\bar{n}$</td>
<td>Negligible</td>
<td>Of order unity</td>
<td></td>
</tr>
<tr>
<td>$n_{nf}/\bar{n}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Some examples in special experiments and approaches

| Most earlier investigations on stellarators, multipoles, mirrors | Most tokamaks, also operating in permeable-impermeable transition region; some stellarators, mirrors | Full-scale reactors; tokamaks at high density; $\beta$-pinches; high-$\beta$ stellarators; some multipoles; most rotating plasmas |

where $\sigma'_i$, $\sigma_i$, $\sigma_{en}$ are the ionization, ion-neutral, and non-ionizing electron-neutral collision cross-sections at the corresponding mutual velocities $w_{en}$, $w_{in}$, and the subscript $\nu = (s,f)$ refers to slow and fast neutrals. The deductions of Eqs (2) and (3) are on the verge of applicability of the macroscopic theory and should only be taken as an order-of-magnitude estimate of the penetration depths.
For a plasma body of finite size $L_b$, we can now define the corresponding critical densities:

$$n_{cs} = \frac{1}{\sigma_{cs} L_b}, \quad n_{cf} = \frac{1}{\sigma_{cf} L_b}$$  \hspace{1cm} (4)

Here recent atomic data [10] yield $1/\sigma_{cf} \approx 100/\sigma_{cs} \approx 2 \times 10^{18} \text{ m}^{-2}$ for hydrogen in the temperature range $10^5 < T < 10^7 \text{ K}$. At lower temperatures even a dense plasma can become permeable to neutral gas, on account of the steep decrease of the ionization rate $\xi$ at decreasing temperatures in the range below some $10^5 \text{ K}$, and of a corresponding increase in the penetration length $L_{nf}$ of fast neutrals as given by Eq.(3). This behaviour must be kept in mind when discussing small-scale experiments run at low plasma temperatures.

Hot plasmas can be divided into three classes: permeable dilute systems, permeable non-dilute systems, and impermeable systems, as defined in Table I. Here slow neutrals penetrate by free streaming in the permeable dilute case and by diffusion in the permeable non-dilute and impermeable cases, whereas fast neutrals penetrate by free streaming and by diffusion in the permeable and impermeable cases, respectively. We also observe that the steady-state penetration lengths of Eqs (2) and (3) include combinations of the mean free paths of ionizing and non-ionizing collisions. In the limit $\xi = 0$ of negligible ionization, $L_{ns}$ remains finite because the slow neutrals are converted into fast neutrals by ion-neutral collisions, but $L_{nf}$ tends to infinity because the fast neutrals should fill the entire plasma body when not being removed by ionization. Further, the steady neutral concentrations inside and outside the plasma body become very different in the three ion density regimes defined by Table I [3]. Some examples from special experiments and applications associated with these three regimes are given at the bottom of this table. In particular, it should be observed that full-scale quasi-steady fusion reactors are expected to operate at mean densities $\bar{n}$ and dimensions $L_b$ leading to values of $\bar{n}/n_{cf} \approx 30$ to $100$, the latter thus being far inside the impermeable ion density regime.

Finally, for small-scale experiments, it is seen from Eq.(4) and Table I that models on plasma/neutral-gas interaction such as those associated with full-scale reactors should be established in terms of a scaling where $\bar{n} \cdot L_b = \text{const}$. These experiments therefore have to be performed at higher ion densities than those of the corresponding full-scale reactors.

3.2. Outline of cold-mantle systems

The class of impermeable plasmas listed in Table I includes many important features which can to a great extent be studied by means of small-scale experiments. Among these, the cold-mantle (gas-blanket) concept has recently attracted
considerable interest [3–9, 11—13]. A quasi-steady impermeable plasma with such a mantle becomes a system of strongly inhomogeneous spatial conditions in which the physical properties of various subregions, such as the hot core and the cool layers near the wall, differ considerably from each other. The situation can, in principle, be illustrated by the simple model of Fig.2, in which a cylindrical plasma column is confined in a strong axial magnetic field $\vec{B}$. In this steady-state model the outermost part of the plasma is defined by a wall or limiter edge at $r = a$ of a cylindrical frame $(r, \varphi, z)$. Neutral gas is present in the plasma boundary layer in the form of slow neutral particles of density $n_{ns}$ in the immediate neighbourhood of $r = a$ and of fast neutrals of density $n_{nf}$ inside the main part of this layer. Thus the plasma density drops from the value $n_b = n(r_b)$ at the inner 'edge' $r = r_b$ of the layer to a value $n(a) \ll n_b$ at the outer wall or limiter edge, whereas the density of fast neutrals drops from the value $n_{na}$ near the outer edge to $n_{nf}(r_b) \ll n_{na}$ at the inner edge. In the following descriptions of cold-mantle systems it is convenient to divide the boundary layer into an outer 'diffusion region' defined by $r_d < r < a$, and an inner 'ionization region' defined by $r_b < r < r_d$.  

---

**FIG.2.** Crude outline of a quasi-steady cold-mantle system [1, 8] consisting of a cylindrical plasma column confined by an axial magnetic field $\vec{B}$. Local plasma density and temperature are $n$ and $T$; the densities of slow and fast neutrals are $n_{ns}$ and $n_{nf}$. For a fusion reactor the reaction rate is larger than the bremsstrahlung loss in the region defined by $r < r_t$. 

---
In the diffusion region the neutral density $n_{nf}$ is too high for a large ionization rate $\xi$ to be permitted by the available power input into the layer. In other words, this region is defined by an ionization rate $\xi$ much smaller than both its 'saturation' (maximum) value $\xi_{\text{max}} \approx 10^{-14}$ m$^3$ s$^{-1}$ and the ion-neutral collision rate $\xi_{\text{in}}$. Thus, the plasma temperature $T$ must be much smaller than $10^5$ K in the diffusion region. Here the plasma/neutral balance is governed mainly by an outward diffusion of plasma across the magnetic field, balanced by an inward diffusion of neutral gas through the plasma. The thickness of this region is $x_d = a-r_d$.

In the ionization region, the neutral density is low enough for an ionization rate of the order of $\xi_{\text{max}}$ to be permitted by the available power input. Here the 'residual' part of the incoming neutral gas flux becomes converted into plasma, in a combined ionization-diffusion process with the e-folding penetration length $L_{\text{nf}}(T_b,n_b)$ corresponding to the temperature $T(r_b) = T_b$ and density $n_b$. Thus, the ionization region has a thickness of the order of $L_{\text{nf}}(T_b,n_b)$. In this connection it should be observed that $L_{\text{nf}}$ decreases steeply with increasing $T$ in the range $10^2 < T < 3 \times 10^4$ K, and then stays nearly constant in the range $3 \times 10^4 < T \lesssim 10^7$ K. Thus, $T = T_b \approx 10^5$ K becomes a rather good first approximation of the temperature which defines the ionization region and the inner 'edge' of the partially ionized boundary layer of a hot plasma [8].

Consequently, we speak of a 'well-defined' or 'fully developed' cold mantle as a system in which there is a low-temperature diffusion region of width at least exceeding $L_{\text{nf}}(T_b,n_b)$. As will be seen later, such a situation can only be realized under special conditions.

Inside the radius $r = r_b$ there is a fully ionized plasma whose temperature has a maximum at $r = 0$. In a reactor system, the temperature becomes high enough for the thermonuclear reaction rate to exceed the bremsstrahlung loss in an inner region defined by $r < r_t$.

From this description it can be understood that cold-mantle systems have a number of interesting features and potentialities [3–5]. Some examples are: wall protection from high-energy particle interaction [11, 12]; refuelling and ash removal in reactor systems as well as impurity transport between the plasma and the walls; control of element and isotope concentrations in the hot plasma core by means of a corresponding control of the concentrations in regions near the wall; and alternative and modified methods of plasma stabilization due to the presence of a partially ionized boundary layer. Small-scale model experiments can easily be made on most of these important features, without the necessity of reaching thermonuclear temperatures in the central parts of the plasma body.

We now examine more closely the partially ionized boundary layer in Fig. 2, in the limit of a low-$\beta$ approximation. The average ion density in the
FIG. 3. Relation between normalized ion density $n_b$ at inner edge of the partially ionized region and normalized neutral gas density $n_{na}$ at outer edge of the same region, defined by a wall or limiter surface [1, 8]. Branches corresponding to the asymptotic cases of Section 3.2.1 are indicated by (I) and (II).

layer is chosen far inside the impermeable range, i.e. so that the layer can be occupied mainly by fast neutrals [3], and to allow the penetration depth $L_{nf}(T_b, n_b)$ of Eq.(3) to become smaller than the layer thickness $x_b$. Since the boundary layer thickness $x_b$ turns out to be much smaller than the characteristic plasma dimension $L_b$, the balance inside the layer can be treated with good approximation as a one-dimensional problem in a local rectangular frame $(x,y,z)$, $x$ being perpendicular to the magnetic field $\mathbf{B} = (0,0,B)$ and the layer surfaces (see Fig.2).

3.2.1. Particle and momentum balance of boundary layer

The particle balance yields a flux $\Gamma(x) = -n v_x = n_n v_{nx}$ through the boundary layer, where $v_x$ and $v_{nx}$ are the macroscopic fluid velocities of plasma and neutral gas along $x$ (we assume $n_{ns} \ll n_{nf}$, and drop subscripts $s$ and $f$ in this section). The real temperature distribution $T(x)$ is further
replaced by a constant mean value $T_m$ in the region $0 \leq x \leq x_b$ of Fig.2, and the notation $\Gamma(0) = \Gamma_a$ at $x = 0$ is introduced, as well as

$$s = \left[ \frac{\Gamma_a}{2kT_m} \right]^{1/2} \left( \frac{m_i + m_n}{m_i m_n} \right) \left( \xi_i + \xi_n \right)$$

(5)

$$n_B = B \left[ \left( \frac{\Gamma_a}{2kT_m} \right)^{1/2} \left( \frac{m_i + m_n}{m_i m_n} \right) \right]^{1/2} = k_B B$$

(6)

related to the momentum balance. Here $s$ is a dimensionless coordinate, $n_B$ represents a characteristic density related to the steady processes of plasma/neutral-gas 'counter-diffusion' in the layer, and $n$ stands for the resistivity.

In cases of classical diffusion, the latter has the conventional value associated with the perpendicular directions of $B$, but anomalous diffusion can also be described semi-empirically by this model, through an anomalous resistivity $\eta$.

The latter would then imply that $n_B$ is decreased at given values of the rest of the parameters in Eq.(6). Introducing the normalized plasma and neutral densities, $N = n/n_B$ and $N_n = n_n/n_B$, the differential equations of the momentum balance lead to solutions $N(s)$ and $N_n(s)$ inside the boundary layer of Fig.2 [8, 9, 14]. Especially at the boundaries $x = (0, x_b)$ the corresponding plasma and neutral densities

$$n_b = n(x = x_b) = n_B n_{no} \text{ and } n_{na} = n_n(x = 0) = n_B n_{no}$$

have asymptotic values related by

$$N_n = 2(N_b^3 - N^3)/3; \quad N_{na} \approx 2N_B^3/3 \quad \text{for } N_B << 1 \quad (I)$$

$$N_n = 2(N_b - N); \quad N_{na} = 2N_B \quad \text{for } N_B >> 1 \quad (II)$$

(7)

In fact, the transition from case (I) to case (II) occurs within a rather narrow region around $N_B = 1$, as demonstrated in Fig.3. In case (I) the plasma pressure gradient is mainly balanced by diffusion due to Coulomb collisions, whereas the same gradient is mainly balanced by plasma/neutral-gas friction in case (II).

The situation of classical collision-dominated diffusion with $k_B = k_{BC}$, associated with a corresponding resistivity through Eq.(6), finally yields the density relations:

$$n_{na} = 2n_B^3/3k_{BC}^2 B^2 \quad (I)$$

$$n_{na} = 2n_B \quad (II)$$

(8)
3.2.2. Heat balance of boundary layer

For the heat balance of the boundary layer, integration of the corresponding balance equation over the interval $0 < x < x_b$ yields \([8, 9, 14]\)

$$\Delta Q_b = Q_b - (C_\xi + C_s)/x_b - (C_\eta + C_R)x_b$$

(9)

where $Q_b$ is the heat per unit area and time flowing into the layer at $x = x_b$ from the fully ionized region; $C_\xi$ represents the plasma particle loss due to diffusion across the layer which is compensated by a corresponding ionization and heating work on the incoming neutral gas; $C_s$ stands for the heat shunted away across the layer; $C_\eta$ represents the excitation radiation loss, and $C_R$ the bremsstrahlung and cyclotron radiation losses. Detailed deductions of the coefficients in Eq.(9) are given elsewhere [8, 9, 14].

The equilibrium state of the heat balance is expressed by $\Delta Q_b = 0$. When $\Delta Q_b > 0$ there is a heating power excess which tends to increase the degree of ionization and to shrink the layer thickness $x_b$, whereas $\Delta Q_b < 0$ leads to
a heating power deficit which tends to decrease the degree of ionization and to expand the layer thickness $x_b$.

At fixed values of $Q_b$ and of the $C$-coefficients of Eq.(9), the equilibrium leads to the two solutions:

$$x_b = (x_{b1}, x_{b2})$$

$$= \frac{Q_b}{2(C_\eta + C_R)} \left\{ 1 + \frac{1}{4(C_s + C_\xi)} \left( \frac{C_\eta + C_R}{Q_b^{1/2}} \right)^2 \right\}$$

of the layer thickness. Only the solution $x_{b1}$ given by the minus sign yields a stable heat balance and will therefore be considered henceforth [8, 9]. It corresponds to an enthalpy loss-dominated branch in $(Q_b, x_b)$-space where $x_b$ increases with decreasing values of $Q_b$ towards a maximum layer thickness $x_{b\text{max}}$, as outlined in Fig.4 and explained in the next paragraph. Two limits determine the range of the power input $Q_b$:

(a) A lower power input limit is defined by

$$Q_{b\text{min}} = 2 \left[ (C_s + C_\xi) (C_\eta + C_R) \right]^{1/2}$$

When $Q_b < Q_{b\text{min}}$, the losses cannot be balanced by the power input, and the partially ionized layer grows thicker, thus tending to 'swallow up' the entire plasma body. The limit $Q_{b\text{min}}$ corresponds to a maximum possible layer thickness:

$$x_{b\text{max}} = x_b (Q_b = Q_{b\text{min}}) = \left[ (C_s + C_\xi) / (C_\eta + C_R) \right]^{1/2}$$

(b) On the other hand, when $Q_b$ increases in the range $Q_b > Q_{b\text{min}}$, there is a corresponding decrease of $x_b$ in the equilibrium state $\Delta Q_b = 0$. At a sufficiently large power input, $x_b$ thus approaches the limit

$$x_{b\text{min}} \approx r_d - r_b \approx L_{nf}(T_b, n_b)$$

When passing this limit, the cold diffusion region is instead 'swallowed up' by the hotter ionization region in Fig.2 and there is no longer a well defined cold mantle. The power input $Q_b$ then becomes too large to be balanced by the
losses inside a cold partially ionized boundary layer. The limit of Eq.(13) is thus defined by Eq.(9) with $\Delta Q_b = 0$ and leads to an upper power input limit:

$$Q_{b\text{max}} = Q_b(x_b = x_{b\text{min}}) \quad \text{where} \quad T_b \approx 10^5 K \quad (14)$$

and

$$x_{b\text{min}} = x_{b1\text{min}} = L_{n\text{f}}(T_b, n_b) = 1/n_b \sigma_{cf}(T_b) \quad (15)$$

Even in some cases where $Q_b$ slightly exceeds $Q_{b\text{max}}$, the temperature $T_b$ may remain at a tolerably low level, resulting in moderately strong plasma/wall interaction. Such a marginal situation may be of interest in some cases, but will not be further discussed in this context; nor will the case of cooling of a fully ionized plasma edge region by impurity radiation be treated here.

The limits of the power input $Q_b$ and of the layer thickness $x_b$ have maximum and minimum values according to Eqs (11—15), and are related through the expressions

$$F = \frac{Q_{b\text{max}}}{Q_{b\text{min}}} = (f/2) + (1/2\tilde{f}) \quad (16)$$

and

$$\tilde{f} = \frac{x_{b\text{max}}}{x_{b\text{min}}} = \left[\frac{(C_s + C_\xi)}{(C_n + C_R)}\right]^{1/2} n_b \sigma_{cf}(T_b) \quad (17)$$

where $f \gg 1$, by definition, and $F \gg 1$. For a cold-mantle state to exist in a wide domain of parameter space it is therefore necessary that $f \gg 1$ and $F \gg 1$. Then $x_{b\text{min}} \approx (C_s + C_\xi)/Q_{b\text{max}}$.

In this connection it should be observed that the ratio $f$ and the corresponding width of the cold-mantle domain in parameter space become reduced at decreasing $N_b$ and $\beta$ values $\beta \propto n/B^2$, as well as when the radiation losses are enhanced by impurities. Consequently, when $f$ and $F$ approach unity, this domain shrinks to zero and a fully developed cold-mantle state ceases to exist. Comparatively high average $\beta$-values, of some 10% or more, are required in an efficient fusion reactor. This is expected to result in relatively high densities $n_{na}$ and large values of $f$, making the fully developed cold-mantle model of Fig.2 applicable in the case of classical or nearly classical transport inside the boundary layer. These results and their application to experiments are discussed further in Sections 4 and 5.
Anomalous transport phenomena are expected to broaden the parameter ranges of \((x_{b\min}, x_{b\max})\) and \((Q_{b\min}, Q_{b\max})\). This should facilitate attempts to obtain a cold mantle in such systems as tokamaks. In the case of classical transport, and at the present low \(\beta\)-values of the latter, the high-temperature region of the plasma would otherwise become situated too close to the wall surfaces. This in turn should lead to heavy interaction between the walls and those fast neutrals which are subject to charge-exchange collisions with the hot outer plasma layers.

3.2.3. Stability properties in the presence of a boundary layer

Not only the magnetic field geometry but also the plasma distribution in phase space and the boundary conditions determine the stability of every special confinement system. Thus, the simplified picture adopted earlier of a fully ionized plasma body limited by a sharp vacuum boundary must be abandoned in most quasi-steady situations of interest to fusion research. Instead, the plasma body will be subject to spatially inhomogeneous conditions, and to neutral-gas interaction. For impermeable plasmas with a cold mantle, these circumstances are of special importance. The presence of a fully developed partially ionized boundary layer situated between the fully ionized plasma body and the surrounding walls can change the stability situation radically and even seems to open up new possibilities for stabilization of a number of electrostatic and electromagnetic modes [1, 3, 7, 15–22]. Thus, in the case of flute-type interchange modes, the low temperature and high resistivity, together with the finite pressure gradient in the boundary layer, may render minimum-average-B stabilization inefficient, especially at the long magnetic connection lengths in full-scale reactor systems. On the other hand, in a situation like that outlined in Fig.2, other stabilizing mechanisms arise owing to the special features of the boundary layer. Further, the equilibrium plasma pressure \(p\) will differ from zero, and the corresponding 'normalized' gradient \(|\nabla p|/p\) becomes finite and comparatively smaller at the internal 'interface' \(r = r_b\) of the cold-mantle system than at the boundary of a hot plasma which is in direct contact with a limiter or wall surface at \(r = a\). The described properties can lead to overall stabilization of the plasma body, both when the hot plasma interior is stabilized by conventional methods like those provided by minimum-average-B fields and when alternative stabilization methods become available. This should be the situation in the presence of a sufficiently dense cold mantle (e.g. in the experiments described in Section 5), as well as that expected to exist in a full-scale reactor with a sufficiently high \(\beta\)-value. The low-density boundary layers in present tokamak experiments seem, however, to have only minor effects on plasma stability.
Here we limit the stability discussion to moderately high $\beta$-values, corresponding to the equilibrium model of Fig. 2. To simplify the treatment, the boundary layer and the fully ionized regions are treated separately, but it should be understood that a rigorous non-localized approach must connect the instability modes by the boundary conditions at $r = r_b$ and the values at $r = (0, a)$. The following features and results should be mentioned; they are described in more detail elsewhere \[1, 15-22\].

(a) Owing to the comparatively high neutral gas concentration and low temperature in the main part of the boundary layer, a 'compound' plasma/neutral-gas viscosity from ion-ion, ion-neutral, and neutral-neutral collisions becomes an important layer feature, as well as finite resistivity and pressure. These features give rise to a special 'boundary layer stabilization' mechanism. For flute-type disturbances this can be illustrated by an analysis of low-frequency modes. We consider a partially ionized cylindrical plasma shell, with its axis along $z$ of a frame $(r, \phi, z)$ and where there is an immersed magnetic field $\vec{B} = [0, B(r), 0]$. This frame should not be confused with that of Fig. 2. The unperturbed plasma pressure $p = 2nkT$, density $n$, and temperature $T$ thus have gradients only in the radial direction, as well as the corresponding neutral gas parameters $p_n, n_n, T_n$. A simple localized perturbation analysis is imposed on the particle and momentum balance equations of plasma and neutral gas, in terms of localized normal modes of the form $\exp\left[i(\omega t + \kappa r + \kappa z)\right]$ and where the ionization rate and finite Larmor radius effects are neglected. The dispersion relation of low-frequency flute-type mode then reduces to

$$\omega = \pm \left((V + D) + \sqrt{\Gamma + (V - D)^2}\right)^{1/2}$$

where

$$\Gamma = 2\left[pB'/mnBe\right]B'_P - 2\gamma B_B$$

$$D = \gamma kT e_i k^2 / m \omega_i \omega_e$$

$$2V(\text{strong}) = k^2 / 2(\text{mn} + \text{nn} / n_n) \left[ n_n T_n \left< \sigma_{in} w_{in} > / n_n + n_n \sigma_{in} w_{in} > / n_n \right] + \frac{9nT_v i i 1}{16\omega^2} \right]$$

$$2V(\text{weak}) = \frac{\nu}{n} = n_n \sigma_{in} w_{in}$$

$$f_{\theta}(\text{strong}) = 1 + (mn + \text{nn} / mn); f_{\theta}(\text{weak}) = 1$$
for strong and weak frictional coupling between the plasma and neutral gas motions, as defined earlier [15]. In these equations \( \kappa^2 = \kappa_r^2 + \kappa_z^2 \); *a prime denotes derivation with respect to \( r \); \( \gamma \) is the ratio between the specific heats in the case of adiabatic changes of state; and \( \gamma = 1 \) in the case of isothermal changes; \( \nu_{ei} \) is the electron-ion collision frequency; \( \omega_e \) and \( \omega_i \) are the Larmor frequencies of electrons and ions; and \( w_n \) the thermal velocity of neutral particles. In Eqs (18)—(23) the driving force of flute-type instability is included in \( \Gamma \); viscous forces due to ion-neutral, ion-ion and neutral-neutral collisions are represented by \( V \), and plasma diffusion due to finite resistivity and pressure by \( D \). The stability condition becomes

\[
4VD \geq \Gamma 
\]

The physical explanation of this result is that the undisturbed growth rate of the instability, represented by \( \sqrt{\Gamma} \), is reduced by the compound plasma/neutral-gas viscosity represented by \( V \), thus putting a brake on the \( E/B \) fluid motion. When this brake becomes large enough, the diffusion due to finite resistivity and pressure, represented by \( D \), is able to smooth out the flute perturbations, and the plasma becomes stabilized. This stabilizing mechanism affects not only the flute-type modes but also a number of other instabilities partly of electromagnetic type [19—21]. It should become important in the boundary layers of certain cold-mantle model experiments as well as full-scale reactor systems [13, 14].

(b) Stability is expected to be achieved for an impermeable plasma body with a dense cold mantle by combining boundary layer stabilization with conventional methods, e.g. those due to minimum-average-\( B \) fields. The latter should become operative in the hot plasma interior where the resistivity is small and magnetic line-tying remains efficient.

(c) Even if conventional methods can be used for stabilizing the plasma interior, as described in the previous section, such methods restrict the possible types of field geometries to be used. Therefore it also becomes important to combine boundary layer stabilization with alternative methods to be applied to the fully ionized plasma interior. This possibility is demonstrated here by 'magnetic gradient stabilization', which is of particular interest in inhomogeneous and mainly poloidal magnetic fields, but is perhaps also important to some other field geometries. As an illustration we apply the cylindrical shell model under (a) above to a fully ionized plasma including finite-ion Larmor radius effects. Localized normal-mode analysis of electrostatic flute-type disturbances yields the stability condition [16, 17]:

\[
-\frac{p'}{p} \leq -2\gamma \left( \frac{B'}{B} \right) \left( 1 + \left( \frac{\gamma}{4} \right) \left( \frac{B'}{B} \right) - 1 \right) \left[ 1 + \frac{1}{4\gamma} \left( \frac{B'}{B} \right) - 1 \right]^2 \left( \frac{\kappa_r^2}{\kappa_z} + \frac{\kappa_z^2}{\kappa_r} \right)^2 \]  

\[
(25)
\]
where $a_i$ is the ion Larmor radius. Within the limits of applicability of condition (25), it is seen that the plasma becomes flute-stable for all $\kappa_1, \kappa_2$ when the pressure $p(r)$ decreases more slowly with increasing $r$ than $B^{2\gamma} \sim r^{-2\gamma} (= r^{-10/3}$ for $\gamma = 5/3)$. In addition, a strong stabilizing contribution due to finite Larmor radius effects arises at large wavenumbers. The stability condition $- \frac{p'}{p} \leq - 2\gamma B'/B$ of expression (25) in the limit $a_i = 0$ can be considered as the result of a 'maximum-B' stabilization mechanism, which is more pronounced the larger $- 2\gamma B'/B$ becomes as compared to $- \frac{p'}{p}$. Forms equivalent to condition (34) have been derived earlier in terms of both the energy principle [23] and plasma dynamics [17]. The physical explanation of this result is as follows.

To preserve the total magnetic flux during flute-type perturbations in the inhomogeneous field $B(r)$, a fluid element of smaller volume $\delta V_1$ which is closer to the axis $r = 0$ has to be interchanged with an element of larger volume $\delta V_2$ further away from the axis. Compression work is needed to move plasma of initial pressure $p_2$ from the larger volume $\delta V_2$ into the smaller volume $\delta V_1$, and expansion work is released by moving plasma of initial pressure $p_1$ from the smaller volume $\delta V_1$ into the larger volume $\delta V_2$. In this case

$$\delta V_2 > \delta V_1, \delta V \propto \phi d l / B \propto r^2 \quad \text{and} \quad \delta p \propto (\delta V)^{-\gamma} \propto r^{-2\gamma} \propto B^{2\gamma}$$

for adiabatic changes of state. It is thus seen that the compression work becomes larger than that of the expansion, provided that the unperturbed pressure distribution $p(r)$ is 'flat' enough, as expressed by the derived stability condition.

(d) Condition (25) suggests that the present stabilizing effect becomes enhanced in systems with larger average magnetic gradients than those of the straight conductor case. In fact, it has also been shown that configurations with a magnetic separatrix can be made flute-stable even in the presence of considerable pressure gradients. This result is obtained from the stability criterion [18]:

$$\frac{d}{dv} (pu^\gamma) \geq 0 \quad (26)$$

where $u = \phi d l / B$, and $v$ is a coordinate specifying the volume enclosed by the magnetic surfaces. In systems with a magnetic separatrix there is a large variation of $u$ across the plasma body, leading to strong maximum-average-$B$ stabilization.

(e) An additional stabilizing mechanism which may become important under certain experimental conditions is based on the joint effects of magnetic line-tying at a metal limiter, such as the cathode plate in Figs 4 and 5, a finite-
FIG.5. Outline of a rotating plasma system [26]. The plasma is confined in the poloidal field $\mathbf{B}$ inside the dotted region. A potential difference is applied between the anode rings on one hand and the cathode plate on the other. The corresponding electric field $\mathbf{E}$ and its associated electrode current put the plasma into rotation at velocity $\nabla$ round the axis of symmetry.

ion Larmor radius, and a large plasma density gradient in a boundary region of limited thickness [24].

(f) A recent extension of the MHD theory on plasmas in axisymmetric poloidal fields has shown that stabilization becomes possible under rather general conditions [22], provided that the pressure does not vanish at the edge of the fully ionized region [1, 16]. For a plasma surrounded by a conducting wall, interchange modes limit the achievable pressure gradient in the low-$\beta$ limit, whereas internal kink modes limit the achievable plasma current density. The latter can be increased by increasing the total magnetic field strength. Kink modes of top-bottom symmetry with respect to the geometrical mid-plane make the marginally stable pressure gradient depend also on current
density, magnetic field strength and the \( \beta \)-value. Finite Larmor radius effects should improve the stability in the case of short-wave perturbations.

(g) Parameter ranges in the cold-mantle regime have been found where plasma/neutral-gas interaction can have a stabilizing effect not only on flute-type electrostatic modes but also on modes of the kink, ballooning, current-convective, drift and velocity-driven types [19—21].

4. SOME GENERAL FEATURES OF POLOIDAL FIELD SYSTEMS

Plasma confinement schemes based on a main toroidal magnetic field lead to a number of limitations and so far unsolved problems, such as those associated with complicated transport mechanisms and not fully understood scaling laws, with a limitation to rather low \( \beta \)-values and high cyclotron radiation losses, as well as with field geometries involving long magnetic connection lengths. Irrespective of the important experimental results so far being obtained with these schemes, there is nevertheless justification for conducting efficient research on other types of magnetic geometries also.

Some experiments are described here which are based on confinement in a purely poloidal magnetic field. In such systems, attention should be drawn to the following general features:

(a) Confinement in the presence of a strong externally imposed poloidal (vacuum) field leads, in principle, to plasma transport without banana diffusion, high available \( \beta \)-values, and minimized cyclotron radiation losses.

(b) At least in cases where the mechanisms described in Section 3.2.3 become operative, stability should be promoted by the relatively short magnetic connection lengths and deep magnetic wells which become available in poloidal field systems.

(c) Purely poloidal field configurations make it possible to apply the rotating plasma technique of crossed electric and magnetic fields. With this technique, impermeable fully ionized plasmas of relatively high ion density can be generated and preheated, even with the limited resources of a small laboratory.

(d) For high-frequency heating, poloidal field systems provide a number of interesting and useful features. These are due to the rather strong spatial inhomogeneities of the magnetic field geometry and to the presence of a well defined field structure and its associated high-frequency resonance surfaces [25].
(e) These systems are also convenient for basic research and for small-scale model studies on a number of important problems, e.g. those of the cold-mantle state. At the comparatively high $\beta$-values achievable in the systems, there is a wide range of the ratios $f = x_{\text{bmax}}/x_{\text{bmin}}$ and $F = Q_{\text{bmax}}/Q_{\text{bmin}}$ in Eqs (16) and (17) for which cold-mantle operation becomes possible. This should make poloidal field systems more convenient for cold-mantle experiments than present-day tokamaks. The latter mostly have parameters in the range close to $f = 1$, where the conditions for a cold mantle are marginal. On the other hand, the parameters of the experiments described here, as well as those of a number of other poloidal field configurations, allow for operation within a rather wide range of $f$-values above $f = 1$ [13, 14].

5. ROTATING PLASMAS

As an example of experiments in this context we now discuss the crossed-field systems of rotating plasmas in open magnetic bottles. The principle of these systems is outlined in Fig.5, where a poloidal magnetic field $\mathbf{B}$ confines a fully ionized impermeable plasma inside the dotted region [26]. The latter is bounded by partially ionized layers in the transverse direction of $\mathbf{B}$. These layers have the features shown in Fig.2. In the longitudinal direction, the plasma region ends on solid non-conducting walls (end insulators) with corresponding thin partially ionized wall layers [9, 26]. An electric field $\mathbf{E}$ is imposed across the magnetic field, by applying a potential difference and a corresponding electric current across a set of anode rings and a cathode plate, as shown in Fig.5. Other electrode positions than those shown in Fig.5 are, of course, also possible. Owing to the comparatively small potential differences that can occur along $\mathbf{B}$, the surfaces defined by the magnetic field lines nearly become equipotentials, and $\mathbf{E}$ is then almost perpendicular to $\mathbf{B}$ within the fully ionized part of the plasma body. The electric current between the anodes and the cathode plate thus produces a torque with the field $\mathbf{B}$, thereby putting the plasma into rotation at a velocity $v_\phi$ directed round the axis of symmetry. The corresponding angular velocity $\Omega = v_\phi/r$ becomes nearly uniform along each field line in the fully ionized region, i.e. Ferraro's isorotation law is satisfied in the first approximation.

The balance between the centrifugal force and the plasma pressure gradient along $\mathbf{B}$ further results in a reduced ion density in the regions close to the axis of symmetry. This centrifugal confinement effect leads to the ratio [6, 27, 28]:

$$\frac{n_w}{n_0} = \exp\left(-\frac{m_i \Omega^2 r_0^2/4}{kT} \left[1 - \left(r_w/r_0\right)^2\right]\right)$$  \hspace{1cm} (27)
between the densities $n_w$ and $n_0$ at the radial positions $r = r_w$ and $r = r_0$ along a given field line in Fig. 5.

Finally, owing to the high thermal conductivity along $\mathbf{B}$ provided by the electrons, the temperature of a sufficiently dense plasma also becomes nearly constant along $\mathbf{B}$, at least in a quasi-steady state.

5.1. Main technical features of a rotating plasma device

Figure 6 is an example of the design of a rotating plasma device [29], where the magnetic field is generated by a main coil and a pair of Helmholz-type auxiliary coils. Other examples of similar devices are given elsewhere [26]. The electrode system consists of two anode rings, placed at the metal casing of the main coil, and a cathode plate whose position can be adjusted in the
axial direction so as to vary the volume and width of the plasma-confinement region. The latter is indicated by the hatched area and is defined by the electrode positions, the magnetic field lines, and the end insulators placed on the casing of the main coil. Owing to the centrifugal force of a fully developed rotating plasma, the magnetic field becomes 'pushed out' radially, as indicated by the displacement between the dashed and full lines near the equatorial plane in the right-hand part of Fig.6. The strongly curved field geometry has been chosen in order to make efficient use of the centrifugal confinement effect [6,28] and to minimize the loss area at the end insulators [26], i.e. to reduce as far as possible the end losses along the magnetic field. The linear dimensions of this device are given in Fig.6. Average magnetic field strengths $B_0$ in the equatorial plane up to 1 T have been reached by pulsing the coil system for some tenths of a second with powers up to about 1 MW, but relevant experimental results have also been obtained within a range $0.3 \leq B_0 \leq 0.5$ T. The discharges have been run mainly with hydrogen at filling densities in the range $3 \times 10^{20} \leq n_0 \leq 10^{22} \text{ m}^{-3}$. An example of the arrangement of the bank, its switching equipment, and other facilities, is shown in Fig.7.

5.2. Some characteristics of impermeable rotating plasmas

We now describe the main features of a rotating plasma confined in a device of the type shown in Fig.6, at ion densities within the impermeable density range defined in Section 3.1. Detailed descriptions appear in earlier reviews [26, 29].
5.2.1. The start-up process

At the transverse linear plasma dimensions of a small-scale experiment, the ion density of an impermeable plasma defined in Section 3.1 must usually be chosen in a range above some $10^{21} \text{ m}^{-3}$. For a laboratory with limited technical resources, the creation of a fully ionized plasma in this range is no trivial problem. If sophisticated and expensive methods are not being used for a gradual build-up of the ion density, such as by injection methods, the question arises how to establish a fully ionized state by 'burning out' a mass of neutral gas of corresponding particle density. Even in devices of moderate size, this requires burn-out powers of several MW, which are not easily available in methods based on induced low-frequency or high-frequency currents. In this connection the rotating plasma technique is convenient for plasma start-up by means of simple technical facilities. This method can be applied to configurations such as those shown in Figs 5 and 6 and in many other cases.

An example of corresponding plasma behaviour during the burn-out process is given by the experiments with the FI device of Fig.6 [29]. At a neutral-gas filling density $n_{n0} \approx 10^{21} \text{ m}^{-3}$, discharges were run with one condenser bank at varying initial voltages $\phi_{b0}$. The discharge is started at time $t = 0$, and a short circuit is applied across the electrodes at a later time $t_s = 0.6 \text{ ms}$. The behaviour of the electrode current $J_{12}$, electrode voltage $\phi_{12}$, and azimuthal current $J_\phi$ is demonstrated in Figs 8 and 9 as a function of time. When the voltage $\phi_{b0}$ and the corresponding power input is chosen at $\phi_{b0} = 8 \text{ kV}$, i.e. below a certain burn-out level, as shown in Fig.8, the plasma remains in a partially ionized state all the way up to the time $t_s$ of short circuit.

This is confirmed by the behaviour of the electrode current $J_{12}$ in Fig.8(a); this indicates no detectable recovered angular momentum which would otherwise appear in the form of a reversed current pulse (crowbar) at time $t_s$. Also, the azimuthal current $J_\phi$ in Fig.8(b) shows no measurable mean value during the entire course of the discharge. The regular spikes of this current recording are, in the present case, due to one single narrow plasma 'spoke' which rotates at the average $E/B$ speed, as has also been revealed in other experimental studies. Such spoke-shaped structures become especially pronounced so long as the rotating plasma remains partially ionized, and they seem to be closely connected with the ionization process. However, when the initial bank voltage is increased to $\phi_{b0} = 10 \text{ kV}$, and the power input slightly exceeds the burn-out level, the plasma behaves as shown in Fig.9. Here the ionization process becomes limited mainly to the first 0.2 ms, after which a fully ionized plasma is formed in the main part of the confinement volume. This leads to a large recovered charge at the time $t_s$ of short circuit, as shown by the trace of $J_{12}$ in Fig.9(a). It also reveals itself in the form of a considerable 'centrifugal pressure' as indicated by the recording of $J_\phi$ in Fig.9(b), as well as by the absence of spokes in the same recording at times $t \gtrsim 0.3 \text{ ms}$.
FIG. 8. Electrode voltage $\phi_{12}$, electrode current $J_{12}$, and azimuthal current $J_{\psi}$ in the FI device for a hydrogen discharge when initial bank voltage is kept at $\phi_{00} = 8$ kV and power input remains below the burnout level [29].

FIG. 9. Same as Fig. 8 but with an increase to $\phi_{00} = 10$ kV and power input chosen slightly above the burnout level [29].
The results of Figs 8 and 9 suggest that there is a well defined transition from a low to a highly ionized plasma state associated with a certain threshold (burn-out) power. This is also consistent with the following theoretical points [26]:

(a) The equivalent electric circuit of a rotating plasma contains, among other things, an equivalent electric capacity $C_{12}$ whose electrostatic energy represents the stored angular momentum of the plasma. A simplified theoretical approach of a fully ionized state yields

$$C_{12} = (r_{o2} - r_{o1}) \frac{L_u n_o m}{\pi} (r_{o2}^2 - r_{o1}^2) \frac{B_0^2}{2}$$

(28)

where $m = m_i + m_e$; $r_{o1}$ and $r_{o2}$ are the radial extensions of the plasma confinement region in the equatorial plane as indicated in Fig.6; $L_u = V/\pi(r_{o2}^2 - r_{o1}^2)$ is the effective extension of this region along the magnetic field $B$ with $V$ indicating the plasma volume; and $n_o$ and $B_0$ are the mean values of the ion density and the magnetic field strength in the equatorial plane. Thus, an abrupt short circuit between the electrodes at time $t_s$ results in a recovered electric charge $Q_e = C_{12} \phi_{12}$ ($t_s - 0$) representing the stored angular momentum of the rotating plasma and revealing itself in the form of a reversed current pulse, as shown in Fig.9(a).

(b) The azimuthal current $J_\varphi$ is mainly balanced by the centrifugal force in the present plasma experiment. In a fully ionized state it is in the first approximation given by

$$J_\varphi = 2m n_o L_u \phi_{12}^2 / (r_{o2}^2 - r_{o1}^2) \frac{B_0^3}{2}$$

(29)

(c) During the earlier parts of the transition from a low-ionized to a fully ionized state, the charged plasma particles are moving at velocity $v_\varphi \equiv |\mathbf{E} \times \mathbf{B}|/B^2$ through a background of immersed neutral gas nearly at rest. As long as the plasma has not become fully ionized, the rotation is limited by Alfvén's critical velocity [30]:

$$v_c = \left(2 e \phi_i / m_i \right)^{1/2}$$

(30)

where $\phi_i$ is the ionization energy. There is a further loss of angular momentum from the plasma which is mainly due to collisions between ions and neutrals at the rate $\xi_{in} = \langle \sigma_{in} w_{in} \rangle$. This loss must be balanced by the force arising from the electrode current $J_{12}$ and the magnetic field $B$, also in a quasi-steady state where the plasma rotation does not become accelerated. The frictional force
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FIG. 10. Equivalent capacity $Q_e/\phi_{12} \propto \bar{n}_0$ (dots) and the quantity $J_{e}/\phi_{12} \propto \bar{n}_0$ (crosses) as functions of initial bank voltage $\phi_{b0}$ of FI [29].

becomes proportional to the product $n_n$ of the ion and neutral particle densities, $n$ and $n_n$, thus having a maximum at a certain ionization degree. When the initial bank voltage $\phi_{b0}$ and the electrode current $J_{12}$ become large enough for this maximum to be exceeded, we should therefore expect a transition to take place all the way to a fully ionized state. The corresponding threshold (burn-out) current is roughly equal to

$$J_{12c} \approx \bar{n}_0 \wedge_n w c \left(1 + V/V_w\right) / 4 (r_{o2} - r_{o1}) \bar{B}_o$$

In Eq.(31), $V_w$ is the volume of the spacing between the shaded region of Fig.5 and the walls; $n_n$ stands for the initial filling density of neutral gas; $w_n w \propto (8kT_{nw}/\pi m_n)^{1/2}$ ($T_{nw}$ being the average neutral gas temperature in the volume $V_w$), and the fluxes of fast and slow neutrals which circulate between the volumes $V$ and $V_w$ have been taken into account.

(d) When the theory of Eqs (28)—(31) is applied to the series of experiments on plasma start-up shown by Figs 8 and 9, the result is as shown by Fig.10. In the latter the filling density has been chosen at $n_n = 2 \times 10^{21}$ hydrogen atoms. Figure 10 shows the quantities $C_{12} = Q_e/\phi_{12} \propto \bar{n}_0$ and $J_{e}/\phi_{12} \propto \bar{n}_0$ measured at the time $t = t_s$ of short circuit as functions of the applied initial bank voltage $\phi_{b0}$. The result clearly indicates that a transition takes place from a low ionized plasma state with a small ion density to a fully ionized state with a high and nearly constant plasma density, as soon as the bank voltage is increased beyond a sharply defined burn-out level, given by $\phi_{b0c} \approx 5.2$ kV in the case of Fig.10. The average density of the fully ionized plasma obtained from Eqs (36) and (39) at bank voltages $\phi_{b0c} < \phi_{b0} < 17$ kV becomes $\bar{n}_o \approx 1.6 \times 10^{21}$ m$^{-3}$. This value is consistent with the measured
value of \( n_{n0} \) within the limits of experimental error and of the present theoretical approximations. Recent measurements by laser interferometry also confirm the estimated ion densities of the fully ionized state [31]. The results of Figs 8—10 therefore suggest that the neutral gas becomes completely ionized in the cross-hatched region of Fig.6, as soon as \( \phi_{b0} > \phi_{b0c} \) and the burn-out current \( J_{12c} \) of Eq. (31) is being exceeded. The theoretical estimate of the latter also agrees within a factor of about 1.5 with experimentally observed data. The corresponding burn-out power \( \phi_{12} J_{12} \) at \( J_{12} = J_{12c} \) is then of the order of 2 MW, even for such a small device as FI.

Other mechanisms may, in principle, exist which can also give rise to limits during the burn-out process, such as those due to the ionization and heat balance. In any case, the limit of Eq. (31), which is caused by the momentum balance, appears to be consistent with the experiments described here.

5.2.2. The quasi-steady state of a fully ionized plasma

As soon as the burn-out process is completed and a fully ionized state reached in the main part of the plasma body, more quiescent conditions are found to develop. This is reflected in a regular trace of the voltage \( \phi_{12} \) and
in the disappearance of a pronounced spoke structure as shown by Fig.9, as well as in a strongly decreased disturbance level in the measuring circuits including the probe and laser interferometer recordings. By using a condenser bank of large capacity as power supply, such a quasi-steady ‘holding mode’ of the rotating plasma has in some cases been sustained for tens of milliseconds in FI [26].

A convenient way of investigating the balance of a rotating plasma is to apply a sudden cut-off from the energy source and study the corresponding ‘free-wheeling’ mode. This gives information on both the over-all momentum balance and the confinement. An example is given in Fig.11, where the discharge is started at time $t = 0$ by means of a first condenser bank [29]. During the fully ionized state which follows after burn-out, the plasma is further accelerated by a second bank, switched on at time $t_2$. Finally, at time $t_f$ the current supply is switched off and a free-wheeling mode starts. Both the electrode voltage $\phi_{12}$ of Fig.11(a) and the azimuthal current $J_\varphi$ of Fig.11(b) are seen to decay nearly exponentially during this mode. The data from a single shot have been plotted in Fig.12 to determine the free-wheeling e-folding times $\tau_v$ and $\tau_\varphi$ of the voltage $\phi_{12}$ and the azimuthal current $J_\varphi$ [29]. We observe that $\tau_\varphi \approx \tau_v/2$, as expected from the fact that the plasma velocity

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**FIG.12. Semilogarithmic plot of free-wheeling mode given by Fig.11 [29].**
FIG. 13. Free-wheeling time $\tau_v$ as a function of average magnetic field strength $B_0$. Dots refer to experiments with FI [29] and crosses to the Ixion device [32]. The curves have been calculated from theory for different values of the effective length $L_\parallel$ of the plasma.

should be nearly proportional to $\phi_{12}$ and the centrifugal force to $\phi_{12}^2$. A series of measurements on the free-wheeling time $\tau_v$ of the electrode voltage and the corresponding average rotation velocity is given in Fig. 13, where dots refer to the FI device and crosses to former experiments with the Ixion device [32].

The experiments on the quiescent holding mode (Figs 11—13) agree rather well with a theoretical model deduced from macroscopic balance equations, details of which are described elsewhere [9, 26]. Thus, the free-wheeling time $\tau_v$ of the velocity field and of $\phi_{12}$ is given by

$$\frac{1}{\tau_v} = \frac{1}{\tau_h} + \frac{1}{\tau_\perp}$$

where $\tau_h$ is due to the losses of angular momentum along $\vec{B}$ to the end insulators, and $\tau_\perp$ includes the superimposed effects of plasma viscosity and diffusion across the magnetic field. The rotational velocity $v_\phi$ and the plasma temperature $T$ become related to the balance between the heating power $P$ and the heat losses $\Lambda$ as represented by

$$P = P_s + P_\mu + P_\eta = \Lambda = \Lambda_s + \Lambda_\lambda + \Lambda_\eta + \Lambda_R$$
In Eq.(33), $P_s$ stands for a 'frictional' heating which is mainly due to the interaction between the rotating plasma and the end insulators; $P_\mu$ is the viscous heating power by ion-ion collisions in the sheared velocity field $v_\phi$; and $P_\eta$ represents Ohmic heating due to the sum of all electric currents that flow in the plasma. $\Lambda_s$ is mainly due to the heat loss along $\overrightarrow{B}$ to the end insulator regions; $\Lambda_\lambda$ arises from the heat conduction across $\overrightarrow{B}$; $\Lambda_\eta$ stands for the corresponding loss from plasma diffusion; and $\Lambda_R$ is the radiation loss due to all sources, with the exception of excitation radiation from electron-neutral collisions already included in $\Lambda_s$ and $\Lambda_\eta$. The theoretical results obtained from a combination of Eqs (32) and (33) lead to the set of curves given in Fig.13. For FI, the best agreement between experiments and theory is obtained for curve (c) in Fig.13, i.e. with an effective length $L_\parallel \cong 0.3$ m of the plasma along $\overrightarrow{B}$. This appears to be a reasonable value for $L_\parallel$ in the device of Fig.6. A rather good agreement is also obtained for Ixion, as given by curve (e) in Fig.13. From this figure it is seen that the end losses, and the corresponding contribution from $\tau_\parallel$ in Eq.(32), dominate the free-wheeling time $\tau_\parallel$ in FI when the average field strength exceeds some 0.8 T. In Ixion the longitudinal losses and $\tau_\parallel$ become even more pronounced owing to the nearly straight magnetic field geometry of this device.

The earlier theoretical model on the plasma balance [9,26] has further predicted angular velocity and temperature distributions $\Omega = v_\phi/r$ and $T$ in the plasma which are nearly constant along $\overrightarrow{B}$, have maximum values $\Omega_0$ and $T_0$ in the central parts of the fully ionized region, and which decrease in the direction across $\overrightarrow{B}$ towards the partially ionized boundary regions of Figs 5 and 6. The heat balance of Eq.(33) results in the approximate relation:

$$T_0 / v_0^2 = \theta_0 = \text{const}$$

between $T_0$ and the associated maximum rotational velocity $v_0$ in the equatorial plane. In FI we then obtain $\theta_0 \cong 10^{-5}$ K $\cdot$ s$^2$ $\cdot$ m$^{-2}$. These features have also been largely confirmed by experiments. Thus there are indications of velocity and temperature distributions with maximum values of the order of $v_0 = 10^5$ m $\cdot$ s$^{-1}$ and $T_0 = 10^5$ K in FI, as obtained from spectroscopic measurements [33] and from laser light scattering [34], laser interferometry [14,31,35], and resistivity measurements relating $T$ to $v_\phi$ [36].

At an increased rotational velocity, centrifugal force, and azimuthal current $J_\phi$, the magnetic field lines become pushed radially outwards, as shown in Fig.6 [29]. This corresponds to an increase in the equivalent $\beta$-value [26,37]:

$$\beta_c = 2 \mu_0 m_0 \Omega^2 r_0 (r_{02} - r_{01}) / B_0^2$$

(35)
where the subscript \( o \) stands for quantities in the equatorial plane, the superscript \(*\) for the displaced field lines as indicated in Fig.6, and a bar on top of a symbol denotes mean value formation over the range \( r_{02} < r_0 < r_{02} \). Within the limits of the quiescent holding mode, \( \beta \)-values have been obtained in FI at different neutral filling densities as shown by Fig.14 [37]. It is seen that appreciable \( \beta \)-values, up to \( \beta_c \approx 0.3 \), can be reached during this mode. Several series of probe measurements have also been made to study the corresponding radial displacements of the plasma boundary in the equatorial plane, with probe positions as indicated in Fig.6 [26, 29, 38]. One example is given in Fig.15 where the saturation current \( J_s \) flowing to a cylindrical double probe is plotted as a function of the radial distance \( r_0 \) in the equatorial plane, at the equivalent \( \beta \)-values 0.03 and 0.25 [38]. Here the cathode plate was in a higher vertical position than in Fig.6. All these measurements are at least in qualitative agreement with the theoretically expected displacements which would occur under macroscopically stable conditions. At the parameter data of FI those displacements range up to some 20 mm [26, 29, 38].

When the power input \( P \) is gradually decreased (e.g. as during a slowly decaying holding mode), the situation becomes as shown in Figs 16 and
Fig. 15. Ion saturation current $J_s$ of a cylindrical double probe as a function of the radial distance $r_0$ [38]. Estimated electron temperatures are indicated at points of measurement.

Here a sudden transition takes place at time $t_m$ from a fully ionized to a low-ionized state, as is also indicated by the measured equivalent capacity $C_{12}$ which drops to a low value in the vicinity of $t_m$. Thus the plasma can only be sustained in a fully ionized state as long as the power input exceeds a rather well defined 'minimum power' level $P_m$, of the order of 0.6 MW in the experiments of Figs 16 and 17. The minimum power required to sustain an already established fully ionized impermeable plasma becomes substantially less than the burn-out power required to create the same plasma from a mass of neutral gas. These circumstances are further explained by the theory on the plasma balance already outlined in connection with Eqs (32) and (33) and described in detail elsewhere [9, 26, 39]. Here we notice that the fully ionized plasma body of Fig. 5 can be sustained and surrounded by partially ionized boundary layers and end-wall layers as long as the maximum temperature $T_0$ (and its associated rotation velocity $v_0$) exceeds a certain minimum value $T_{om}$. 
FIG. 16. Transition at time $t_m$ from a fully ionized to a low-ionized state during a slowly decaying holding mode of a rotating plasma [39].

FIG. 17. Same as Fig. 16, with measured equivalent capacity $C_{12}$ indicated by dots and power input $P$ included [39].
equal to about $3 \times 10^4$ K in the present experiments. The minimum power $P_m$ is due to the overall balance of the plasma body including all partially ionized layers and boundary conditions. Thus $P_m$ is related but not identical to the minimum power $Q_{b\text{min}}$ of the cold-mantle model of Fig.2. The latter is, among other things, limited to plane geometry where there are no losses along the magnetic field lines.

**FIG.18.** Limitation of electrode voltage $\phi_{12}$ at a critical voltage level $\phi_{12c}$ [41]. Three shots are shown at equal start-up conditions but with different initial voltages $\phi_{b20}$ of a second condenser bank which is switched on at time $t = t_2$. Bank voltages $\phi_{b20}$ are (a) 6 kV; (b) 10 kV; (c) 16 kV.
5.2.3. Critical velocity limitation

The range of validity of Eq.(34) is not only limited by a minimum temperature $T_{0m}$ but also by an upper bound on $v_0$ and $T_0$. In fact, all rotating plasma experiments so far performed within the impermeable range of ion densities indicate that there is a sharply defined critical voltage limit $\phi_{12c}$ [26, 40, 41]. One example of this behaviour is given by Fig.18 [41]. Here the start-up of the plasma takes place during the time interval $0 < t < t_2$, under equal conditions throughout Figs 18(a)–(c).

At time $t_2$ a second bank is connected between the electrodes, with an initial voltage $\phi_{b20}$ of varying magnitude from shot to shot. To approach the critical voltage $\phi_{12c}$ slowly from below, i.e. at quasi-steady conditions, an inductance was introduced in series with the second condenser bank. At low bank voltages $\phi_{b20}$ the plasma behaved regularly with a high degree of reproducibility, as shown in Fig.18(a). At higher voltages, however, there was an abrupt change to an irreproducible discharge with large losses and a high degree of impurity as soon as the electrode voltage reached the critical level $\phi_{12c}$ shown in Figs 18(b) and (c). Heavy damage of the end insulator surfaces and a corresponding release of gas were also observed under these conditions.

In all experiments with fully ionized impermeable quasi-steady rotating plasmas, this voltage limitation seems to be associated with a velocity limitation at Alfvén's critical value $v_c$ as given by Eq.(30). There are clear indications that the underlying mechanism is due to plasma/neutral-gas interaction within the thin, partially ionized wall layers at the end insulators of Figs 5 and 6 [26, 41]. Thus the isorotation law limits the corresponding rotational velocity in the equatorial plane to

$$v_{\phi \text{max}} = v_c \left( \frac{r_o}{r_w} \right)$$

where $r_0$ is the radial position in the equatorial plane of a field line which cuts the insulator surface at the axial distance $r_w$ as indicated in Fig.5. Several theoretical attempts have been made to explain the critical velocity phenomenon [26, 42, 43], but some of its features are still open for discussion.

The importance to fusion research of rotating plasmas would be much increased if critical velocity limitation could be avoided under stable conditions and at ion densities of thermonuclear interest. Among the attempts to remove the mechanism of the critical velocity, it has been suggested that concentric metal rings be placed at the end insulator surfaces in order to short-circuit the azimuthal electric field which appears to be associated with this mechanism [26]. In experiments with impermeable plasmas and in the presence of floating rings, no such suppression has yet been achieved [44]. However, a series of experiments has been conducted at the Novosibirsk laboratory with different types
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of devices provided with sets of metal rings which are kept on controllable potential. In these experiments the critical velocity has been exceeded by far, i.e. when the ion density is chosen in the permeable regime and clean vacuum conditions are established [45, 46]. These results must be considered an important and interesting achievement, but their consequences are still difficult to estimate owing to the comparatively short energy containment times obtained and owing to some still unsolved stability questions probably related to the permeable ion density regime. Finally it should be mentioned that the critical velocity phenomenon is likely to be avoided in spiral coil traps of the "Tornado" type [47, 48].

5.2.4. Summary of features of permeable rotating plasmas

The main features of rotating plasmas with ion densities in the permeable range can now be summarized as follows:

(1) The crossed-field technique provides an efficient mechanism for creation of fully ionized plasmas in the permeable ion density regime. In this regime burn-out powers of several MW are required under conditions typical of small-scale experiments.

(2) The same technique also provides a convenient heating mechanism for such plasmas, in the form of viscous heating due to the sheared velocity field which arises in configurations of the type shown in Figs 5 and 6.

(3) Even in schemes making use of a strong centrifugal confinement effect, such as those of Figs 5 and 6, the end losses along the magnetic field become important in the case of permeable plasmas with nearly isotropic temperature distributions.

(4) The plasma can be sustained in a fully ionized state so long as the power input exceeds a certain rather sharply defined minimum power level. In devices of the type described here, this minimum power is substantially smaller than the burn-out power of a corresponding initial cloud of neutral gas.

(5) The velocity of rotation and the associated plasma temperature are subject to critical velocity limitation in experiments so far carried out on permeable plasmas. The viscous shear-heating then becomes an efficient pre-heating mechanism, but must be replaced by other heating methods when higher temperatures are to be reached. Only in the special Novosibirsk experiments [45, 46], in which metal rings are placed at the end insulators, has it become possible to reach high velocities under special conditions. Also, the Tornado traps could provide conditions for super-critical rotation [47, 48].

(6) At first sight, the centrifugal force and the field geometry in schemes of the type shown in Figs 5 and 6 are expected to become macroscopically unstable. Such behaviour has also been observed for the free boundary of a
rotating plasma in straight theta-pinch compression experiments performed earlier. On the other hand, the quasi-steady experiments described in Section 5.2.2 at least indicate that the plasma becomes stable with respect to macroscopic modes. Thus quiescent holding and free-wheeling modes are established with containment and free-wheeling times two orders of magnitude longer than the time of growth of a corresponding flute-type instability. The plasma balance and its dependence on relevant parameters such as magnetic field strength, plasma density and linear dimensions, are also consistent with classical theory. Furthermore, this classical behaviour has also been observed to hold at appreciable \( \beta \)-values and corresponding magnetic field perturbations. These stability properties can, so far, be explained by the mechanisms described in Section 3.2.3 for an impermeable plasma surrounded by a cold mantle.

5.3. Cold-mantle investigations

The cold-mantle concept has recently attracted appreciable interest in fusion research. Here we describe some investigations carried out by means of the FI device sketched in Fig.6.

5.3.1. Existence of a fully ionized plasma core

There are several experimental indications for the existence of the cold-mantle model of Figs 2 and 5, i.e. where there is a fully ionized plasma core surrounded by partially ionized layers and neutral gas regions. The following results are relevant:

(1) The burn-out power of Figs 8—10 supports this model, both in the sense that there is a sharply defined transition at \( \phi_{b0} = \phi_{b0c} \) and that there is no measurable further increase in the estimated ion density when \( \phi_{b0} > \phi_{b0c} \).
(2) The minimum power effect of Figs 16 and 17 is also characterized by a steep decrease in the equivalent capacity $C_{12}$ when the power input $P$ approaches $P_m$ from above.

(3) The observed free-wheeling times of Figs 11–13 are far too long to permit any appreciable amount of neutral gas to be present in the main rapidly rotating parts of the plasma body. Furthermore, in a low-ionized state, such as in Fig.8, and with $\phi_{b0}$ below the burn-out level $\phi_{b0c}$ in Fig.10, the free-wheeling time $\tau_T$ does not become measurable.

(4) Time-integrated electron density and electron temperature data were obtained by interferometry and spectroscopy, from which the neutral particle profiles were also calculated by numerical integration of the Boltzmann equation [31, 35]. These measurements were made as a function of time in a

---

**FIG. 20.** Dependence of equivalent capacity $C_{12}$ on average magnetic field strength $\overline{B}_0$ and neutral filling density $n_{no}$ in FI [8, 26]

(a) Variation with $\overline{B}_0$ at $n_{no} = 1.8 \times 10^{21} m^{-3}$

(b) Variation with $n_{no}$ at $\overline{B}_0 = 0.41 T$

Dashed lines correspond to behavior predicted by cold-mantle theory as represented by Eq. (37).
decaying rotating plasma discharge in FI. The calculated neutral profiles are shown in Fig. 19 for a set of power inputs. Here the neutral density drops steeply in the direction towards the plasma core when there is a high power input $P$, whereas the same density increases in the inner parts of the plasma at lower power inputs, becoming nearly constant across the plasma body when the input has dropped to the level $P = 0.12 \text{ MW}$. The plasma core thus becomes permeable to neutrals at $P \lesssim 0.4 \text{ MW}$ in these experiments.

(5) Spectroscopic investigations were made on the Doppler shifts of the lines emitted by different kinds of impurity atoms [33]. The results are in agreement with the theoretical model of velocity and temperature distributions which have maximum values inside the plasma core and decrease to low values at the plasma boundaries. In particular, no measurable Doppler shifts were recorded from neutral atoms, and this is consistent with a picture where such atoms are localized to the boundary layer of Figs 5 and 6.

(6) Various probe measurements indicate that radial ion density distributions in the boundary regions are at least in qualitative agreement with the model of Fig. 2 [38] (see also Fig. 15).

5.3.2. Density relationships in the boundary layer

The density relationships of Eqs (5)—(18) form a simple but important part of the cold-mantle scaling laws. The following experimental results have been obtained:

(a) The equivalent capacity of Eq. (28) was measured as a function of the average magnetic field strength $B_0$ and the neutral gas filling density $n_{n0}$ as shown by Fig. 20 [8, 26]. This behaviour is largely consistent with the cold-mantle model of Fig. 2. Thus, the density relationship of Eq. (8) in case (1) yields the capacity:

\[
C_{12} \approx \frac{(r_{02} - r_{01}) L \mu (3k_{BC}^2 /2)^{1/3}}{\pi (r_{02}^2 + r_{01}^2) n_{no}^{1/3} \over B_0^{4/3}}
\]  \hspace{1cm} (37)

in a first approximation. Here we have put $\overline{n}_n \approx n_b$ and assumed $n_{na} \approx n_{n0}$ because the total volume of the discharge chamber is substantially larger than the plasma confinement volume in the present experiments. Introducing the experimental data, the result of Eq. (37) becomes as represented by the dashed lines in Fig. 20. Agreement with experiments is as good as can be expected from the present approximations and error limits of the measurements.
(b) To obtain a direct picture of the density profiles in the boundary layer, a localized experimental analysis was recently made by laser interferometry, Thomson scattering of laser light, and spectroscopy [14, 31, 34, 35]. The results were obtained in a plasma at typical peak ion densities of $3.5 \times 10^{21} \text{ m}^{-3}$ and temperatures of 0.5–7 eV. This temperature range, with its modest lower limit, includes the transition from a permeable to an impermeable plasma even at the present relatively high prevailing ion densities. Plasma profiles were studied in the equatorial plane of the FI device, as demonstrated by Fig.21 for several series of decreasing power inputs $P$. In the experiments of Fig.21 the 'vacuum boundary' defined by the cathode plate and its associated field line in Fig.6 were kept at the radial position $r_{02} = 0.265 \text{ m}$. The transition from impermeable to permeable plasma was found in the case of Fig.21 to take place at a power input $P^* \equiv 0.4 \text{ MW}$. For $P > P^*$ the penetration length $L_{nf}$ of Eq.(3) fell below $5 \times 10^{-3} \text{ m}$. Within this range of the power...
input $P$ the magnitude of the measured peak density $n_{pk}$ of Fig.21 was found to be nearly independent of $P$. In all the reported experiments the peak density $n_{pk} \approx n_b$ was further found to scale as $n_b \propto n_{na}^{1/3}$ with the neutral wall density $n_{na} \approx n_{n0}$ as shown by Fig.22, and with the magnetic field strength $B \approx B_0$ as $n_b \propto B^{2/3}$ according to Fig.23. With the points already raised in paragraph (a) above, these results confirm the scaling law (8) of case (I) which applies to the present experiments. Consequently, the neutral density recordings represented by Fig.19 and the electron density and temperature profiles of Fig.21 provide a picture of the boundary layer that is in rather good agreement with the simple model of Fig.2.

5.3.3. Heat balance of the boundary layer

As indicated by the theory of Section 3.2.2, the heat balance of the boundary layer becomes related to its characteristic thickness $x_b$. Measurements of the temperature $T_{pk}$ associated with the peak density $n_{pk}$ defined in Fig.21 are shown in Fig.24 [14]. The figure also includes the estimated boundary layer thickness...
$x_{pk}$, defined here as the distance from the density peak $n_{pk}$ to the outer plasma edge. In the permeable range where $P < P^* \approx 0.4 \text{ MW}$, it is seen from Fig.24 that both $n_{pk}$ and $T_{pk}$ increase steeply with $P$. However, the impermeable range of $P > P^*$ is characterized by a nearly constant density $n_{pk}$, a slightly increasing temperature $T_{pk}$ with $P$, and a decreasing layer thickness $x_{pk}$ with increasing input $P$.

These features are at least in qualitative agreement with the theory on heat balance presented in Section 3.2.2 for a simplified plane model.

In addition, the decreasing layer thickness $x_{pk}$ at increasing power input $P$ in Fig.24 seems to be consistent with the enthalpy loss-dominated branch of Fig.4. Consequently, there is qualitative agreement as far as can be expected from a comparison between the plane model of Figs 2–4 and the device of Fig.6 with reduced but not negligible end losses along the magnetic field lines.

Finally, we observe that the rather high $\beta$-values obtainable in the FI device and the corresponding available ranges $P^* < P \leq 5 P^*$ and $1 < f \leq 10$, allow for rather wide limits of cold-mantle operation [13,14]. The present confinement scheme therefore provides convenient means for initial studies of the cold-mantle problem.

6. CONCLUSIONS

In summing up the main contents and conclusions of this paper, it should first be noticed that fusion research has made steady and significant progress during the last decades. Theory and experiments have been brought closer together and answers have been found to a number of important questions about such things as plasma confinement, stability and heating. Nevertheless many crucial problems are still far from solution. Future efforts in fusion research must therefore include both experimental and theoretical work spanning a broad field, all the way from pure plasma physics to fusion technology and system studies.

Within this complex of activities there is also the need for small- and medium-scale experiments, which would provide efficient and flexible means of investigating a number of important subproblems as well as for backing up less flexible large-scale experiments in the form of model studies and development of diagnostic methods.

In this connection the importance of small-scale experiments in testing new ideas and principles should be stressed. It is not yet certain which scheme (or schemes) will provide the final solution for the fusion reactor, nor has it been proved that the thermonuclear parameter conditions of a very hot plasma cannot be realized by more efficient schemes than those at present being developed and studied. We cannot therefore rule out the possibility that schemes can exist
which could even satisfy such conditions in an experiment no larger than a writing desk. Let us hope that the small laboratories will accept this challenge and convert it into a multitude of successful results!

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INTERACTION OF HIGH-CURRENT RELATIVISTIC ELECTRON BEAMS WITH PLASMA

Physical nature of the phenomenon and its application in microwave electronics

A.A. RUKHADZE
Lebedev Physical Institute,
Moscow,
USSR

Abstract

INTERACTION OF HIGH-CURRENT RELATIVISTIC ELECTRON BEAMS WITH PLASMA: PHYSICAL NATURE OF THE PHENOMENON AND ITS APPLICATION IN MICROWAVE ELECTRONICS.

Pulsed high-current electron beams with characteristic parameters: electron energy $10^5 - 10^7$ eV, electron current $10^3 - 10^6$ A, pulse duration $10^{-8} - 10^{-6}$ s, beam energy $10^2 - 10^6$ J and power $10^8 - 10^{13}$ W, are widely used in different branches of science and technology such as controlled thermonuclear fusion, relativistic microwave electronics, powerful semiconductors, chemical and gaseous lasers, new principles of heavy-ion acceleration, and long-distance energy transmission. The paper discusses a new branch of science — pulsed high-current electronics, which has its own experimental technique and methods of theoretical analysis. Parts I and II determine what is meant by “high current” in an electron beam and calculate the maximum obtainable current values; these calculations are made for the simplest geometrical configurations realizable in practice. Current methods for theoretical analysis of high-current electron beam physics are described, together with classification of current experimental devices for generating such beams according to high-current parameters. The stability of electron beams is discussed and the concept of critical currents is introduced. Part III gives a detailed account of plasma-beam instability which occurs on the interaction of a high-current electron beam with high-density space-limited plasma. The linear and non-linear stages of beam instability are considered. The given theory is used for calculations for amplifiers and microwave generators of electromagnetic radiation. Finally, the experimental achievements in high-current relativistic microwave electronics are reviewed.

Part I

PHYSICAL PARAMETERS OF HIGH-CURRENT RELATIVISTIC ELECTRON BEAMS

1. INTRODUCTION

Let us begin by describing the simplest device for generating high-current electron beams (Fig.1). It has three main parts: the pulsed voltage generator,
FIG.1. Sketch of device for generating high-current electron beams.

The diode and the drift tube. The pulsed-voltage generator is an energy storage device (usually a capacitor bank). The pulsed voltage applied to the diode is formed by gaps and forming line. A typical pulsed voltage shape is given in Fig.2. Here we have \( \tau_f \), duration of voltage increase (usually 5—10 ns); \( \tau_b \), duration of voltage decrease; \( \tau_d \), duration of flat-top (usually \( \approx 20-1000 \) ns for high-current accelerators) [1—4].

The diode is one of the most important and complicated parts of the accelerator (Fig.3). Cold cathodes of explosive emission are mainly used in present-day accelerators. Their geometrical forms are very different but we shall examine the simplest form: a flat diode with transverse dimensions much greater than the distance between cathode and anode. The flat diode consists of a metallic cathode, to which a negative potential \(-v_0\) is applied, and an anode made from metal foil or net, transparent to fast electrons. Under the influence of high voltage, autoelectronic emission occurs from the cathode surface, which induces the explosion of microspikes and the formation of a cathode plasma. Autoelectronic emission then changes to explosive emission from the free plasma surface, which in turn spreads slowly with velocity \( v_0 \approx 1-2 \times 10^6 \) cm\( \cdot \)s\(^{-1}\) and overlaps the diode gap. The duration of the pulse cannot therefore be longer than

\[
\tau \approx \frac{d(0)}{v_0}
\]

(1.1)

Here \( d(0) \) is the distance between the cathode and the anode. The density of the electron current \( j \) in the flat diode during the process of its overlapping by plasma may be obtained from the solution of the Poisson equation for the potential:

\[
\frac{d^2 \phi}{dx^2} = -\frac{4\pi j}{c} \left[ 1 - \left( \frac{c\phi}{mc^2} \right)^2 \right]^{-1/2}
\]

(1.2)
with the following boundary conditions:

\[ \phi \big|_{x=d(t)} = -v_0 \quad \text{and} \quad \frac{d\phi}{dx} \big|_{x=d(t)} = \phi \big|_{x=d(0)} = 0 \]  

(1.3)

This equation can be analytically solved in the opposite limits:

**Non-relativistic:** \( \frac{eV_0}{mc^2} \approx \gamma - 1 \ll 1 \)

**Ultrarelativistic:** \( \frac{eV_0}{mc^2} \approx \gamma - 1 \gg 1 \)

Thus we have for the critical current density the following equations:

\[
J = \frac{mc^3}{e} \frac{1}{2\pi d^2(t)} \begin{cases} 
\frac{2\sqrt{2}}{g} (\gamma-1)^{3/2} & \text{if} \quad \frac{eV_0}{mc^2} \ll 1 \\
\gamma & \text{if} \quad \frac{eV_0}{mc^2} \gg 1
\end{cases} \]  

(1.4)

These expressions may be approximately interpolated into only one formula:

\[
J = \frac{mc^3}{e} \frac{(\gamma^{2/3} - 1)^{3/2}}{2\pi d^2(t)} \]  

(1.5)

This expression is the same as (1.4) when \( \gamma \gg 1 \); and, in the non-relativistic case, when

\[ \gamma \approx 1 + \frac{u^2}{2c^2} \]
it differs from (1.4) by a factor \( \sqrt{3} = 1.73 \). Note that during plasma overlapping, the diode gap \( d(t) \) decreases:

\[
d(t) = d(0) - v_0 t
\]

and as a result, according to (1.5), electron current density increases. For short-pulsed accelerators with \( \tau \lesssim 10^{-7} \) s and for \( d(0) \approx 1 \) cm, the current density increase is not significant, although it is significant for long-pulsed beams of \( \mu s \) duration. The current density (1.5) for the vacuum diode is limited by the space charge of the electrons, while cathode emissivity in high-current electron accelerators with explosive emission is practically unlimited and does not influence the density value (1.5). The possibility therefore arises of increasing the diode current by means of partial or even complete neutralization of the space charge of electrons in the diode (this will be discussed later). We now give an estimate of current density values, noting that \( mc^3/e \approx 17 \) kA and assuming that \( d(t) \approx d(0) \approx 1 \) cm. If \( V_0 \approx 10^5 \) V we have \( j \approx 80 \) A \( \cdot \) cm\(^{-2} \), and if \( V_0 \approx 5 \times 10^6 \) V, \( j \approx 27 \) kA \( \cdot \) cm\(^{-2} \). We could say that high densities in the vacuum diodes are obtained at high voltage \( V_0 \) and small diode gaps \( d(0) \).

The drift tube is an equipotential metal tube whose length \( L \) is much greater than its radius \( R \). The electron beam generated in the diode is injected into the drift tube through a homogeneous metal foil (or wire net). Then it shocks the collector. In the drift tube an electron beam of the required dimensions is formed and focused, so, naturally, there are a large number of drift tubes in powerful electron accelerators.

Usually both the form of the focusing magnetic field and the type of drift tube filled with neutral gas or plasma depend on the need to transport the whole of the electron current injected by the diode. However, in order to classify present-day accelerators according to current, the maximum diode currents are usually given. We shall consider the simplest and most often used type of drift tube: a cylindrical vacuum tube placed in a homogeneous longitudinal magnetic field \( B_0 \) of sufficient force to transport the electrons. The electrons move along the lines of the magnetic field with velocity \( v_\parallel \) and rotate round them with velocity \( v_\perp \). The latter depends on the injection angle of electrons into the drift tube, on current density, magnetic field \( \vec{B}_0 \), and beam geometry. Neglecting a small region near the diode (with length of the order of \( R \)), we can consider the electron beam in the drift tube to be longitudinally homogeneous. For its radial distribution, we consider the beam to be annular with current distribution as shown in Fig.4. For example, if \( a = r_0 \), we have a continuous cylindrical beam. We shall classify high-current electron accelerators according to the current propagating through such a drift tube.

The electron-beam space-charge potential prevents injection of the electrons into the drift tube, and therefore some maximum value of the beam current exists.
The value for the annular beam, shown in Fig. 4, can be easily calculated, assuming a strong axial magnetic field, when we ignore the Larmor radius of the rotation of electrons and consider them to be "stuck" to the magnetic field forces. So we have to solve the equation for the potential:

\[
\frac{1}{2} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) = -\frac{4\pi j_{||}}{v_{||}} \begin{cases} 
0 & \text{if } r < r_0 - a \\
\frac{j_{||}}{v_{||}} & \text{if } r_0 - a \leq r_0 \leq r_0 \\
0 & \text{if } r_0 < r \leq R 
\end{cases} 
\]  

(1.7)

with the boundary conditions

\[
\phi \bigg|_{r = r_0 - a} = \phi_0 \quad \quad \phi \bigg|_{r = R} = 0
\]

(1.8)

here \( \gamma = (1 - u^2/c^2)^{-1/2} \) is the relativistic factor and \( u \) is the absolute electron velocity at the injector. Taking into account the continuity of the potential and its derivative on the external \( (r = r_0 - a) \) and internal \( (r = r_0) \) surfaces of the annular beam and its limit on the drift-tube axis, we obtain the value of the total maximum current in the drift tube [5]:

\[
J_0 = 17 \frac{\gamma}{\gamma_{||}} \frac{\left( \gamma_{||}^{2/3} - 1 \right)^{3/2}}{1 + 2\xi_n \frac{R}{r_o} + 2 \frac{(r_0 - a)^2}{r_0^2 - (r_0 - a)^2} \ln \left( 1 - \frac{a}{r_0} \right)} \quad \text{(kA)}
\]

(1.9)

where \( \gamma_{||} = (1 - u_{||}^2/c^2)^{-1/2} \).
In solving this equation, we take into account the integrals of motion of the electrons in the strong magnetic field and in the space-charge field:

\[ v_1 \left( 1 - \frac{v_\parallel^2 + v_\perp^2}{c^2} \right)^{-1/2} = \gamma u_1 \]  

Equation (1.9) may be simplified in the case of a continuous cylindrical beam of \( r_0 \) radius (i.e. if \( a = r_0 \)):

\[ J_0 = 17 \frac{\gamma}{\gamma_\parallel} \frac{(\gamma_\parallel^{2/3} - 1)^{3/2}}{1 + 2\xi n r_0} \text{ (kA)} \]  

For a thin annular beam, when \( a \ll r_0 \), we have

\[ J_0 = 17 \frac{\gamma}{\gamma_\parallel} \frac{(\gamma_\parallel^{2/3} - 1)^{3/2}}{\frac{a}{r_0} + 2\xi n r_0} \text{ (kA)} \]  

Now we can write down the condition when the Larmor radius of electron rotation may be neglected, and (1.9) is therefore valid. Thus we may neglect the change of the perpendicular velocity of the electrons due to their rotation in the radial field of the beam space charge and in the longitudinal magnetic field:

\[ \frac{1}{\gamma} \gg \frac{\left| \nabla \phi \right|}{B_0} \approx \frac{2\pi ja}{cB_0} \approx \frac{J}{\gamma r_0 B_0} \]  

where \( j \) is the electron current density, and \( J \) the total current. Even if \( j \approx 10^4 \text{ A} \cdot \text{cm}^{-2} \), which could easily be obtained in the vacuum diode when \( \gamma \gg 5 \) and \( a \approx 0.3 \text{ cm} \), this inequality is valid for the fields \( B_0 \gg 20-30 \text{ kG} \). The inequality (1.13) makes it possible to neglect the self-magnetic field of the current as compared with the external longitudinal magnetic field, which we shall assume later. The condition (1.13) therefore allows us to neglect the defocusing of the electron beam during its transmission through the drift tube.

Another obvious lower limit of the magnetic field force must be mentioned. To make expressions (1.9)–(1.12) valid, the beam thickness \( a \) should be greater
than the Larmor radius of electron rotation in the longitudinal magnetic field $u_L(\gamma/\Omega)$ where $\Omega = eB_0/mc$, and we have

$$B_0 > \frac{mc^3}{e} \frac{(\gamma^2 - \gamma_\|^2)^{1/2}}{\gamma_\|} \frac{1}{ca}$$

(1.14)

The inequalities (1.13) and (1.14) and all expressions given here are written in the CGSE system. They give an upper bound on the magnitude of the maximum beam current for the finite values of the magnetic field force $B_0$.

The quantity $J_0$ determines the maximum obtainable current in the drift tube. If the tube is filled by neutral gas or plasma, there may be partial or even complete neutralization of the beam space charge because the surplus charge is pushed out towards the tube ends. Without going into the details of the beam charge neutralization (it is discussed in Ref. [6], for example), we only note that for the strong magnetic field the neutralization time is

$$\tau_0 > \frac{R}{u} = \frac{R}{c} \frac{\gamma}{\sqrt{\gamma^2 - 1}}$$

(1.15)

The maximum current is greater than $J_0$ when neutralized to its beam charge. In fact, if we introduce the neutralization degree $f = n_i/n_b$ (here $n_b$ is the electron beam density and $n_i$ is the neutralizing ion density) it could be easily demonstrated that the maximum current of the neutralized beam is equal to

$$J_{\text{neutral}} = J_0 \frac{1}{1 - f}$$

(1.16)

i.e. it is $(1 - f)^{-1}$ times greater than in the unneutralized vacuum beam. The expression (1.16) could be easily obtained, noting that the current $j$ in Eq. (1.7) should be displaced by $j(1 - f)$ if neutralization occurs.

Equation (1.16) shows that the maximum beam current increases with the increase of neutralization degree $f$ and it becomes infinite when $f \rightarrow 1$ (complete neutralization). And now the limits depending on beam stability become significant (see Part II).

We shall briefly discuss another limit on the application of the expressions (1.9) and (1.16), on the assumption that the beam injection is stationary. The transitional processes depending on the beam magnetic field formation may be essential for the pulse systems. These processes are obviously significant when the magnetic field energy of the current is compared with the kinetic energy of
electrons in the drift tube. For the annular beam shown in Fig. 4 it is easily found that when the beam current is greater than

\[ J_m = 17 \frac{\gamma (\gamma - 1)}{\sqrt{\gamma^2 - 1}} \left( \frac{1}{\gamma} \right) \left( \frac{1}{r_0 - r_1^2} \right)^{1/2} \left( \frac{1}{l} \right) \left( \frac{1}{4} \right) \left( \frac{1}{r_0^2 - r_1^2} \right)^{1/2} \left( \gamma l_0 \right) \left( \frac{r_0 - r_1^2}{r_1} \right)^{1/2} \] (kA) (1.17)

(where \( r_1 = r_0 - a \)) the magnetic energy of the beam current becomes greater than the electron kinetic energy. In this case the injector power will be spent for rather a long time on the creation of the magnetic field of the current, and the shape of the current momentum should differ strongly from that of the voltage lagging behind. The duration of this stage is

\[ \tau_m \geq \frac{L}{u_||} \approx \frac{L}{c} \frac{\gamma}{\sqrt{\gamma^2 - 1}} \] (1.18)

much longer than the time of charge neutralization \( \tau_0 \) (1.15), and therefore it is just \( \tau_m \) that should be considered the minimal duration of the current establishment. Obviously, the pulse duration \( \tau \) should be much greater than \( \tau_m \) for formation of the stationary beam.

Finally, we note that we cannot refer to \( J_m \) as the beam current characteristic for vacuum drift tubes because

\[ \frac{J_m}{J_0} \approx \frac{\gamma^2 (\gamma - 1)}{\gamma \sqrt{\gamma^2 - 1} (\gamma^2 - 1)^{3/2}} \geq 1 \] (1.19)

The current \( J_m \) is an important characteristic only for the neutralized beams of \( J_n >> J_0 \), when \( J_m < J_n \), but in this case we should exclude the situations when plasma is of high density for which \( n_p >> c/a, n_b \). For such plasma, as shown in Ref. [6], both charge and current neutralization exist during time less than \( \tau_0 \), and therefore this transitional process has no influence.

In addition to the currents \( J_0, J_n \) and \( J_m \), we introduce the Alfvén current \( J_A \) [7]:

\[ J_A = 17 \beta || \frac{r_0 - a}{a} \gamma \] (kA) (1.20)

where \( \beta || = u_|| / c \). In our case of transmitting electron beams through the drift tube in the presence of a strong longitudinal magnetic field, the current \( J_A \) has
no physical meaning, but we shall in any case use the concept of the Alfvén current for classifying high-current electron beams since it is widespread in the literature. The current is determined as a maximum equilibrium current of a charge-neutralized electron beam without strong longitudinal magnetic field, and its magnitude could be taken from the equality of transverse dimensions of the beam $a$ and the Larmor radius of electron rotation in the self-magnetic field of the current.

It is assumed throughout that electron beams injected into the drift tube are monoenergetic, i.e. there is neither transverse nor longitudinal variation in the momenta. In other words, the distribution of injected electrons is such that

$$f_0 = \frac{n_b(r)}{2\pi p_{10}} \delta(p_\perp - p_{10}) \delta(p_\parallel - p_{\parallel 0})$$  \hspace{1cm} (1.21)

Here $n_b(r)$ is the electron density with spatial distribution the same as the beam geometry (Fig. 4):

$$n_b(r) = \begin{cases} 
0 & \text{if } r < r_0 - a \\
n_b & \text{if } r_0 - a \leq r \leq r_0 \\
0 & \text{if } r_0 < r 
\end{cases}$$  \hspace{1cm} (1.22)

and $p_{10}$ and $p_{\parallel 0}$ are transverse and longitudinal electron momenta, respectively, and

$$p_{\perp 0} = m\gamma u_\perp \quad p_{\parallel 0} = m\gamma u_\parallel$$  \hspace{1cm} (1.23)

Electrons in reality always have the impulse variation essential for the high-current electron beams. Therefore the distribution function of electrons (1.21) should be more correctly written as

$$f_0 = \frac{n_b(r)}{2\pi p_{10} \mathcal{P}_\perp \mathcal{P}_\parallel} \exp \left\{ -\frac{(p_\perp - p_{\perp 0})^2}{2\mathcal{P}_\perp^2} - \frac{(p_\parallel - p_{\parallel 0})^2}{2\mathcal{P}_\parallel^2} \right\}$$  \hspace{1cm} (1.24)

Here $\mathcal{P}_\perp$ and $\mathcal{P}_\parallel$ are the average statistical transverse and longitudinal momentum deviations of the electrons. It is also assumed that the inequalities

$$T_\perp = \frac{\mathcal{P}_\perp^2}{m} \ll mc^2 \quad T_\parallel = \frac{\mathcal{P}_\parallel^2}{m} \ll mc^2$$  \hspace{1cm} (1.25)
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<th>Accelerator</th>
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TABLE I. PARAMETERS OF SOME PRESENT-DAY HIGH-CURRENT ACCELERATORS
are valid, i.e. that the momentum deviation is considered to be non-relativistic. This condition is apparently fulfilled for all modern operating accelerators.

The statistical deviation of momenta, particularly the longitudinal one, leads to another important condition. We should have the Debye radius of electrons smaller than the transverse dimensions of the beam if we consider the beam to be monoenergetic to a large extent. This leads to a lower limit for the beam current, which in our geometry could be written:

\[
J > J_{\text{min}} = 17\beta_\parallel \frac{r_0^2 - (r_0 - a)^2}{4a^2} \frac{T_\parallel \gamma}{mc^2} \quad \text{(kA)}
\]

\[
\approx 8.5 \frac{r_0}{a} \frac{T_\parallel \gamma}{mc^2} \quad \text{(kA)}
\]

2. CLASSIFICATION OF PRESENT-DAY HIGH-CURRENT ELECTRON ACCELERATORS

Now we pass to the classification of present-day working accelerators and compare their achieved injected currents (diode currents) with \(J_0\), \(J_m\) and \(J_A\). It is obvious that if the beam current \(J_b\) is negligibly small compared to \(J_0\) and \(J_m\) we can neglect both the space charge of the electrons and the magnetic field of the current while considering the transport of such a beam through the drift tube. Such beams we call low-current, as opposed to high-current for which \(J_b \geq J_0\) or \(J_b \geq J_m\).

While comparing beam current with \(J_0\), \(J_m\), \(J_A\), the latter would be taken for vacuum drift tubes completely filled by an electron beam, i.e. for the case \(R - r_0 = a\). Also, the beam is considered to be strictly linear when \(u_\perp = 0\). In these conditions

\[
J_0 = 17(\gamma^{2/3} - 1)^{3/2} \quad \text{(kA)}; \quad J_m = 17 \frac{4\gamma \sqrt{\gamma - 1}}{\sqrt{\gamma + 1}} \quad \text{(kA)}; \quad J_A = 17 \sqrt{\gamma^2 - 1} \quad \text{(kA)}
\]

(1.27)

Let us introduce the notion of the "linear" electron, which is written for the continuous cylindrical beam of radius \(r_0\) as follows:

\[
\nu = n_b \pi r_0^2 \frac{e^2}{mc^2} = N_b r_K^2
\]

(1.28)
Here $n_b$ is the total number of electrons per beam length unit, and $r_K = e^2/mc^2$ is the classical electron radius. The electron current may be written:

$$J = 17 \frac{\nu}{\gamma} \sqrt{\gamma^2 - 1} \text{ (kA)}$$

(1.29)

From Eqs (1.27) and (1.28),

$$\frac{J}{J_0} = \frac{\nu}{\gamma} \frac{\sqrt{\gamma^2 - 1}}{(\gamma^{2/3} - 1)^{3/2}}; \quad \frac{J}{J_m} = \frac{\nu}{\gamma} \frac{\gamma + 1}{4\gamma}; \quad \frac{J}{J_A} = \frac{\nu}{\gamma}$$

(1.30)

Table I gives the basic physical parameters of some of the best-known present-day high-current accelerators working in the USA and the USSR and classifies electron beams according to the ratio of beam current to $J_0, J_A$. Parameters are taken from Ref. [1].

Part II

STABILITY OF HIGH-CURRENT ELECTRON BEAMS
THE PROBLEM OF CRITICAL CURRENTS

1. INTRODUCTION

The simplest equilibrium configuration of high-current relativistic electron beams was discussed in Part I, where we defined the maximum currents that could be realized in this configuration both with and without neutralization of the beam charge. Realization of such beam currents depends in practice on the solution of one of the most important problems of high-current beam physics: the stability of equilibrium configurations.

We shall analyse the stability of the simplest configuration of high-current beams discussed in Part I, restricting the analysis to high-frequency pure electron perturbations only, having in mind the short pulse length of such beams.

The most characteristic high-frequency instabilities of electron beams are the following:

(a) Pierce instability, caused by the finite-longitudinal dimensions of the system.
HIGH-CURRENT REB-PLASMA INTERACTION

(b) Budker-Buneman instability, caused by the relative motion of the electron and ion in the completely or partially neutralized beams.
(c) Convective instabilities, i.e. the type of current-convective and slipping instabilities of the beam.
(d) Cherenkov and cyclotron-type beam instabilities, caused by beam radiation of the electromagnetic waves.

These instabilities are more dangerous for pulsed electron beams, focused by a sufficiently strong longitudinal magnetic field, determined by the expression:

$$\frac{B^2_0}{8\pi} \gg n_b mc^2 \gamma$$

(2.1)

If this inequality is valid, it also enables us to neglect the beam self-magnetic field compared with the external magnetic field. As shown in Part I, it is in such strong fields that, when the beam charge is neutralized, the equilibrium configuration can be realized with practically unlimited longitudinal currents. This gives rise to the very important question of the stability of such equilibrium configurations. As we shall see later, the stability requirements limit the beam current from above by certain values, which we shall call critical. For strong instabilities with fast aperiodic growth, critical currents in fact appear to be maximum, because equilibrium configurations with such a critical current would collapse for a short time. However, slowly increasing periodic instabilities are interesting from the point of view of amplification and generation of electromagnetic waves by electron beams, which will be discussed later.

2. PIERCE INSTABILITIES OF NEUTRALIZED ELECTRON BEAMS

One of the most dangerous instabilities of a neutralized beam with regard to its charge is the pure-electron Pierce instability, caused by the finite-longitudinal dimensions of the system [8, 9]. Let us consider this instability on the example of a continuous cylindrical beam, completely filling the metallic waveguide (i.e. $R = r_0 = a$) of finite length $L \gg R$.

At the beam input and output we assume the presence of thin metal foils (input net and output collector), transparent to electron beams. The beam is considered completely neutralized and its parameters are therefore homogeneous along its section. Taking into account the inequality (1.1) we shall discuss only the electric (potential) field perturbation, also neglecting the electron velocity perturbation across the magnetic field. We consider the electron beam to be monoenergetic with a momentum distribution function of the form given in (1.21), i.e. in the condition that neither longitudinal $u_\parallel$ nor transverse $u_\perp$ is
equal to zero. The system of equations for the small perturbations can be written as follows:

\[
\frac{\partial \delta \psi}{\partial t} + u_1 \frac{\partial \delta \psi}{\partial Z} = -\frac{e}{m \gamma \gamma_b^2} \frac{\partial \phi}{\partial Z}
\]

\[
\frac{\partial \delta n}{\partial t} + \frac{\partial}{\partial Z} \left( n_b \delta v_1 + u_1 \delta n \right) = 0
\]

where \( \delta n \), \( \delta v_1 \) and \( \phi \) are the perturbations of the electron density, their velocity and field potential, respectively (we should bear in mind that in neutralized beams the equilibrium potential is absent). The simultaneous equations (2.2) have to be supplemented by the boundary conditions:

\[
\phi|_{r=R}=\phi|_{Z=0,L}=0, \quad \delta n|_{Z=0} = \delta v_1|_{Z=0} = 0
\]

(2.3)

corresponding to the injection conditions of an unperturbed beam in the plane \( Z=0 \).

The simultaneous equations (2.2) are easily reduced to a single equation for the potential \( \phi \):

\[
\left(-i \omega + u_1 \frac{\partial}{\partial Z}\right)^2 \Delta \phi + \frac{\omega_b^2}{\gamma_1 \gamma} \frac{\partial^2 \phi}{\partial Z^2} = 0
\]

(2.4)

the solution of which, taking into account cylindrical symmetry and the first of the boundary conditions (2.3), has to be sought in the form:

\[
\phi = J_\ell \left( \mu_{\ell s} \frac{r}{R} \right) \sum_{n=1}^{4} \phi_n \exp(iK_n Z)
\]

(2.5)

Here, \( \mu_{\ell s} \) are the roots of the Bessel function; \( J_\ell(\mu_{\ell s}) = 0 \); and \( K_n \) is determined by the characteristic equation of the fourth degree:

\[
\frac{\mu_{\ell s}^2}{R^2} + K^2 \left[ 1 - \frac{\omega_b}{\gamma \gamma_b^3(\omega - Ku_1)^2} \right] = 0
\]

(2.6)
Taking an interest in the development of the Pierce instability threshold, we consider the frequency range of $\omega \ll \omega_b$ in which the roots of Eq.(2.6) are the following:

$$K_{1,2} \approx \pm \frac{1}{u_1} \sqrt{\frac{\omega_b^2}{\gamma\gamma_i^2} - K_i^2 u_1^2 + \frac{\omega\omega_b^2}{\gamma\gamma_i^2 u_1} \left( \frac{\omega_b^2}{\gamma\gamma_i^2} - K_i^2 u_1^2 \right)}$$

(2.7)

$$K_{3,4} \approx \omega \left( u_\parallel \pm \frac{\omega_b}{K_i \gamma_i \sqrt{\gamma}} \right)$$

Here $K_i = \mu_{qs}/R$. Now it is not difficult to use the boundary conditions (2.3) and to obtain the dispersion equations for defining the spectrum of the eigenvalues $\omega$:

$$\left( \frac{\omega_b^2}{\gamma\gamma_i^2} - K_i^2 u_1^2 \right)^{3/2} \left( \exp(iK_1L) - \exp(iK_2L) - \frac{2\omega\omega_b^2}{\gamma\gamma_i^2} \left( \exp(iK_1L) + \exp(iK_2L) \right) 

- \exp(iK_3L) - \exp(iK_4L) \right) + \frac{\omega\omega_b^2}{\gamma\gamma_i^2 \sqrt{\gamma} K_i u_1} \left( \frac{\omega_b^2}{\gamma\gamma_i^2} + K_i^2 u_1^2 \right) \times \left( \exp(iK_3L) - \exp(iK_4L) \right) = 0,$$

(2.8)

It may be seen that Eq.(2.8) has increasing solutions with $J_n \omega > 0$ in the interval:

$$\left( 2n - 1 \right) \frac{\pi u_\parallel}{L} < \sqrt{\frac{\omega_b^2}{\gamma\gamma_i^2} - K_i^2 u_1^2} < 2n \frac{\pi u_\parallel}{L}$$

(2.9)

Here, $n = 1, 2, 3$; assuming $n = 1$ and $K_i = \mu_{qs}/R = 2.4/R$, we can find the critical beam current above which Pierce instability develops in the electric beam:

$$J_{cr1} \approx 24 \frac{\gamma}{\gamma_i} (\gamma_i^2 - 1)^{3/2} (kA)$$

(2.10)

The maximum increment of instability development can be achieved for the small excess of the threshold (2.10) and it is equal to

$$J_m \omega_{max} \approx \frac{u_\parallel}{L}$$

(2.11)
It should be noted that the Pierce instability is purely aperiodic and develops with the formation of a virtual cathode and the shutting down of the upper critical beam current.

If we compare (2.11) with the maximum vacuum beam current (1.13) with \( R = r_0 \), we see that

\[
\frac{J_{cr1}}{J_0} \approx \left( \frac{\gamma_1^2 - 1}{\gamma_1^{2/3} - 1} \right)^{3/2}
\]

(2.12)

In the case of an ultra-relativistic beam, i.e. with \( \gamma \gg 1 \), the Pierce critical current may be much greater than the vacuum beam current \( J_{cr1} \approx \gamma_1^2 J_0 \gg J_0 \); and for the non-relativistic beam \( J_{cr1} \approx 5J_0 \).

However, if (2.11) is compared with the maximum current of the partially neutralized beam (1.16), we may conclude that until

\[
\left( \frac{\gamma_1^{2/3} - 1}{\gamma_1^2 - 1} \right)^{3/2} \frac{1}{1-f} < 1
\]

(2.13)

the maximum beam current is determined by the equilibrium, i.e. by the expression (1.16). For the opposite condition, the maximum achieved current is caused by the requirement of stability, i.e. by the expression (2.11). When Eq.(2.13) is valid, the notion of beam instability loses sense; it becomes unstable.

In conclusion, we note that this calculation of the critical current is easily generalized for the case of random annular beam geometry, Eq.(2.12) being valid.

3. BUDKER-BUNEMAN INSTABILITY

So far, we have completely neglected the motion of the ions which neutralize the electron beam charge. Thus the ions were considered to be of infinite mass. Let us now take into consideration the finite mass of the ions and show that the neutralized electron beam, with the ion motion taken into account, may become unstable against potential (electrostatic) perturbations when the system is of infinite length. This instability, which is due to the relative motion of electrons and ions of finite mass, is called after Budker and Buneman [9, 10]. To study such an instability in an abstract way, we shall consider an infinitely long system in the form of a metallic waveguide, filled with a completely neutralized electron beam and placed in a strong longitudinal magnetic field, for which condition (2.1) is fulfilled. However, we shall not consider ions to be unmagnetized, assuming the inequality \( \omega_L^2 \gg \Omega_i^2 \) to be valid; here \( \omega_L \) is the ion plasma frequency, and \( \Omega = eB_0/\mu_0 \) is the ion Larmor frequency.
For these limits, the system of equations for small electrostatic perturbations of the compensated electron beam can be written down as follows:

\[
\Delta\phi = -4\pi (e\delta n_e + e_i\delta n_i)
\]

\[
\left(\frac{\partial}{\partial t} + u_i\right) \delta v_{ei} = -\frac{e}{m\gamma_i^2} \frac{\partial \phi}{\partial Z}, \quad \frac{\partial}{\partial t} \delta v_i = -\frac{e_i}{M} \frac{\partial \phi}{\partial r}
\]

\[
\frac{\partial \delta n_e}{\partial t} + \frac{\partial}{\partial Z} (n_b \delta v_{ei} + u_i \delta n_e) = 0, \quad \frac{\partial \delta n_i}{\partial t} + \text{div} n_{0i} \delta v_i = 0
\]

(2.14)

where \(\delta n_e\) and \(\delta n_i\) are the perturbations of electron and ion densities, whose non-equilibrium values are \(n_e\) and \(n_{0i}\), respectively; \(\delta v_e\) and \(\delta v_i\) are perturbations of their velocities; \(M\) is the ion mass, and \(e_i\) is their charge. Taking into account the longitudinal infinity of the system, Eqs (2.14) must be supplemented by the single boundary condition,

\[\phi|_{r=R} = 0\]

(2.15)

We write down the general solution of the equations (2.14) for the field oscillation potential, valid for the boundary condition (2.15):

\[\phi(r) = A J_0 \left(\frac{\mu k_s}{R}\right) r \exp (-i\omega t + iK_Z Z) \]

(2.16)

Putting (2.16) into (2.14) leads to the system of homogeneous equations with non-trivial solutions, while the following dispersion equation is valid:

\[
\frac{\mu^2 k_s}{R^2} \left(1 - \frac{\omega^2}{\omega_{Li}^2}\right) - \frac{\omega_b^2 K_Z^2 \gamma_i^2}{(\omega - K_Z u_i)^2} = 0
\]

(2.17)

The solutions of this equation, corresponding to the unstable oscillations, exist only in the frequency range \(\omega < K_Z u_i\). From the conditions of

\[
\frac{\omega_b^2}{\gamma_i^2 u_i^2} \left[1 + \left(\frac{\mu k_s}{R^2} \frac{\omega_{Li}^2 \gamma_i^2}{\omega_b^2 K_Z^2}\right)^{1/3}\right] \geq \frac{\mu k_s}{R^2}
\]

(2.18)

\[
\omega = K_Z u_i \left(\frac{\mu k_s}{R^2} \frac{\omega_{Li}^2 \gamma_i^2}{\omega_b^2 K_Z^2}\right)^{1/3} > \omega_{Li}
\]
we obtain the critical Budker-Buneman current for a neutralized electron beam, completely filling the metallic waveguide:

\[ J_{cr2} = 24 \frac{\gamma}{\gamma_1} \frac{(\gamma_1^2 - 1)^{3/2}}{(1 + \omega/KZu_\parallel)^3} \text{ (kA)} \] (2.19)

It follows from the inequality \( \omega \ll KZu_\parallel \) that the critical Budker-Buneman current (2.19) differs little from the Pierce current (2.10): it is insignificantly smaller. It may be shown that this statement remains valid also for annular electron beam geometry, so that, to a great degree of accuracy, we have

\[ J_{cr2} \sim J_{cr1} \] (2.20)

Therefore all the conclusions about Pierce instability in the last paragraph remain valid for the Budker-Buneman instability.

The difference between the physical nature of the Budker-Buneman instability and that of the Pierce instability is essentially shown in the growth rate, which for the beam current \( J > J_{cr2} \) is of the order:

\[ \text{Im } \omega \approx \frac{\sqrt{3}}{2} KZu_\parallel \left[ \frac{m}{M} \left( \frac{2.4}{R} \right)^2 \frac{\gamma_1^2}{KZ^2} \right]^{1/3} \] (2.21)

i.e. independent of the system length. It exceeds the ion plasma frequency. Therefore one may say that in the systems with \( L < u_\parallel /\omega_{Li} \) the maximum beam current obtained would be dependent on the circumstances of the Pierce instability development, but for \( L > u_\parallel /\omega_{Li} \) it is dependent on the Budker-Buneman instability. This all agrees with taking the Pierce instability to be an electrostatic instability of a neutralized electron beam, caused by the positive feedback through the external electric circuit [9]. That is why its growth rate depends on the system length. At the same time, the Budker-Buneman instability is also the same electrostatic instability, in which positive feedback is, however, provided by the ions. Consequently the growth rate depends on the ion mass.

The Pierce and Budker-Buneman instabilities lead to the formation of a fluctuating virtual cathode and to shutting down the beam current that exceeds the critical currents (2.10) and (2.19). The development time of these instabilities is extremely short — of the order of the transit time in the system (Pierce instability) or even shorter (Budker-Buneman instability), and they are therefore dangerous for practically all infinitely short-pulsed electron beams.
4. CONVECTIVE ELECTRON BEAM INSTABILITIES

These instabilities can develop for practically all infinitely strong magnetic fields. Therefore the obtained critical currents $J_{c1,2}$ are independent of magnetic field force. If we take into account the finite magnetic field force, another type of extremely dangerous electrostatic instability appears, the convective instabilities, caused either by the transverse inhomogeneity with regard to beam density [11] (current-convective instability) or by the directed velocity [12] (slipping instability). Convective instabilities are dangerous because they may manifest themselves with beam charge not completely neutralized for currents less than critical ((2.10) and (2.19)). Owing to the space charge, the density and electron velocity distributions in the cross-section are always inhomogeneous in such beams. In addition, even neutral beams may have essential inhomogeneity. It is sufficient to note that beams that are finite in the radial direction (e.g. annular beams) are always inhomogeneous with regard to their density. Great difficulty arises during analysis of inhomogeneous beam instability because the differential equations, describing small perturbations of such beams, have variable coefficients. They can be solved analytically only for a limited class of functions, characterizing the dependence of these coefficients on coordinates, usually without describing the real situation.

For analysis of convective instabilities of relativistic electron beams we therefore make some simplifying assumptions. First, the external longitudinal magnetic field will, as a rule, be considered sufficiently strong for the inequality (2.1) to be valid. It permits us to restrict ourselves to the analysis of the electrostatic (potential) oscillations of the system only and to neglect the inhomogeneity of the longitudinal magnetic field. Second, we shall analyse the long-wave perturbation limit, when beam parameters could be represented by piecewise homogeneous radial distributions (annular beam with the sharp boundaries shown in Fig.4), and the problem could be solved for the individual homogeneous intervals, by putting the obtained solutions together in the interval boundaries.

If we take into account the sufficiently strong but finite magnetic field, the system of equations (2.14) become somewhat difficult, so we have to consider the perturbations of the electron transverse velocity, which, for our assumption, is written down as follows:

$$\delta v_{e1} = \frac{c}{B_0^2} \left[ B_0 \times \nabla \phi \right]$$

This changes both the equation of motion for $\delta v_e$ and the continuity equation:

$$\left( \frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial Z} \right) \delta v_{e1} + (\delta v_{e1} \nabla) u_1 = -\frac{e}{m \gamma_1^2} \frac{\partial \phi}{\partial Z}$$

$$\frac{\partial \delta n_e}{\partial t} + \frac{\partial}{\partial Z} \left( n_b \delta v_{e1} + u_1 \delta n_e \right) + \text{div} \left( n_b \delta v_{e1} \right) = 0$$

(2.23)
From the simultaneous equations (2.14), (2.22) and (2.23), considering all the perturbed values to be dependent on time and coordinates, like

\[ \phi(\vec{r}) = \phi(r) \exp \{-i\omega t + i\varphi + iKZz\} \]  

(2.24)

we can obtain the equation for the small oscillation potential of the radially inhomogeneous electron beam with partially compensated charge:

\[ \frac{1}{2} \frac{\partial}{\partial r} \left[ r \left( 1 - \frac{\omega_{Li}^2}{\omega^2} \right) \frac{\partial \phi}{\partial r} \right] - \frac{q^2}{r^2} \left( 1 - \frac{\omega_{Li}^2}{\omega^2} \right) \phi - \frac{q}{r} \phi \frac{\partial}{\partial r} \left( \frac{\omega_b^2 \Omega_e^{-1}}{\omega - KZu_l} \right) - KZ \left( 1 - \frac{\omega_{Li}^2}{\omega^2} \right) \frac{\omega_b^2 \gamma^{-1} \gamma_{||}^{-2}}{(\omega - KZu_l)^2} \phi = 0 \]  

(2.25)

Here \(\Omega_e = eB_0/mc\) is the Larmor frequency of the electron rotation in the homogeneous magnetic field, and the quantities \(\omega_{Li}, \omega_b, \gamma, \gamma_{||}\) are considered dependent on the radial coordinate (r), while beam charge compensation may be incomplete, but \(\omega_{Li}^2 \gg \Omega_e^2\).

Equation (2.25) is valid both inside and outside the beam, and the boundary conditions may therefore be obtained by integrating Eq.(2.25) proper over the infinitely thin layer near the free beam surface. As a result, we find

\[ \phi|_{r=R} = 0, \ \{\phi\}_{r=r_0, r_0-a} = 0 \]  

(2.26)

\[ \left\{ 1 - \frac{\omega_{Li}^2}{\omega^2} \frac{\partial \phi}{\partial r} - \frac{q}{r} \frac{\omega_b^2 \phi}{\Omega_e (\omega - KZu_l)^2} \right\}_{r=r_0, r_0-a} = 0 \]

Without giving details of the solution of this problem (they can be found in Ref.[5]), we shall give only the final result for critical current. First, we note that in the case of a homogeneous beam, completely filling the metallic waveguide, the convective instabilities cannot develop and the critical current is caused by the development of either the Pierce or the Budker-Buneman instability.

For a continuous cylindrical beam separated from the waveguide walls (i.e. \(R > r_0, a = 0\)), we find, from the condition of convective instability development,

\[ J_{cr3} \approx 34 \frac{(\gamma^2 - 1)^{3/2}}{1 + \frac{4\gamma \gamma_{||} u_l L}{\pi r_0^2 \Omega_e}} (kA) \]  

(2.27)
For a thin annular beam, when $a \ll r_0$, the possibility of convective instability development in the system determines the following critical current:

$$J_{cr3} \approx 21 \frac{\gamma}{\gamma_i} \frac{r_0}{a} \frac{(\gamma_i^2 - 1)^{3/2}}{1 + \frac{\gamma_i u_\parallel L}{\pi a r_0 \Omega_e}} \text{(kA)}$$

(2.28)

It is seen from relations (2.27) and (2.28) that convective instabilities may develop not only in systems of finite values of longitudinal magnetic field force, but also in systems of finite length. We should note that they are more dangerous in long than in short systems, in the sense that they develop more easily. For convective instabilities, development is necessary:

$$l < \frac{\gamma_i u_\parallel L}{\pi r_0 \Omega_e} \times \begin{cases} 4/r_0 & \text{for cylindrical beams} \\ 1/a & \text{for annular beams} \end{cases}$$

(2.29)

When these inequalities are valid and if critical current $J_{cr3}$ turns out to be less than (2.10) and (2.19), convective instabilities have to be determined in the system. It should be noted that when the inequalities (2.29) are valid, this advantage of beam charge neutralization loses its force. The question is about the increase of the critical currents (2.10) and (2.19) caused by Pierce and Budker-Buneman instabilities, with electron energy $\gamma_i^3$. While (2.29) is valid for $J_{cr3}$, such an increase does not take place and, moreover (and this should be particularly emphasized), in very long systems $J_{cr3}$ may become even less than maximum vacuum beam current $J_0$. If we add to this that convective instabilities, in the same way as the Pierce and Budker-Buneman instabilities, develop very quickly, with growth rate $\text{Im} \omega > \omega_{L1}$, and, since they are electrostatic, they lead to the formation of a fluctuating virtual cathode, shutting down the current, it then becomes easy to understand their danger for long-pulsed as well as short-pulsed electron beams.

5. ELECTRON BEAM INSTABILITY FOR DENSE PLASMA

We have deduced that, in neutralized beams, stable currents may be obtained which are many times higher than the vacuum maximum currents of unneutralized beams. In reality, it is very difficult to use this advantage, particularly if sufficiently long systems are involved. Very strong magnetic fields are necessary to put down the convective instabilities that develop in long systems for very small beam currents (in principle, even less than vacuum maximum currents). In the opposite case, the instability, corresponding to excitation of the axially non-symmetric electrostatic oscillation modes, may bring this advantage of neutralized beams almost to nought.
To obtain high-current electron beams in long systems, it seems better to transport such beams through dense plasma. During injection of an electron beam into dense plasma, as noted in Part I, complete neutralization of the beam charge and current is going on, and therefore the limits, depending on the space-charge potential and on the self-magnetic field of the current, are absent. In addition, owing to the presence of very many light and mobile plasma electrons, the electrostatic instabilities of the Pierce, Budker-Buneman and convective types, which lead to the formation of a fluctuating virtual cathode and to shutting down the current, cannot develop. This should undoubtedly increase the upper limit of electron beams obtained in dense plasma if new and dangerous instabilities limiting the beam current do not appear.

We shall consider stability of the plasma-beam system for the conditions of complete charge and beam-current neutralization, restricting ourselves to the case when the drift tube is completely filled with the beam and plasma and is placed in an infinitely strong longitudinal magnetic field. The continuous equations for small oscillations of plasma and beam electrons are then written down as follows:

\[
\left( \frac{\partial}{\partial t} + u_\parallel \frac{\partial}{\partial Z} \right) \delta v_\parallel \hat{Z} = \frac{e}{m \gamma^2} \left( \hat{E} + \frac{1}{c} [\hat{u} \times \hat{B}] \right)
\]

\[
\frac{\partial \delta u}{\partial t} + \frac{\partial}{\partial Z} (n_0 \delta v_\parallel + u_\parallel \delta n) = 0
\]

\[
\text{rot} \hat{E} = -\frac{1}{c} \frac{\partial \hat{B}}{\partial t}
\]

\[
\text{rot} \hat{B} = \frac{1}{c} \frac{\partial \hat{E}}{\partial t} + \frac{4\pi}{c} (n_0 \delta v_\parallel + u_\parallel \delta n) \hat{Z}
\]

(2.30)

These equations are written for electrons of both beam and plasma, in equilibrium plasma, electron density \(n_0\) being equal to \(n_p\), and for the beam electron \(n_0\) equal to \(n_b\). Plasma electrons have no equilibrium velocity, and the perturbed field is not considered to be potential.

Having written the solution of the system of equations (2.30) as follows:

\[
E_Z = E_0 J_e \left( \frac{\mu q_s}{R} \right) \exp \{ -i \omega t + i \phi + i K_Z A \}
\]

(2.31)

and having made valid the boundary conditions on the surface of the metallic waveguide:

\[
E_Z \big|_{r=R} = E_\phi \big|_{r=R} = 0
\]

(2.32)
we obtain the following dispersion equation of oscillation:

$$\frac{\mu_{b_5}^2}{R^2} + \left( K_Z^2 - \frac{\omega^2}{c^2} \right) \left[ 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2 \gamma_{\parallel}^{-2}}{(\omega - K_Z u_\parallel)^2} \right] = 0$$

(2.33)

where $\mu_{b_5}$ are the roots of the Bessel function $J_0(\mu_{b_5}) = 0$. For a small-density beam, when $n_b \ll n_p$, Eq.(2.33) has unstable solutions only in the region $\omega \approx K_Z u_\parallel$. The instability corresponds to the Cherenkov excitation in the plasma waveguide of self-electromagnetic waves [13] (and see [14] also), the following inequality for excitation of the mode with radial wavenumber being necessary:

$$\omega_p^2 > \frac{\mu_{b_5}}{R^2} \gamma_{\parallel}^2 u_\parallel^2$$

(2.34)

In this case the frequency and growth rate of the excited $E$-type electromagnetic wave are equal to:

$$\omega^2 = K_Z^2 u_\parallel^2 = \omega_p^2 - \frac{\gamma_{\parallel}^2 u_\parallel^2 \mu_{b_5}^2}{R^2}$$

$$\text{Im} \frac{\omega}{\omega} = \frac{\sqrt{3}}{2} \left[ \frac{n_b}{2n_p \gamma_{\parallel}^2} \left[ 1 + \frac{\mu_{b_5}^2 \gamma_{\parallel}^2 u_\parallel^2 (\gamma_{\parallel}^2 - 1)}{R^2 \omega_p^2} \right]^{-1} \right]^{1/3}$$

(2.35)

It follows from the inequality (2.34) that this beam instability could increase only when the plasma density is greater than some critical value, which is equal to

$$n_{pcr} \approx 2 \times 10^{-9} \frac{\gamma_{\parallel}^2 u_\parallel^2}{R^2}$$

(2.36)

When the plasma density is less than critical, the beam is stable within the density range $n_b \ll n_p < n_{pcr}$. Hence it follows that the critical current of the recompensated beam, steadily transmitted through the plasma with density less than $n_{pcr}$, is equal to

$$J_{cr4} \approx \frac{24}{\gamma_{\parallel}} \frac{(\gamma_{\parallel}^2 - 1)^{3/2}}{\left[ 1 + \left( \frac{n_b \gamma_{\parallel}^{1/2} \gamma_{\parallel}^{-2}}{n_{pcr}} \right)^{1/3} \right]^3} (\text{kA})$$

(2.37)

This current is somewhat lower than the maximum Pierce current $J_{cr1}$ (and the Buneman current $J_{cr2}$), but it could be essentially higher than $J_m$, particularly in the range of the relativistic electron energies, when $\gamma_{\parallel}^2 \gg 1$. 
However, as follows from (2.35), even for the conditions when the beam instability cannot develop (i.e. when $n_p > n_{pcr}$), its relative growth rate is always small because

$$\frac{\text{Im } \omega}{\omega} \approx \left( \frac{n_b}{n_p \gamma^4} \right)^{1/3} \ll 1$$

(2.38)

Therefore, beam instabilities caused by the development of instability are always small (of the order $\text{Im } \omega/\omega$) and are caused by Cherenkov radiation of electromagnetic waves in plasma followed by small beam modulation. This in its turn means that such an instability does not in practice influence current transport of the electron beam through dense plasma and is not dangerous in this sense.

Everything written above is also valid for the beam of annular geometry shown in Fig. 4. The critical plasma density for such a beam, above which the development of beam instability is possible, turns out to be equal to

$$n_{pcr} \approx \frac{m}{2ne^2} \frac{u_1^2 \gamma_1^2}{a^2} \left( 1 + 2 \ln \frac{R}{r_0} \right)^{-1}$$

(2.39)

The growth rate also changes as a result of the geometrical factor but, as in the case of $\text{Im } \omega/\omega \ll 1$, the main conclusion on the small influence of the beam instability on the beam parameters remains valid. This means that in dense plasma there are, in fact, no limits on the beam current; it is possible to transport through dense plasma practically any electron beam current with slight distortion of its parameters. But if we insist on complete beam stability, then, from the inequalities $n_b < n_p < n_{pcr}$ in the annular beam as well as in the continuous beam, according to (2.38) the only electron beam with current less than the Pierce current $J_{cri}$ may be strictly stable.

Along with the Cherenkov-type beam resonance instability in dense plasma, a resonance cyclotron instability may also develop [15] (see [14] also). In contrast to the Cherenkov instability developed in the conditions when the longitudinal beam velocity is close to the phase velocity of the electromagnetic waves in the system, i.e. $\omega \approx K Zu_1$, the cyclotron instability may also develop when $\omega > K Zu_1$, if only the resonance condition $\omega \approx \Omega_c / \gamma$ is valid, and the latter may be valid only for the finite forces of the beam confining the external magnetic field. Therefore, when analysing this instability, we shall consider the external magnetic field to be finite but sufficiently strong for $\Omega_c^2 > \gamma^2 \omega_p^2$. We also restrict ourselves, for simplicity, to the case of a complete waveguide (drift
tube) filled with the electron beam. In these conditions, the following dispersion equations for small oscillations can be easily obtained [16]:

$$\frac{\mu_{qs}^{2}}{R^2} = \frac{\omega^2}{c^2} \left[ 1 + \sum_{n} \frac{\omega_b^2}{rc^2} \frac{u_{I}^2 J_n^2 \left( \frac{\mu_{qs}^{2} u_{I}^2 \gamma^2}{R^2 \Omega_e^2 (\omega-n \gamma)} \right)}{\left( \omega-n \gamma \right)^2} \right]$$

(2.40)

$$\frac{\mu_{qs}'^{2}}{R^2} = \frac{\omega^2}{c^2} \left[ 1 + \sum_{n} \frac{\omega_b^2}{\gamma \omega^2} \frac{\omega^2 \frac{u_{I}^2}{c^2} - \left( \omega-n \frac{\Omega_e}{\gamma} \right)^2}{\left( \omega-n \frac{\Omega_e}{\gamma} \right)} J_n^2 \left( \frac{\mu_{qs}'^{2} u_{I}^2 \gamma^2}{R^2 \Omega_e^2 (\omega-n \gamma)} \right) \right]$$

the first of which describes the $\mu$-wave and the second the E-wave. The values $\mu_{qs}$ and $\mu_{qs}'$ are the Bessel function and its derivative roots, respectively, i.e. $J_n(\mu_{qs}) = 0$ and $J_n'(\mu_{qs}) = 0$.

It follows from Eqs (2.40) that when the following resonance conditions are valid:

$$\omega = n \frac{\Omega_e}{\gamma} \left\{ \begin{array}{ll}
\mu_{qs} c / R & \text{for } \mu\text{-waves} \\
\mu_{qs}' c / R & \text{for } E\text{-waves}
\end{array} \right. \tag{2.41}$$

the oscillations become unstable, their growth rates being respectively equal to

$$\frac{\text{Im } \omega}{\omega} = \frac{\sqrt{3}}{2} \left( \frac{\omega_b^2 \gamma}{2 \Omega_e^2} \right)^{1/3} \left\{ \begin{array}{ll}
\left[ \frac{u_{I}}{c} J_n' \left( \frac{u_{I}^2}{c^2} \right) \right]^{2/3} & \text{for } \mu\text{-waves} \\
\left[ \frac{u_{I}}{c} J_n \left( \frac{u_{I}^2}{c^2} \right) \right]^{2/3} & \text{for } E\text{-waves}
\end{array} \right. \tag{2.42}$$

This stability, which is caused by the transverse velocity of beam electrons, is called cyclotron (gyrotron) instability. It is significant that it takes place in both neutralized and unneutralized beams, being in general independent of plasma density when $\Omega_e \gg \omega_b \gamma$.

Finally, it should be noted that both cyclotron and Cherenkov instability, in the conditions of magnetic focusing of the beam, when the self-magnetic field force of the current is much lower than the external longitudinal magnetic field force (i.e. $\Omega_e^2 \gg \omega_b^2 \gamma$) is weak, $\text{Im } \omega / \omega \ll 1$. This instability might therefore lead to only small distortions of the beam parameters, and in this sense it is not
dangerous either. The cyclotron instability, unlike the Cherenkov instability, may develop infinitesimal beam currents, and even when there is no plasma (if the beam is monoenergetic to a sufficient degree) it is necessary only that there should be a transverse velocity component $u_x$ of electrons for the resonance condition (2.41) to be valid.

It follows that Cherenkov and cyclotron beam instabilities for high-current electron beams are not dangerous — they may lead only to small perturbations of the beam. On the other hand, these particular small perturbations may be of great interest for transformation of kinetic beam energy directed into the energy of the electromagnetic radiation, i.e. for generating electromagnetic waves. Part III will therefore be devoted to the study of the problem of generating electromagnetic waves with the help of high-current electron beams.

The instabilities we have discussed do not, of course, exhaust all the possible instabilities of high-current beams; they cover only the quickly developing high-frequency type of instability and they may therefore develop in short-pulsed beams. There are many slow instabilities which are dangerous for long-pulsed beams: for example, centrifugal, interchange, hose and dissipative instabilities, the pinch instability, etc. We shall not consider those hydrodynamic instabilities which are not dangerous for short-pulsed electron beams.

Part III

PLASMA MICROWAVE ELECTRONICS

1. INTRODUCTION

This concluding part is devoted to one of the most important and interesting problems of applied high-current electronics: the utilization of the beam instability discussed earlier for amplifying and generating electromagnetic (e.m.) waves of microwave range. In other words, we shall now deal with plasma microwave electronics. We shall give the general electrodynamic theory of plasma amplifiers and generators of e.m. waves, using high-current electron beams, and describe their characteristics. We shall restrict ourselves to the Cherenkov and cyclotron generator with straightforward beams. For two concrete examples: the Cherenkov generator of space plasma $E$-wave and the cyclotron generator (gyrotron) of the $\mu$-wave, the generation frequency spectrum will be calculated, as well as the threshold currents of generator excitation and optimum conversion efficiencies of generation. Finally we shall briefly discuss achievements during recent years in the experimental realization of powerful pulsed radiation generators, using high-current electron beams.
2. FORMULATION OF THE PROBLEM: INITIAL EQUATIONS

We shall formulate the theory of plasma amplifiers and generators of e.m. radiation on the basis of the general formalism of electrodynamics in material media as opposed to in vacuo. Following this formalism, we shall consider any such amplifier or generator to be a spatially limited medium consisting of plasma and relativistic electron beams (REB). In such a non-equilibrium medium the excitation of e.m. waves is possible, i.e. small initial perturbations may become unstable and increase with time. The phenomena existing in the system are described in terms of normal waves in the linear approximation, i.e. waves not interacting in space, but mutually transforming on the system boundaries.

Thus we have a classical initial-value problem of the linear electrodynamics in a spatially limited medium, for the solution of which the equations of the e.m. field in the medium:

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad \text{div} \mathbf{D} = 0$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \quad \text{div} \mathbf{B} = 0$$

(3.1)

together with the material equation:

$$D_i(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \epsilon_{ij}(\mathbf{r}, \mathbf{r}', t, t') E_j(\mathbf{r}', t')$$

(3.2)

should be supplemented by boundary conditions. Such electrodynamic boundary conditions are usually obtained directly by integrating over the simultaneous equations (3.1) and (3.2) infinitesimal boundary layers near the medium surfaces, which is only possible when the single material equation (3.2) is valid for the whole system. Thus the problem arises of deducing the material equation (3.2) from a concrete underlying medium model. For this we shall take the model of a cold pure-electron plasma and monoenergetic electron beam, i.e. the equilibrium distribution of plasma electrons should be considered as follows:

$$f_{op} = n_p \delta(\mathbf{p})$$

(3.3)

and the distribution of the beam electrons has the form of (1.21). For such a model it is not difficult to obtain permittivity and to write down the material
This is done by solving the linearized kinetic Vlasov equation for plasma and beam electrons:

\[
\frac{\partial \delta f}{\partial t} + \vec{v} \frac{\partial \delta f}{\partial \vec{r}} - \frac{\Omega_c}{\gamma(\nu)} \frac{\partial \delta f}{\partial \phi} = e \left[ \vec{E} + \frac{1}{c} \left( \vec{v} \times \vec{B} \right) \right] \frac{\partial f_0}{\partial \vec{p}}
\]

where \( \Omega_c = e B_0 / mc \).

Before we start to analyse the solution of Eq.(3.4), let us briefly discuss the geometry of this system, which will be examined below. The actual device used for generating e.m. waves is a waveguide (with length \( L \) and radius \( R \)) limited in the longitudinal direction (region I in Fig.5), placed between the ab and cd boundaries, and completely filled with plasma. For the ab boundary a metal net or thin foil transparent to electron beams and nontransparent (reflecting) to radiation is used. Region III is a pure vacuum, from which the unperturbed thin annular electron beam (\( a < r_0 \)) is supplied, and region II imitates in a simplified way the transmitting antenna in the shape of a smooth waveguide filled with dielectric \( \epsilon_0 \). The boundaries ab and cd play an essential role in the generation of e.m. radiation; the reflection and transformation of amplified waves take place on them and these waves are suitable for a longitudinally infinite system. As a consequence of the transformation there is a feedback (the information being transmitted from one boundary to the other), which is necessary for any generator of e.m. radiation. However, not every feedback produces generation; for this the system must be non-equilibrium, and for our case the electron beam current value should be higher than some threshold value. Later on, such thresholds will be determined for particular generators of e.m. radiation.

Let us return to Eq.(3.4) and, without going into details of its solution (they may be found in Ref. [17]), we shall give the final form of the material
equation (3.2) for our medium model case of a monochromatic field, depending on the time and coordinates, as follows:

\[ \vec{E} = \vec{E}(r, Z) \exp(-i\omega t + i\varphi) \]

(3.5)

\[ D_{ij}(r, Z) = \hat{\epsilon}_{ij}(\omega, \hat{K}, r, Z) E_j(r, Z) \]

where the tensor operator of the dielectric constant \( \hat{\epsilon}_{ij}(\omega, \hat{K}, \vec{v}) \) is as follows:

\[
\hat{\epsilon}_{11} = \hat{\epsilon}_{22} = \epsilon_0 - \frac{\omega_p^2}{\omega^2 - \Omega_e^2} - \frac{\omega_p^2 \gamma^{-1}}{2\omega^2} \left\{ \frac{2(\omega - \hat{K}_Z \omega)}{(\omega - \hat{K}_Z \omega)^2 - \Omega_e^2} \right\}
\]

\[
+ u_1^2 \left( \frac{\hat{K}_Z^2 - \omega^2}{\gamma} \right) \left\{ \frac{(\omega - \hat{K}_Z \omega)^2 + \Omega_e^2}{\gamma^2} \right\}
\]

\[
\hat{\epsilon}_{12} = -\hat{\epsilon}_{21} = -i \frac{\omega_p^2 \Omega_e}{\omega(\omega^2 - \Omega_e^2)} - i \frac{\omega_p^2 \gamma^{-1}}{2\omega^2} \left\{ \frac{2(\omega - \hat{K}_Z \omega)}{(\omega - \hat{K}_Z \omega)^2 - \Omega_e^2} \right\}
\]

\[
+ 2u_1^2 \frac{\Omega_e}{\gamma} \left( \frac{\hat{K}_Z^2 - \omega^2}{\gamma^2} \right) \left\{ \frac{(\omega - \hat{K}_Z \omega)^2 + \Omega_e^2}{(\omega - \hat{K}_Z \omega)^2 - \Omega_e^2} \right\}
\]

\[
\hat{\epsilon}_{33} = \epsilon_0 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_p^2}{\gamma \gamma_1} \frac{1}{(\omega - \hat{K}_Z \omega)^2}
\]

\[
\hat{\epsilon}_{13} = \hat{\epsilon}_{31} = \hat{\epsilon}_{23} = \hat{\epsilon}_{32} = 0
\]

(3.6)

The operator \( \hat{\epsilon}_{ij} \) is written in cylindrical coordinates with the Oz axis along the waveguide axis and external magnetic field \( \vec{B}_0 \), and the operator

\[ \hat{K}_Z = \frac{1}{c} \frac{\partial}{\partial Z} \]

In region I, \( \epsilon_0 = 1 \), and in region II both plasma and beam are absent and therefore \( \hat{\epsilon}_{ij} = \epsilon_0 \delta_{ij} \). In calculating the tensor (3.6) it was assumed that \( u_1^2 \ll c^2 \), since only such beams may be sufficiently high-current, as shown in Part I.
Equations (3.1) and (3.5), together with (3.6), form a complete system of equations for our generation problem. They should be completed by the additional boundary conditions. The first is zero tangential components of electric field on the metal surfaces bounding the waveguide:

\[ E_Z|_{r=R} = E_\phi|_{Z=R} = 0 \]
\[ E_r|_{Z=0} = E_\phi|_{Z=0} = 0 \]  \hspace{1cm} (3.7)

and the finiteness of the fields on the waveguide axis:

\[ |E|_{r=0} < \infty, \quad |B|_{r=0} < \infty \]  \hspace{1cm} (3.8)

On the right-hand boundary of the waveguide cd (Fig.5) there should be continuous tangential components of the electric and magnetic fields:

\[ \{E_r\}_{Z=L} = \{E_\phi\}_{Z=L} = \{B_r\}_{Z=L} = \{B_\phi\}_{Z=L} = 0 \]  \hspace{1cm} (3.9)

These conditions are obtained by direct integration of the simultaneous field equation over the transitional layer near the plasma surface\(^1\). In the same way, we deduce the boundary conditions connecting the fields from the other sides of the thin annular electron beam:

\[ \{E_Z\}_{r=r_0} = \{E_\phi\}_{r=r_0} = 0, \quad \int_a j_{br} dr = 0 \]
\[ \{B_Z\}_{r=r_0} = -\frac{4\pi}{c} \int_a j_{\phi r} dr = 0 \]  \hspace{1cm} (3.10)
\[ \{B_\phi\}_{r=r_0} = \frac{4\pi}{c} \int_a j_{b\phi} dr = 0 \]

The high-frequency beam current density \( j_b \) is easily obtained with the help of the tensor operator of the dielectric constant (3.6), taking into consideration only the beam contribution, after which in the integration at (3.10) the change \( n_b \to n_b a \) has to take place.

It should be noted that the number of boundary conditions are not enough to solve the generation problem. The missing boundary conditions are supplied

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\(^1\) In reality, during the integration of the field equations near the boundary cd it turns out that the \( B_r \) and \( B_\phi \) components have a break because of radial and azimuthal beam currents. They are, however, essential and do not influence the results given below.
by the absence of beam perturbations on the ab surface and by the cd boundary being transparent to the beam, i.e.

$$\delta f_b |_{Z=0} = 0, \quad \delta f_b |_{Z=L} = 0$$ (3.11)

For the monoenergetic beam, these conditions are reduced to the corresponding conditions for the first moments of the function $\delta f_b$, i.e.

$$\rho_b |_{Z=0} = \bar{J}_b |_{Z=0} = 0$$

$$\{\rho_b\} |_{Z=L} = \{\bar{J}_b\} |_{Z=L} = 0$$ (3.12)

where $\rho_b$ is the density of the high-frequency beam charge.

In conclusion, the obvious condition for the generation problem is the absence of perturbations in regions II and III, falling on region I. This requirement is in fact equivalent to the condition of radiation. The problem is now completely formulated and we can get down to its solution.

3. GENERAL LINEAR THEORY OF PLASMA GENERATORS; EXCITED WAVE FREQUENCY AND THRESHOLD CURRENTS

As a beginning, we note that this is a linear electrodynamic eigenvalue problem. Its solution consists in obtaining the e.m. oscillation frequency spectrum of the system $\omega$. If, among the eigenvalues, such values are found which have $\text{Im} \omega > 0$, the system proves to be unstable, and excitation of small oscillations takes place, which, in the form of e.m. waves, is radiated into region II. The equality $\text{Im} \omega = 0$ determines the threshold conditions of the system excitation.

A method of solving such a problem has been worked out and substantiated in Ref.[18], and we shall briefly describe the basic features of the method.

We shall discuss the non-equilibrium medium, between parallel planes $Z = 0$ and $Z = L$ ($0Z$ axis is normal to these planes). We shall assume that the small perturbation equation permits the separation of variables into the $r$ and $\varphi$ coordinates. The solution of these equations may therefore be represented in the form of the superposition of the medium's normal waves, with the structure:

$$A_i \exp (-i \omega t + i K_{Zi}(\omega)Z), \quad i = 1, 2 \ldots N$$ (3.13)

Here $A_i$ are constants, $\omega$ is the unknown oscillation frequency of the finite system, and $K_{Zi}(\omega)$ is the solution of the normal medium dispersion wave equation $D(\omega, K_Z) = 0$, the number of which is equal to $N$. 
We divide the solutions (3.13) into two groups: one with waves propagating along the positive direction of the OZ axis,

\[ A_i^+ \exp (-i\omega t + iK_{Zi}^+(\omega)Z), \quad i = 1, 2 \ldots n^+ \]  

(3.14)

and the other with waves propagating in the opposite direction:

\[ A_i^- \exp (-i\omega t + iK_{Zi}^-(\omega)Z), \quad i = 1, 2 \ldots n^- \]  

(3.15)

\[ n^+ + n^- = N \]

In the most general form, the boundary conditions for the planes \( Z = 0 \) and \( Z = L \) are as follows:

\[ A_p^+ = \sum_{j=1}^{n^-} \alpha_{pj}^+ A_j^-, \quad p = 1, 2 \ldots n^+ \]  

(3.16)

\[ A_p^- \exp (iK_{Zp}^- (\omega) L) = \sum_{s=1}^{n^+} \alpha_{ps}^+ A_s^+ \exp (iK_{Zs}^+ (\omega) L), \quad p = 1, 2 \ldots n^- \]

Here \( \alpha_{pj}^+ (\omega) \) and \( \alpha_{ps}^+ (\omega) \) are the coefficients of the wave transformation on the boundaries \( Z = 0 \) and \( Z = L \), respectively. Eliminating the constants \( A_i^+ \) and \( A_i^- \) from the system (3.16), we arrive at the following characteristic equation for determining generation frequencies \( \omega \):

\[ \det \left\{ \sum_{p=1}^{n^+} \alpha_{ip}^- \alpha_{pj}^+ \exp \left[ i \left( K_{Zj}^- - K_{ZP}^- \right) L \right] - \delta_{ij} \right\} = 0 \]  

(3.17)

In the general case it is impossible to analyse Eq.(3.17), so we shall just discuss a simple example. Let us consider, in some region \( g \) of the complex region \( \omega \), the following inequality to be valid:

\[ \text{Im} \ K_{Zj}^- - \text{Im} \ K_{Zp}^- < \text{Im} \ K_{Zj}^+ - \text{Im} \ K_{Zp}^- \quad , \quad j \neq j_0, \quad p \neq p_0, \quad \omega \in g \]  

(3.18)

Here \( j_0 \) and \( p_0 \) are fixed wave numbers: generally speaking, the \( g \)-region functions. If conditions (3.18) are valid, then, choosing the system
length $L$ to be sufficiently long, it may be possible to essentially simplify Eq.(3.17):

$$\alpha_{j_0}^+ \alpha_{p_0j_0}^+ \exp \left( i \left( K_{Zj_0}^+ - K_{Zp_0}^- \right) L \right) = 1$$

(3.19)

or

$$K_{Zj_0}^+ - K_{Zp_0}^- = \frac{2\pi M}{L} - \frac{i}{L} \ln \left| \frac{1}{\alpha_{j_0}^- \alpha_{p_0j_0}^+} \right|$$

(3.20)

$$M = \begin{cases} n & \text{if } \alpha_{j_0p_0}^+ \alpha_{p_0j_0}^+ > 0, \quad n = 1, 2, \ldots \\ n + \frac{1}{2} & \text{if } \alpha_{j_0p_0}^+ \alpha_{p_0j_0}^+ < 0, \quad n = 0, 1, \ldots \end{cases}$$

Let us assume for simplicity that the transformation coefficients are slightly dependent on the frequency $\omega$. Then, representing $\omega$ in the form $\omega + i\delta$ and introducing the real and imaginary function parts $K_{Zj_0}^+$ and $K_{Zp_0}^-$, we obtain the following system from (3.20):

$$\text{Re} \left[ K_{Zj_0}^+ (\omega) \right] - \text{Re} \left[ K_{Zj_0}^- (\omega) \right] = \frac{2\pi M}{L}$$

(3.21)

$$\text{Im} \left[ K_{Zj_0}^+ (\omega) \right] - \text{Im} \left[ K_{Zp_0}^- (\omega) \right] = -\frac{1}{L} \ln \frac{1}{|\alpha_{j_0}^- \alpha_{p_0j_0}^+|}$$

For each fixed $M$, the system (3.21) determines the single point on the complex plane $\omega = \omega + i\delta$, otherwise such points cannot exist. If the identified points are located in the upper half-plane of the complex region $\omega$, wave generation takes place within the system.

The following discussion is an essential extension of this general method on the concrete physical systems — plasma-beam resonators. For the system given in Fig.5 it is sufficient to know the field equation solution only in regions I and II, since there is complete e.m. wave reflection on the boundary and there are no perturbations in region III. If we try to find the field-equation solutions in the form $f_n \exp(ik_{Zn} Z)$ and take an interest in the high-frequency waves $\omega^2 \gg \omega_b^2$, it will be possible to show that the characteristic equations in regions I and II are written down in the unified form:

$$D_0(\omega, K_Z) = \frac{A\omega_b^2}{(\omega - K_Z u_{\|} - S \frac{\Omega e^2}{\gamma})^2}$$

(3.22)
where $A$ depends on the system's concrete geometry of the given region; $D_0(\omega, K_Z) = 0$ is the wave dispersion equation where there is no beam in the region, which is considered as longitudinally infinite; and resonant denominators in the right-hand part, $\omega - K_Z u_1 - S(\Omega_e/\gamma)$, characterize the electron beam's interaction mechanism with the wave. When $S = 0$, a Cherenkov beam interaction with the longitudinal field component $E_z$ (i.e. $u_1 E_z \neq 0$) takes place, and, when $S \neq 0$, beam electrons are in cyclotron resonance with the transverse field component $^2 E_\phi$ or with $E_r$ (i.e. $u_1 E_1 \neq 0$).

Region I is the most interesting, because in this region there develops resonance beam electron interaction with the wave and wave amplification. It is necessary to obtain all the solutions of (3.22), $K_{Zn}$, $n = 1 \ldots N$ in this region. The equation $D_0(\omega, K_Z) = 0$ is usually even over $K_Z(\omega)$ and therefore defines the conjugate branches of the system's eigenoscillations in the absence of the beam $\pm K_0(\omega)$. However, the relations $\omega - K_Z u_1 - S(\Omega_e/\gamma) = \pm B\omega_b$ (where $B$ depends on the system geometry) determine the spectra of the fast and slow beam waves. The resonance beam's interaction with the wave takes place in the crossing of these oscillation branches (see Fig. 6), i.e. when the relations

$$D_0(\omega, K_0) = 0, \quad \omega = K_0(\omega) u_1 + S \frac{\Omega_e}{\gamma}$$

In our case of high-current beams, when $u_1^2 \ll c^2$, as seen from the expressions (3.6), resonance cyclotron beam interaction is possible only with the first harmonic ($S = 1$).
are simultaneously valid. The relations (3.23), if $K_0$ is excluded, define the oscillation excited by the beam frequency.

Near the resonance frequencies (3.23) we find four characteristic solutions of Eq. (3.23):

$$K_{Z_1} = K_0(\omega) \left[ 1 + \psi_i \varphi^{1/3} \right], \ i = 1, 2, 3$$

$$K_{Z_\varphi} = -K_0(\omega)$$

Here

$$\psi_i = \left[ -1, \frac{1 \pm i \sqrt{3}}{2} \right] \text{sgn} \frac{A}{\partial D_0 / \partial K_0}$$

$$\varphi = \left| \frac{A \omega_b^2}{\partial D_0 / \partial K_0} \right|$$

The above condition that the beam density be small results in the inequality:

$$\varphi^{1/3} \ll 1$$

(3.26)

It is easy to find the solution of the characteristic equation in region II also, where there is no radiation of the e.m. waves amplified in region I. In this region it is natural to require the beam not to be in resonance with the e.m. wave, i.e. the conditions of the (3.23) form should not be valid. We therefore have two eigenwaves (beamless system) propagating in opposite directions, and two beam waves moving in the direction of the beam. Taking into account that region II in Fig. 5 is a smooth metal waveguide filled with dielectric, and that in this region we are interested only in non-resonance radiated waves moving along the beam, we write down the necessary solutions of the characteristic equation in a direct form:

$$K_{Z_1,2} = \frac{1}{u_1} \left( \omega - S \frac{\Omega_e}{\gamma} \right) \pm \frac{\sqrt{B} \omega_b}{u_1}$$

$$K_{Z_3} = \sqrt{\varepsilon_0 \frac{\omega^2}{c^2} - \frac{\mu_{Q_s}^2}{R^2}}$$

(3.27)

The values $\mu_{Q_s}$ characterize the field radial structure in region II for E-waves, $J'_z(\mu_{Q_s}) = 0$, and for $\mu$-waves, $J'_z(\mu_{Q_s}) = 0$. 
Substituting, further, those solutions into the boundary conditions (3.7), (3.9), (3.11) and (3.14), we obtain a system of algebraic equations for $f_n$ coefficients, the condition for whose solution is the dispersion equation [19, 20]:

$$
\sum_{i=1}^{3} \alpha_i \exp \left[ iK_{Z_1}^I (\omega) L \right] + \frac{L}{\kappa} \exp \left[ K_{Z_4}^I (\omega) L \right] = 0
$$

(3.28)

where $\alpha_i = 1/3$ for $i = 1, 2, 3$. The physical meaning of this equation is very simple. The wave $K_{Z_4}$ in region I transforms into the waves $K_{Z_1}$ on the boundary $ab$ with a transformation coefficient $\alpha_i = 1/3$ for each of them. In their turn, the waves $K_{Z_1}$ on the boundary $cd$ together transform into the wave $K_{Z_4}$ with the transformation coefficient $\kappa$. Thus the quantity $|\kappa|$ has the sense of an e.m. wave reflection coefficient on the boundary $cd$.

**Note:** $\kappa$ includes the medium characteristics in the region II. In particular, when region I is the smooth metal waveguide, completely filled with plasma [20],

$$
\kappa = \frac{K_{Z_3}^{II} - \epsilon_0 K_{Z_4}^I}{K_{Z_3}^{II} + \epsilon_0 K_{Z_4}^I}
$$

It is obvious that the quantity $\kappa$ also essentially depends on the geometry of the real transmitting antenna. The idealization of this region assumed in Fig.5 is far from perfect. The calculation of real transmitting systems meets with great mathematical difficulties. Therefore, $|\kappa|$ is usually obtained experimentally from the "cold measurements", i.e. without beam. We don’t make bad mistakes here; thus, when condition (3.29) is valid, the e.m. energy, carried away by the beam waves, is infinitesimal. If $\kappa$ is considered to be unknown, the whole generation problem, according to the dispersion equation (3.28), is reduced to analysis only in region I, where the resonance beam interaction with the e.m. wave takes place. Now we shall continue, considering the quantity $|\kappa|$ to be demonstrated.

The wave of the transmission coefficient $(1 - |\kappa|)$ goes through the boundary with practically all the energy transforming into the radiated wave $K_{Z_3}^{II}$; a small part of the energy, of order $(\omega_b/\omega)^{2/3}$, transforms into beam waves $K_{Z_1,2}^{II}$, and that energy supplies the beam modulation.

From the three waves running along the beam in region I, according to (3.25) only one is amplified — we shall designate it $K_Z$. It is obvious that we may restrict ourselves to consideration of the amplified wave on the boundary $cd$ when the following condition is valid:

$$
\text{Im} \ K_Z L \sim K_0 \varphi^{1/3} L > 1
$$

(3.29)
on which boundary this wave transforms into the wave $K_{Z4}$, moving in the
direction opposite to the beam and supplying the feedback; let us designate it $K_Z$.
The remaining part of the amplified wave transforms into the $K_{Z3}$
wave and is radiated from the system. For effective wave generation, only such a case is of
interest, Eq.(3.20) being reduced to the form:

$$
\exp \left[ \left( K_Z^* - K_Z \right) \right] = \frac{1}{k}
$$

(3.30)

from which we find the frequency and growth rate of the e.m. wave generated
in the system ($\omega \to \omega + i\delta$):

$$
\omega = \frac{\pi M u_i}{L} \left[ 1 - \frac{\varphi^{1/3}}{4} \sgn \left( \frac{A}{\partial D_0/\partial K_0} \right) \right] + S \frac{\Omega_e}{\gamma}
$$

(3.31)

$$
\delta = \frac{\sqrt{3}}{2} \pi M \left| \varphi^{1/3} - \ln \frac{1}{|k|} \right| L + \frac{L}{W} \frac{1}{v_g_0}
$$

where $M \gg 1$ is an integral number, $v_{g_0} = (-\partial D_0/\partial K_Z)/(\partial D_0/\partial \omega)$ is the wave-group
velocity in a beamless system, and $W$ is the group velocity of the drift during
amplification:

$$
W = \frac{3v_{g_0} u_i}{u_i + 2v_{g_0}}
$$

(3.32)

which is the same for all three resonance waves $K_{Zi}$($i = 1,2,3$); the group velocity
of the fourth non-resonance wave, $K_{Z4}$($\omega$), obviously does not coincide with $v_{g_0}$.

From the condition $\delta \gg 0$ we obtain the threshold current of generator
excitation. Here we should keep in mind that Eqs (3.30) -- (3.32) are valid only
in the condition of (3.29). Therefore this condition also determines the use of
the expression obtained from (3.31) for the threshold current. It is easy to see
that only the inequality (3.29) is compatible with the excitation possibility of
low-quality (and consequently powerful) generators of e.m. radiation, for which
$|k| \ll 1$. At the same time, we should note that when the condition (3.29) is valid,
the width of the generation range

$$
\delta \sim \text{Im} \ K_Z^* u_i \approx \frac{\sqrt{3}}{2} \frac{\pi M}{L} \ u_i \varphi^{1/3}
$$

exceeds the frequency difference between the neighbouring longitudinal modes
$\Delta \omega \approx \pi u_i/L$. It makes possible the excitation of different longitudinal e.m.
oscillation modes in the system. In this sense, in high-current electronics if we don't take any special measures (e.g. preliminary beam modulation), we always deal with several mode generators.

The theory given here of low-quality (and consequently high-current) e.m. wave generators is, precisely speaking, applied only to positive dispersion systems, in which phase and group velocities of waves excited by the beam are parallel. For negative-dispersion systems, i.e. for backward-wave generators, this theory is valid for Eqs (3.21). However, analysis of the roots of the characteristic equation (3.22) proves in this case to be more complicated since it is sufficient to have only two waves, as in Eqs (3.30) and (3.31).

Later on, in concrete examples, we shall only discuss systems with positive dispersion. The backward-wave theory, which is in fact more complicated, may be studied in Ref.[21].

4. EXAMPLES OF HIGH-CURRENT PLASMA GENERATORS

So far, we have intentionally not given the direct expressions $D_0, A, B, \varphi, \kappa, v_{g0}$ and $W$ for concrete systems, in order to avoid overloading the statement of the general electrodynamic theory. Let us consider now several examples widely used in high-current plasma electronics.

(a) We begin with the simplest case of a smooth metal waveguide completely filled with highly magnetized plasma: this is a resonator, i.e. region I of our system [22]. For such a system, in conditions of a very strong magnetic field, when

$$\Omega_c \gg \omega_p, \omega_b \sqrt{\gamma}, \frac{c\gamma}{R}$$

(the infinite-field limit in components (3.6)), Cherenkov beam interaction is only possible with the low-frequency (plasma) branch of e.m. E-waves. The characteristic equation (3.22) for the region I can be written as follows:

$$\left(K_Z^2 - \frac{\omega^2}{c^2}\right)\left(1 - \frac{\omega_p^2}{\omega^2}\right) + \frac{\mu_{\phi s}^2}{R^2} = \left(K_Z^2 - \frac{\omega^2}{c^2}\right) \frac{\omega_b^2 \pi r_0 \alpha \mu_{\phi s} J_\perp^2}{\gamma^2 \gamma R^2 (\omega - K_Z u_\perp)^2} \left(\frac{\mu_{\phi s} I_0}{R}\right)$$

(3.34)
where $\mu_{\text{gs}}$ are the Bessel function roots and $J_\ell(\mu_{\text{gs}}) = 0$. Further, following this general theory, we obtain the frequency of the plasma e.m. waves excited by the beam and the threshold current for their excitation:

$$\omega = \frac{\pi M u_i}{L} = \sqrt{\frac{\omega_p^2}{\mu_{\text{gs}}^2} \frac{u_i^2 \gamma_i^2}{R^2}}$$

(3.35)

$$J_{\text{th}} = \frac{8\omega_p^2 v_{g0} \gamma_i^4}{\omega^2 R \mu_{\text{gs}} J_\ell^2 \left( \frac{r_0}{R} \right)} \left( \frac{R}{L} \frac{u_i}{c} \sqrt{n \frac{1}{|\kappa|}} \right)^3 (\text{kA})$$

The group velocity of the plasma e.m. E-wave in the absence of the beam $v_{g0}$ is in this case given by:

$$v_{g0} = u_i \left( 1 + \frac{\omega^2 R^2}{\mu_{\text{gs}}^2 u_i^2 \gamma_i^4} \right)^{-1}$$

(3.36)

The radial $E_Z$-component structure of the excited wave, with which the Cherenkov interaction of the electron beam in general develops, has the form $E_Z \approx J_\ell(\mu_{\text{gs}}(r/R))$. Therefore, to obtain radial single-mode radiation generation, the annular electron beam should be transmitted when the field is maximum for the $\mu_{\text{gs}}$ mode. But this is insufficient; according to (3.35) the following inequality must be valid:

$$\omega_p^2 > \frac{\mu_{\text{gs}}^2 u_i^2 \gamma_i^2}{R^2}$$

(3.37)

There is another method for radial mode selection (even absolute) for the plasma generator considered. If the inequality

$$(3.8)^2 > \left( \frac{\omega_p R}{u_i \gamma_i} \right)^2 > (2.4)^2$$

(3.38)

is valid, then only the basic axially symmetric oscillation mode will be excited in the generator. The case is different with the longitudinal field structure excited by the beam wave. In the condition (3.29), as already noted, the generation range width exceeds the frequency difference between neighbouring longitudinal modes, and the threshold currents of different longitudinal modes differ slightly from one another in the generation range width. Therefore, without special devices it is virtually impossible to obtain longitudinal single-mode excitation of the e.m. waves. Naturally, in high-current electronics those generators are of interest whose
threshold excitation currents are comparable to or even higher than the corresponding vacuum maximum beam currents, i.e. in our case they are higher than (1.12).

For the plasma generator considered here, this condition is reduced to the form:

$$\ln \left| \frac{1}{|\kappa|} \right| > \frac{\pi M}{\gamma_1} \sqrt{1 - \gamma_1^{-2/3}} \left( \frac{a}{r_0} + 2 \ln \frac{R}{r_0} \right)^{-1/3} > 1 \quad (3.39)$$

and in fact it is equivalent to the inequality (3.29).

We have so far completely neglected the longitudinal velocity variation of beam electrons (due, for example, to angular variation), which does not contradict the condition (3.29), or (which is the same)

$$\frac{u_L}{\omega} \frac{1}{L} \ln \frac{1}{|\kappa|} > \frac{\langle v_T \rangle}{u_L} \quad (3.40)$$

Here \( \langle v_T \rangle \) is the longitudinal velocity variation of the electrons. The inequalities (3.40) limit the beam current from both above and below. It is obvious that the threshold current (3.35) has sense only when it is within this interval.

In conclusion, we give an estimate of the plasma generator parameters in the cm wavelength range. It is easy to show that, with plasma density \( n_p \approx 3 \times 10^{12} \text{ cm}^{-3} \) (\( \omega_p \approx 10^{11} \text{ s}^{-1} \)) and \( R \approx 2.5 \text{ cm} \), an electron beam with energy 1 MeV (i.e. \( \gamma_1 = \gamma \approx 3 \) and \( u_1 = u = 2.8 \times 10^{10} \text{ cm s}^{-1} \)) will excite only the basic radial E-wave mode with frequency \( \omega \approx 6 \times 10^{10} \text{ s}^{-1} \). In the low-quality resonator with \( |\kappa| \approx 10^{-1} \) and length \( L \approx 25 \text{ cm} \), the threshold current of such generator excitation with \( r_0 \approx 2 \text{ cm} \), according to (3.35) and (3.36), proves to be of the order of \( J_{th} \approx 5 \text{ kA} \), which does not exceed the maximum vacuum current (1.12). With increase of \( \gamma \), the threshold current (3.41) increases more than the maximum current (1.12).

(b) Let us make this system somewhat more complicated. We consider the longitudinal magnetic field to be finite, but assume that the following inequalities are valid [22]:

$$\Omega_e > \frac{c}{R} \gamma > \omega_p, \quad \omega_b \sqrt{\gamma} \quad (3.41)$$

Let us assume that plasma again completely fills the waveguide. In these circumstances the Cherenkov generation of plasma E-waves proves to be impossible because the inequality (3.37) is not valid for any radial mode \( \mu \). However, the high-frequency E-wave and \( \mu \)-wave excitation as a result of cyclotron electron interaction with the e.m. wave is probable. The threshold current for \( \mu \)-wave excitation turns out to be lower, and therefore we consider here only the
\( \mu \)-wave for which the characteristic equation (3.22), taking into account the conditions (3.41), may be written as follows:

\[
\frac{c^2 K Z^2 + \frac{\mu_{qs}^2 c^2}{R^2}}{1 - \omega^2} = -\frac{1}{4} \frac{u_{1}^2}{c^2} \frac{\omega_{b}^2 \gamma^{-1}(K_{Z}^2 c^2 - \omega^2)}{\left(\omega - K_{Z} u_{1} \frac{\Omega_{e}}{\gamma}\right)}
\]

\[
\times \frac{\pi \mu_{qs} \frac{r_{0} a}{R^2}}{J_{\gamma}^2} \left(\frac{r_{0}}{R}\right)
\]

(3.42)

Here \( \mu_{qs} \) are already the Bessel function derivative roots; \( J_{\gamma}(\mu_{qs}) = 0 \), the electrons interacting resonantly with the electric field component \( E_{\varphi} \approx J_{\gamma}(\mu_{qs} r/R) \).

Therefore, to obtain radial single-mode \( \mu \)-wave generation we must localize the electron beam in the field component \( - E_{\varphi} \) maximum for the radial mode \( \mu_{qs} \).

The excited wave frequency and the threshold current for generator excitation during this are determined by the relations:

\[
\omega = \frac{\Omega_{e}}{\gamma} + \frac{\pi M u_{l}}{L} = \sqrt{\frac{\mu_{qs}^2 c^2}{R^2} + \frac{\pi^2 M^2 c^2}{L^2}}
\]

(3.43)

\[
J_{th} = \frac{32 \gamma v_{g0} c}{u_{l}^2 \mu_{qs} J_{\gamma}^2} \left(\frac{R}{L} \frac{u_{l}}{c} \frac{r_{0}}{c} \frac{1}{|\kappa|}\right)^3 (kA)
\]

where \( M \) is the integral number exceeding unity and \( v_{g0} \) is the group \( \mu \)-wave velocity in the absence of electron beam:

\[
v_{g0} = c \frac{\pi M c}{L \omega}
\]

(3.44)

Usually, during cyclotron excitation of the \( \mu \)-wave, it is desirable to make the frequency \( \omega \) as equal as possible to the cyclotron rotation frequency of the electron \( \Omega_{e}/\gamma \), which may be obtained by making the following conditions valid:

\[
\omega \approx \frac{\Omega_{e}}{\gamma} \approx \frac{\mu_{qs} c}{R} \approx \frac{\pi M c}{L}
\]

(3.45)

then \( v_{g0} \ll c \). For the derivation (3.43) these inequalities have been taken into account. Let us note that cyclotron \( \mu \)-wave generators, when conditions (3.45) are valid, are called gyrotrons or masers at cyclotron resonance[23]. There is absolutely no density of the plasma-filling waveguide in Eqs (3.43) and (3.44). This is a result of inequalities (3.41); when they are valid, the cyclotron resonance
between the electron beam and the e.m. wave becomes most effective. In this sense the above theory of the plasma cyclotron generator by no means differs from the theory of the vacuum generator [23]; it is given only in the language of the electrodynamics of material media. The role of plasma, however, even in such conditions, may be essential — as a result of the effect of beam space-charge neutralization, plasma might essentially increase the currents used in cyclotron generators, particularly in the case of small fractions $u_2^2c^2$. It is seen from Eqs (3.43) that, with this fraction decrease, the threshold current of the generator increases. If in the vacuum generators the threshold current should be only a small part of the maximum current (1.12), then in a generator filled with plasma the threshold current (3.43) might exceed this limit in the conditions:

$$\ln \left( \frac{1}{|\kappa|} \right) > \left( \frac{L^2 u_\perp}{R^2 c} \right) \frac{\sqrt{1 - \gamma_0^2}}{\gamma} \left( \frac{a}{r_0} + 2 \ln \frac{R}{r_0} \right)^{-1/3}$$

(3.46)

However, we should keep in mind that plasma may shield the radiation excited by the beam. To avoid such shielding it is necessary to limit plasma density in the generator by the quantity [17]:

$$\omega_p^2 < \frac{\Omega_e^2}{\gamma} (1 + \gamma)$$

Using relations (3.43) — (3.45), we give an estimate of the parameters of the plasma cyclotron generator in the cm wavelength range. Since such generators are well adapted for high radial mode excitation, let us assume $\mu_0s = \mu_{13} \approx 8.5$, i.e. we consider the $\mu_{13}$-wave excitation in the resonator with $|\kappa| = 0.1$, radius $R \approx 4 \text{ cm}$ and length $\approx 12 \text{ cm}$, electron beam with energy $1 \text{ MeV}$ and $u_\perp/u_\parallel \approx 1/3$. The average beam radius coincides with the $E_\varphi$ field maximum for this mode with $r \approx 1.6 \text{ cm}$, and excited oscillation frequency is $\omega \approx 7 \times 10^{10} \text{s}^{-1}$ order (i.e. $\lambda = 2.6 \text{ cm}$). With these generator parameters, the resonance magnetic field force is equal to $7 \text{ kG}$, and the threshold current $I_{\text{th}} = 10 \text{ kA}$, which is almost twice as high as the maximum vacuum current (1.12).

In conclusion, we note that neglecting the beam electron longitudinal velocity variation causes the following limits on the application of the above results:

$$\min \left( \frac{u_\perp^2}{2c^2} - \frac{\omega}{\Omega_e^2} \right) \left( \frac{\omega}{\gamma} - \frac{1}{\Omega_e^2} \right) \left( \frac{\ln \left| \kappa \right|}{\gamma} \right) > \frac{u_\parallel}{L} \frac{\Omega_e}{\gamma} \frac{\ln \left| \kappa \right|}{\gamma} > \frac{\langle v_\parallel \rangle}{u_\parallel}$$

(3.47)

Here $\langle v_\parallel \rangle$ is the beam electron longitudinal velocity variation. Like (3.40), the inequalities (3.47) limit the beam current from both above and below but, unlike
(3.40), here the current upper limit turns out to be more important, and from (3.47) it follows that the value \( u_{jc}^2 c^2 \) cannot be very small.

5. NON-LINEAR THEORY OF PLASMA GENERATORS: GENERATOR CONVERSION EFFICIENCY

The linear theory of plasma e.m. wave amplifiers and generators has been developed in the previous sections; their performance is based on the beam instability phenomenon. In the linear approximation it was concluded that generation (field increases infinitely with time) in the electrodynamic system takes place only when the electron beam current exceeds the threshold current. How long the field will increase in the system, what values of generator conversion efficiency (GCE) and output radiation power can be reached and by what means could they be maximized — these are the problems, which can be solved only with the help of non-linear theory.

Such non-linear theory is developed below for the Cherenkov mechanism of beam instability, i.e. for the interaction between a straightforward \((u_j = 0)\) mono-energetic annular electron beam and a low-frequency e.m. plasma E-wave in a smooth metal waveguide completely filled with plasma and placed in a strong longitudinal magnetic field [24]. To solve this problem the Maxwell equations for E-waves (it is only with them that the straightforward electron beam can interact) and the hydrodynamic equations for plasma and beam electrons are taken as the initial model (see, e.g., Ref. [17]).

With small electron beam density, plasma may be considered in the linear approximation. In addition, since the beam is very thin it is naturally assumed that the radial field structure in the waveguide is not perturbed by the electron beam. Therefore solution of the field equations for example, for the \( E_z \)-component should be in the following form:

\[
E_Z = E(Z,t) J_0 \left( \frac{r}{R} \right) \cos [\omega t - K_Z Z + \alpha(Z,t)]
\]  

(3.48)

Here \( \omega \) and \( K_Z \) are bound by the dispersion relation \( D_0(\omega, K_Z) = 0 \), determining the spectra in the beamless system.

Since it is assumed that the beam slightly perturbs the system, the following equations are valid for the amplitude \( E(Z,t) \) and the phase \( \alpha(Z,t) \):

\[
\left| \frac{1}{\omega} \frac{\partial E}{\partial t} \right|, \left| \frac{1}{K_Z} \frac{\partial E}{\partial Z} \right| \ll |E| \quad \left| \frac{1}{\omega} \frac{\partial \alpha}{\partial t} \right|, \left| \frac{1}{K_Z} \frac{\partial \alpha}{\partial Z} \right| \ll 1
\]

(3.49)
Putting (3.48) into the initial equations (taking (3.49) into account) and making them average with respect to the wavelength of the oscillations \( \lambda = 2\pi/K_Z \), we obtain the equations for the slowly changing functions \( E(Z,t) \) and \( \alpha(Z,t) \):

\[
E \left( \frac{\partial}{\partial t} + v_{go} \frac{\partial}{\partial Z} \right) \alpha = 4\pi BJ(\sin) \quad \left( \frac{\partial}{\partial t} + v_{go} \frac{\partial}{\partial Z} \right) E = 4\pi J(\cos)
\]

\[
B = \frac{ar_0}{R^2} \left( K_Z^2 - \omega^2 \right) \left( K_Z^2 - \frac{\omega_p^2}{c^2} - \frac{\omega^2}{c^2} \right)^{-1} J_0 \left( \mu_0s \frac{r_0}{R} \right) J_1^2 (\mu_0s)
\]

\[
J(\cos) = -\frac{en_b}{N_0} \sum_{p=1}^{N} \left\{ \sin [\omega t - K_Z Z_p + \alpha(Z_p,t)] \right\}
\]

\[
\cos [\omega t - K_Z Z_p + \alpha(Z_p,t)]
\]

\[
\frac{dZ_p}{dt} = u_p
\]

\[
\frac{du_p}{dt} = \frac{e}{m} J_0 \left( \mu_0s \frac{r_0}{R} \right) \left[ 1 - \frac{u_p^2}{c^2} \right] E(Z_p,t) \cos [\omega t - K_Z Z_p + \alpha(Z_p,t)]
\]

(3.50)

The system (3.50) is written in a form already suitable for numerical integration by the particle method. Here \( N_0 \) is the number of particles in the unperturbed state and \( N \) is the same but in the perturbed state per one wavelength; \( Z_p \) and \( u_p \) are the coordinate and velocity of the large particle \( p \); \( v_{go} \) is the same as in (3.36); and \( n_b \) is the real beam-electron density in the laboratory coordinate system.

Equations (3.50) should be completed with the boundary and initial conditions.

We shall consider two types of problem. The first is the linear initial-boundary problem, in which case the equations are completed by the conditions:

\[
\begin{align*}
E(0,t) &= \psi(t) & t &> 0 & \quad E(Z,0) &= \varphi(Z), & Z &> 0 \\
\alpha(0,t) &= \alpha(Z,0) = 0 & t, Z &> 0 & \varphi(0) &= \psi(0) \\
\end{align*}
\]

(3.51)

\[
u_p \big|_{Z_p=0} = u_{||} = \frac{\omega}{K_Z}
\]

The second problem deals with the non-linear stage of the Cherenkov plasma generator. Corresponding boundary conditions will be obtained later.
We now pass on to the analysis of the initial-boundary problem. It can be shown that in linear approximation the system (3.50) has two characteristics:

\[ t = \frac{Z}{W}, \quad t = \frac{Z}{u_g} \]  

(3.52)

dividing the plane \((t, Z)\) into three regions (Fig.7). Here \(u_g = \frac{2}{3} u_\parallel + \frac{1}{3} v_{g0}\) is the drift velocity for the case of instability, and \(W\) is the drift velocity for the case of amplification. In the linear approximation, the solution (3.50) corresponds in region III to the solution of the initial problem for a longitudinally infinite system [25], and in region I to the solution of the boundary problem [26]. In region II one solution overlaps with the other.

It is clear from the above that the non-linear solution of the initial-boundary problem is of the greatest interest. We solve this problem in the simplest case, when \(\psi(t) = E_0, \varphi(Z) = E_0\).

Let us now give the results of the numerical solution of this problem for the following concrete parameters: \(\omega_p = 12 \times 10^{10} \text{s}^{-1}; \ R = 4.1 \text{ cm}; \ r_0 = 2 \text{ cm}; \ a = 0.1 \text{ cm}; \ u_\parallel = 2.8 \times 10^{10} \text{ cm} \cdot \text{s}^{-1} (\gamma \approx 3).\) For the numerical solution it was assumed that the basic symmetrical radial mode of E-waves \(\mu_{01} = 2.4\) with frequency \(\omega = 11 \times 10^{10} \text{s}^{-1}\) falls on the boundary \(Z = 0.\) In this case \(K_Z = \frac{\omega}{u_\parallel} = 3.93 \text{ cm}^{-1},\) and \(v_{g0} = 1.69 \times 10^{10} \text{ cm} \cdot \text{s}^{-1}.\) The field \(E(Z,t)\) was measured in the units \(\sqrt{4\pi n_b mc^2(\gamma - 1)}.\)

Results of calculations for two electron-beam density values: \(\omega_b = 3 \times 10^{10} \text{s}^{-1}\) and \(\omega_b = 5 \times 10^{10} \text{s}^{-1},\) are given in Fig.8(a)–(h). It turns out that in the non-linear stage of beam instability development, as in the linear stage, the solutions of our problem are divided into three types over regions I, II and III, respectively (see Fig.7). In particular, at large distances from the boundary, \(Z > u_{gt},\) the solutions coincide with those for the initial problem [23]. They are given in Fig.8 (a) and (b), where the time \(\tau\) is measured in the units \(0.57 \times 10^{-10}\) s. The linear stage for
FIG. 8. Calculations for two electron beam density values (see text for details).
small $\tau$ is characterized by the exponential increase of the wave amplitude with linear increment of beam instability. At the non-linear stage for large $\tau$, beam electron capture takes place in the wave field and the wave amplitude oscillates near the capture value.

Figure 8(c) and (d) give the functions $E(Z)$ and $\alpha(Z)$ for $t > Z/W$ (in this case $Z$ is measured in 80-cm units) when the process of wave establishment is finished and the picture does not change with time. These solutions coincide with those of the boundary problems [26].

Figure 8(e)–(h) show the processes of oscillation establishment with regard to amplitude and phase and demonstrate the transition from solution of the initial problem to solution of the boundary problem (region II). These transitional zones are marked by thick lines in Fig.8(e) for the moments $\tau = 0.34$ and $\tau = 0.67$.

These analyses completely reflect the non-linear dynamics of the initial boundary problem and show that transitional processes are essential in calculations for e.m. radiation generators.

Let us now discuss non-linear theory for plasma generators. We shall try to find the solution of the field equations in the resonator in the form of two waves, one propagating in the direction of the electron beam motion $E_{Z1}(Z,t)$ and the other in the opposite direction $E_{Z2}(Z,t)$. The second wave supplies the feedback:

\begin{equation}
E_{Z1} = E_1(Z,t) \cos [\omega t - K_Z Z + \alpha(Z,t)]
\end{equation}

\begin{equation}
E_{Z2} = E_2(Z,t) \cos [\omega t + K_Z Z + \beta(Z,t)]
\end{equation}

To make the problem much simpler we assume that the $E_{Z2}$-wave does not on average interact with the electron beam. This assumption holds when the large-number wavelength modes may lay down the resonator length and it allows exclusion of the $E_{Z2}$-wave from consideration. Then the boundary conditions may be written:

\begin{equation}
E_1(0,t) = |\kappa| E_1 \left( L, t - \frac{L}{v_{go}} \right)
\end{equation}

\begin{equation}
K_Z = \frac{\pi n}{L} + \frac{\xi}{2L} + \frac{\alpha(L,t) - \alpha(0,t)}{2L}, \quad \xi = \begin{cases} 0 & \text{if } \kappa < 0 \\ \pi & \text{if } \kappa > 0 \end{cases}
\end{equation}

The meaning of the first relation is obvious: this is the feedback equation. In the second relation we may consider $\alpha(0,t) = 0$ without any restriction. Then, if $K_Z$ is known, it is possible to obtain (numerically) $\alpha(L,t) = \alpha(L,t,K_Z)$. Consequently the second equation is equivalent to $K_Z = \varphi(K_Z)$, determining the generator frequencies in the non-linear regime.
This enables us to completely solve the problem. Its mathematical formulation is as follows:

\[
E_t \left( \frac{\partial}{\partial t} + v_{go} \frac{\partial}{\partial Z} \right) \alpha = 4 \pi BJ(\sin), \quad 0 < Z < L, \quad t > 0
\]

\[
\left( \frac{\partial}{\partial t} + v_{go} \frac{\partial}{\partial Z} \right) E_t = -4 \pi BJ(\cos), \quad 0 < Z < L, \quad t > 0
\]

\[
J \left( \frac{\sin}{\cos} \right) = -\frac{e_n}{N_0} \sum_{p=1}^{N} u_p \left\{ \sin[\omega t - K_Z Z_p + \alpha(Z_p, t)] \cos[\omega t - K_Z Z_p + \alpha(Z_p, t)] \right\}
\]

\[
\frac{dZ_p}{dt} = u_p, \quad 0 \leq Z_p \leq L
\]

\[
\frac{du_p}{dt} = -\frac{e}{m} J_0 \left( \mu_0 \frac{r_0}{R} \right) \left( 1 - \frac{u_p^2}{c^2} \right)^{3/2} E_1(Z_p, t) \cos[\omega t - K_Z Z_p + \alpha(Z_p, t)]
\]

\[
K_Z = \frac{\pi n}{L} + \frac{\xi}{2L} + \frac{\alpha(L, t)}{2L}
\]

\[
D_0(\omega, K_Z) = 0, \quad \alpha(0, t) = 0, \quad t \geq 0
\]

\[
\alpha(Z, 0) = -\frac{K_Z}{2} \varphi^{1/3} Z, \quad 0 \leq Z \leq L
\]

\[
Z_p|_{t=0} = \frac{L}{nN_0} p, \quad p = 0, 1, \ldots, nN_0
\]

\[
u_p|_{t=0} = u_t = \frac{\omega}{K_Z(t=0)}, \quad K_Z(t=0) = \frac{\pi n}{L} \left( 1 - \frac{\varphi^{1/3}}{4} \right) + \frac{\xi}{2L}
\]

\[
E_t(0, t) = |\kappa| E_t \left( L, t - \frac{L}{v_{go}} \right), \quad t > \frac{L}{v_{go}}
\]
\[ E_1(Z,0) = E_0 \exp \left\{ \frac{\sqrt{3}}{2} \frac{\pi n}{L} \varphi^{1/3} + \delta \left( t - \frac{Z}{W} \right) \right\} \quad 0 \leq Z \leq L \]

\[ E_1(0,t) = E_0 e^{\delta t} \quad 0 \leq t \leq \frac{L}{v_{g0}} \] (3.55)

The quantity \( \varphi^{1/3} \) is determined in (3.25) and \( \delta \) in (3.31).

The solutions of (3.55) describe all the processes in the plasma generator, as well as the establishment processes in particular. It can be shown that this problem has stationary solutions which, if they exist, can be obtained from the following system of equations:

\[
\begin{align*}
\frac{v_{g0}}{L} \frac{\partial E_1}{\partial Z} &= 4\pi BJ(\sin) \quad \alpha(0) = 0 \\
\frac{v_{g0}}{L} \frac{\partial E_1}{\partial Z} &= -4\pi BJ(\cos), \quad E_1(0) = |\kappa|E_1(L) \\
K_Z &= \frac{\pi n}{L} + \frac{\xi}{2L} + \frac{\alpha(L)}{2L}
\end{align*}
\] (3.56)

Let us give the conditions of existence of the stationary solutions, i.e. the solubility of (3.56). It is obvious that the fields at \( Z = L, \ E_1(L) \), depend on the field at \( Z = 0, \ E_1(0) = E^* \); i.e. \( E_1(L) = E_1(L,E^*) \). Stationary solutions exist when the equation \( E^* = |\kappa|E_1(L,E^*) \) has a solution for \( E^* \) less than the capture field [27]. In the linear approximation this equation has a solution only when the beam current is equal to the threshold current, which is naturally the stationary condition in the linear regime. In the non-linear regime the stationary solutions exist if the beam current is more than the threshold current. Since the stationary solutions are of the greatest interest, we shall discuss them only.

One of the main problems of the generator theory is the optimization of the corresponding solutions for all parameters determining the conditions necessary to obtain the maximum generator conversion efficiency (GCE) and output radiation power. We shall not give such a wide programme here because the number of independent parameters in the system is large. We shall fix the beam and plasma parameters and choose the system length \( L \) and reflection coefficient \( |\kappa| \) to give maximum GCE.

We give the results of the solution for the following parameters: \( \omega_p = 12 \times 10^{10} \text{s}^{-1} \); \( R = 4.1 \text{ cm} \); \( r_0 = 2 \text{ cm} \); \( a = 0.1 \text{ cm} \); \( u_i = 2.8 \times 10^{10} \text{ cm}\text{s}^{-1} \); \( n = 3.93 \text{ L}/\pi \); \( \omega_p = 3 \times 10^{10} \text{s}^{-1} \) and \( 5 \times 10^{10} \text{s}^{-1} \).
The calculations show that generation may be optimal with regard to maximum GCE and output radiation if the field at the resonator output \( Z = L \) is equal to the capture field, which seems fairly obvious.

The optimal functions \( L(\omega) \) are shown in Fig.9 (continuous lines). The threshold currents in the plasma resonator of such \( L \) and \( \omega \) turn out to be less than \( J_b = J_{th} = 140 \text{ A} \) (for \( \omega_b = 3 \times 10^{10} \text{ s}^{-1} \), \( J_b = 1.59 \text{ kA} \)) and \( J_{th} = 4 \text{ kA} \) (for \( \omega_b = 5 \times 10^{10} \text{ s}^{-1} \), \( J_b = 4.45 \text{ kA} \)). This shows that generation in the system would take place even when minimum resonator length for a given beam current is determined from the condition \( J_b = J_{th} (L, \omega) \), where \( J_{th} \) is determined from the linear theory (3.35); for example, during \( \omega \) fixation the resonator length would be less than that given in Fig.9 by continuous lines. The corresponding threshold functions \( L(\omega) \) are given in Fig.9 by dashed lines.

Finally, we obtain GCE and output radiation power for the system with the above parameters. By definition the radiation power is

\[
P_Z = \frac{c}{4\pi} (1 - |\omega|^2) \int \int \int d\Omega [\vec{E} \times \vec{B}]_Z |_{Z=L}
\]

(3.57)

and GCE

\[
\eta = \frac{P_Z}{2\pi r_0 a n_b u_0 m e^2 (\gamma - 1)}
\]
The results of the calculations are:

\[
\eta = (1 - |\kappa|^2) \times \begin{cases} 
31\% & \text{if } \omega_b = 3 \times 10^{10} \text{ s}^{-1} \\
36\% & \text{if } \omega_b = 5 \times 10^{10} \text{ s}^{-1} 
\end{cases}
\]

\[
P_Z = (1 - |\kappa|^2) \times \begin{cases} 
0.4 \times 10^9 \text{ W} & \text{if } \omega_b = 3 \times 10^{10} \text{ s}^{-1} \\
1.2 \times 10^9 \text{ W} & \text{if } \omega_b = 5 \times 10^{10} \text{ s}^{-1} 
\end{cases}
\]

(3.58)

For this electron beam this is the maximum possible GCE and radiation power. During beam current increase, generation power would periodically change, oscillating near the maximum value, and GCE would decrease, but this is valid only in the case of single-mode generation. In high-current generators, as we have repeatedly noted, a large number of longitudinal modes are excited and therefore the oscillations of full radiation power are not expected with the increase of beam current for such multimode generators, although redistribution of power over the spectrum within the generation range is possible.

It is possible to calculate the GCE of other plasma generators in a similar way, particularly the plasma gyrotron discussed above. However, in the plasma gyrotron, the presence of the plasma plays no significant role (it is brought in to charge neutralization and to increase the beam current) and therefore the GCE of the plasma gyrotron does not differ from that calculated in Refs [21] and [23] for vacuum gyrotrons, being \( \approx 30\% \).

6. ACHIEVEMENTS IN HIGH-CURRENT UHF PLASMA ELECTRONICS

Although we are supposed to be dealing with present-day theoretical investigations, we shall, in conclusion, briefly discuss experimental successes in high-current UHF electronics. The review of experiments will be restricted to UHF generators using only REBs and based on Cherenkov and cyclotron radiation mechanisms, the theory of which has been given above.

Immediately after the appearance of the first high-current electron accelerators in the USA and the USSR, attempts were made to use them for generating high-power radiation pulses. The first experiments of the 1970s should, however, be considered as failures, since their effective radiation power was very low — GCE did not exceed 1\%. The weakness in these experiments was the complete absence of generator parameter calculations for optimal work, and this explains their comparative ineffectiveness.

The first successful experiment is considered to be the one performed in 1973 at the Lebedev Physical Institute of the USSR Academy of Sciences together
with the Gorky Radiophysical Institute [28]. According to the general theory, the
Cherenkov generator was designed with a slowing-down system in the form of a
rippled waveguide for the $E_{01}$-mode with wavelength $\lambda = 3.1$ cm, generator length
$L = 12$ cm, inner radius $1.6$ cm, ripple period $1.6$ cm and depth $0.4$ cm. The
whole system was placed in a strong magnetic field of order $20$ kG and vacuumized
up to a pressure of $2 \times 10^{-5}$ torr, the latter excluding plasma creation during beam
injection time $\tau \approx 30$ ns. The electron beam current transmitted through this
system was up to $8$ kA with energy $\epsilon = 670$ keV. Maximum generation was
obtained with $J_b \approx 5$ kA while the threshold current was $J_{th} \approx 3$ kA. GCE achieved
the values $\eta \approx 15\%$ and radiation power $P \approx 400$ MW. The radiation pulse duration
$\tau_I \approx 20$ ns and relative generation line width was less than $5\%$. The independence of
radiation power on magnetic field for $B_0 > 3$ kG proved that it was a vacuum
Cherenkov generator with a slowing-down wave (if $B_0 < 3$ kG the beam electrons
are lost intensively). The authors of Ref. [29] recently achieved radiation power
of $10^9$W with GCE $\eta \approx 30\%$ by increasing the external magnetic field to $20$ kG.

This experiment was repeated in the USA in 1974 [30], where radiation
power of $500$ MW was obtained, corresponding to GCE $\eta \approx 17\%$.

In 1975 two works [31, 32] were simultaneously published dealing with research
on Cherenkov generators using REBs. The first [31], performed at the Physical-
Technical Institute, Kharkov, USSR, dealt with research on a $3$-cm radiation
generator using a beam ($\epsilon \approx 1$ MeV, $J_b \leq 50$ kA and $\tau \approx 30$ ns) injected into the
slowing-down system of length $L = 70$ cm. The whole system was placed in a
longitudinal magnetic field up to $12$ kG and filled with gas in the pressure range
$10^{-5} - 10^{-2}$ torr. At low pressure ($p_0 < 10^{-4}$ torr), when plasma could not be
created, the vacuum Cherenkov generator worked, and maximum radiation power
was $200$ MW with pulse duration $\approx 20$ ns. The limiting vacuum current was
$J_b \approx 12$ kA, and GSE was $\eta \approx 3\%$. At high pressure ($p_0 \approx 10^{-3}$ torr) the beam
had enough time for gas ionization, and the current through the generator
increased to $J_b \approx 20$ kA, i.e. it became almost twice as high as the vacuum-limiting
current. In this case the generation power increased by a factor 3, turning out to be $600$ MW with $\eta \approx 7\%$. At very high gas pressure ($p_0 > 10^{-2}$ torr) generation
decreased owing to the creation of very high density plasma, $\omega_p > \omega$, which
shielded the radiation.

In a later work [33] the same authors repeated their experiment with a beam
$\epsilon = 0.7$ MeV, $J_b \approx 5$ kA (fixed current). In the gas pressure range $p_0 \leq 10^{-4}$ torr,
the radiation power with wavelength $\lambda \approx 3.3$ cm was $P \approx 300$ MW and exceeded
$P \approx 700$ MW when the pressure increased up to $p_0 \approx 10^{-2}$ torr (GCE in this case
was very high: $\eta \approx 22\%$). When $p_0 < 10^{-2}$ torr, generation decreased, apparently
as a result of radiation shielding by dense plasma. To interpret the experimental
results, the authors used the idea of double resonance, when the excitation of
both high- and low-frequency (plasma) E-waves is possible in the system and their
frequencies are slightly different. Their suggestion is confirmed by the fact that
maximum power of radiation is observed when $\omega_p \approx \Omega_e$. 
The second experiment was performed in 1975 in the USA [32]. There an annular beam was used with internal radius 0.8 cm and thickness 0.3 cm, beam energy $e \simeq 450$ keV, and current up to 7 kA with pulse duration $\tau \simeq 50$ ns. The beam was injected into the vacuum ($p_0 < 10^{-5}$ torr); the slowing-down system had length $L \simeq 100$ cm. Maximum radiation was obtained for current $J_b \approx 5$ kA and the power was $P \approx 600$ MW. The GCE in this case was $\eta \approx 20\%$ and radiation pulse duration $\tau \approx 40$ ns.

This review of experiments dealing with vacuum Cherenkov generators using straightforward beams concludes with work performed in the USSR (Lebedev Institute together with the Institute of Applied Physics [34]), where an 8-mm wavelength generator was achieved, in which $E_{\Pi}$-mode generation was obtained in the slowing-down system in a vacuum ($p_0 < 2 \times 10^{-5}$ torr), using an electron beam of energy $e = 670$ keV and current up to $J_b = 500$ A, with pulse duration $\tau \approx 30$ ns. Radiation power was 10 MW and GCE $\eta \approx 3\%$. Compared to the generators discussed earlier, this one has low efficiency, due to the very low beam current that barely exceeded the threshold current. If it had been possible to increase beam current to 1 kA, we should have expected an essential increase of GCE of at least 10–15%, and the radiation power would have been $\approx 100$ MW. This proved impossible because at present we are unable to obtain beam current densities exceeding $10^4$ A·cm$^{-2}$ with the highly mono-energetic electrons which are necessary for Cherenkov radiation.

The highly effective generators with cyclotron mechanism of generation using REBs appeared somewhat later. In the work done at the Lebedev Institute in 1975 [35], it was shown that when a REB was injected into the smooth metal waveguide ($e \approx 400$ keV; $J_b \leq 10$ kA; $R \approx 4$ cm) at an angle $\theta \neq 0$ to the longitudinal magnetic field, an intensive radiation appears with frequency $\omega \approx \Omega_c/\gamma$, testing the cyclotron radiation mechanism.

Later [36], the same authors made the calculations for a cyclotron generator (gyrotron) for the $M_{13}$-mode of wavelength $\lambda \approx 3$ cm. This is a smooth waveguide of diameter 7 cm and length 12 cm, through which an annular electron beam with average radius 1.5 cm and 1 mm thick was transmitted. The angle between the electron velocity and the magnetic field direction was $\approx 45^\circ$. The maximum generation power in the vacuum case was $P \approx 25$ MW when $J_b \approx 0.5$ kA, $e \approx 350$ keV, $\tau \approx 40$ ns, and GCE $\eta \approx 25\%$. When this system was filled with plasma [37], the beam current increased and the power also increased proportionally to the current. Thus the generation power increased to $P \approx 70$ MW with $\eta \approx 20\%$. This was the first generator in the world with supervacuum current (the beam current is twice as high as the vacuum-limiting current $J_0$).

The superpower vacuum gyrotron was realized in the USA in 1975 [38]. An electron beam of energy $e \approx 3.3$ MeV and current $J_b \approx 80$ kA, with pulse duration $\tau \approx 70$ ns, was injected into the smooth waveguide at an angle $7^\circ$ to the strong magnetic field. The generation power obtained was $P \approx 10^9$ W (wavelength
\( \lambda \approx 6 \text{ cm} \). Although this value is quite impressive, it should be noted that the generator worked in the obviously non-optimum regime, since \( \eta \approx 1\% \).

Essentially more optimum is the superpower gyratron (\( \lambda \approx 10 \text{ cm} \)), realized in 1975 at the Institute of Nuclear Physics, Tomsk, USSR [39]. There an electron beam with \( e \approx 1-1.2 \text{ MeV} \) and \( J_b \approx 30 \text{ kA} \), \( r_0 \approx 2 \text{ cm} \), \( \tau \approx 60 \text{ ns} \), was injected into a smooth waveguide of radius 9.8 cm. The generator length and threshold current were calculated for generating M_{11}-mode e.m. waves. The injection angle was determined by beam scattering in titanium foil 5 \( \mu \)m thick. Optimum generation for the vacuum was obtained when \( e \approx 1 \text{ MeV} \), \( J_b \approx 8 \text{ kA} \) and \( P \approx 2 \times 10^9 \text{ W} \), i.e. \( \eta \approx 30\% \). When the system was filled with gas at pressure \( p_0 \gtrsim 10^{-2} \text{ torr} \), plasma was created and radiation power decreased.

All published literature on high-current relativistic generators of the Cherenkov and cyclotron types using straightforward REBs is included in the references given here. In all the generators discussed here, the plasma compensates the beam space charge and increases the transmitting current but it does not determine the generation frequency. Pure plasma generators with plasma wave excitation by REBs have not yet been realized. Investigations of non-relativistic beams are described in the review Ref. [40]; they are not touched on here.

High-current REBs are successfully used in generators of different types, using curvilinear beams, e.g. in magnetrons, ubitrons, etc. Important results, comparable with those of Cherenkov and cyclotron generators, have also been obtained with these devices.

So we may conclude that relativistic electronics has already moved from the pure research stage and is now widespread. If we take into account the data on the successful generation experiments in the frequency regime [41], we are justified in announcing the appearance not only of a new branch of science, but a new branch of energetics — relativistic UHF energetics.

REFERENCES


## Appendix

**PROGRAMME OF THE COLLEGE**

**FIRST TWO WEEKS: 16 – 26 October (Basic Course)**

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<tr>
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<td>Macroscopic plasma properties</td>
</tr>
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<td>M.H.A. Hassan</td>
<td>Plasma kinetic theory</td>
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<tr>
<td>S. Cuperman</td>
<td>Microinstabilities and waves in homogeneous magnetized plasmas</td>
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<td>R.K. Varma</td>
<td>Plasma equilibrium and stability in magnetic confinement devices</td>
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<tr>
<td>A.A. Skorupski</td>
<td>Linear waves and instabilities in fluid plasmas</td>
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<td>A. Sen</td>
<td>Non-linear phenomena and parametric processes</td>
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<td>A. El Nadi</td>
<td>Microscopic phenomena in fusion devices</td>
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**TUTORIALS**

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<td>N. Sato</td>
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<td>Non-linear phenomena related to particle trapping</td>
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<td>Plasmas in simple <em>non-uniform</em> magnetic fields</td>
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<tr>
<td>B. McNamara</td>
<td>Computing made easy</td>
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<td>M.C. Wirth</td>
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<td>Automatic numerical analysis</td>
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<tr>
<td>J.H. Malmberg</td>
<td>Non-neutral plasma (3 lectures)</td>
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<tr>
<td>G. Culler</td>
<td>Modern mini-computers (3 lectures)</td>
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<tr>
<td>B. Lehnert</td>
<td>Experiments in poloidal field systems (2 lectures)</td>
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<tr>
<td>N. Rostoker</td>
<td>Plasma confinement with hot electrons</td>
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<tr>
<td>B. McNamara</td>
<td>Single particle motion by Lie transforms</td>
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<td>A. Nishida</td>
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<td>H.L. Berk</td>
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<td>O.N. Krokhin</td>
<td>Principles of laser fusion (2 lectures)</td>
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<tr>
<td>M.N. BUSSAC</td>
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<tr>
<td>A.A. OFFENBERGER</td>
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<tr>
<td>A.A. RUKHADZE</td>
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<td>K. NISHIKAWA</td>
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<td>T. SATO</td>
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<td>T. HATORI</td>
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<td>A. BUFFA</td>
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<td>A.N. LEBEDEV</td>
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<td>V.S. VLASENKOV</td>
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<td>F. ENGELMANN</td>
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<td>A.A. GALEEV</td>
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<td>D.E. DUCHS</td>
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<td>A. NISHIDA</td>
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<td>R.N. SUDAN</td>
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<td>S. MAHAJAN</td>
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<td>B. COPPI</td>
<td>High-field tokamaks (2 lectures)</td>
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<tr>
<td>J. KILLEEN</td>
<td>Linear and non-linear calculation of resistive MHD instabilities</td>
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<td>Fokker-Planck models of neutral-beam-driven plasmas</td>
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<td>G. HAERENDEL</td>
<td>Observations on reconnection of the Earth's field</td>
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<td>Theory of primary auroral particles</td>
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<td>M.H. BRENNAN</td>
<td>Auroral theory applied to solar flames</td>
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<td>Magnetoacoustic oscillations in current-carrying plasma (2 lectures)</td>
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<tr>
<td>J.D. CALLEN</td>
<td>Magnetic islands and tokamak transport</td>
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<td>The last word on plasma transport</td>
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</table>
FACULTY

DIRECTORS OF THE COLLEGE

An asterisk indicates that lectures are published in these Proceedings

B.B. Kadomtsev
Kurchatov Institute for Atomic Energy, Academy of Sciences of the USSR, Moscow, USSR

B. McNamara
Lawrence Livermore Laboratory, University of California, Livermore, California, USA

K. Nishikawa*
Hiroshima University, Hiroshima, Japan

D. Pfirsch
Max-Planck-Institut für Plasmaphysik, Garching, Federal Republic of Germany

M.N. Rosenbluth
School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey, USA

R.K. Varma*
Physical Research Laboratory, Navrangpura, Ahmedabad, India

EDITOR

M. Lewis
Division of Publications, IAEA, Vienna
LECTURERS

An asterisk indicates that lectures are published in these Proceedings

A.S. Bakai
Kharkov Physical-Technical Institute,
Kharkov, USSR

H.L. Berk*
Lawrence Livermore Laboratory,
University of California,
P.O. Box 808, Livermore, CA 94550, USA

M. Bornatici*
Istituto di Fisica Applicata,
Università degli Studi di Pavia,
Via A. Bassi, 6, I-27100 Pavia, Italy

M.H. Brennan
School of Physical Sciences,
Flinders University,
Bedford Park, SA 5042, Australia

A. Buffa
Centro di Studio sui Gas Ionizzati,
Associazione Euratom-CNR,
Via Gradenigo, I-35100 Padua, Italy

M.N. Bussac*
Ecole polytechnique,
Centre de physique théorique,
Plateau de Palaiseau,
91128 Palaiseau Cedex, France

B. Buti
Physical Research Laboratory,
Navrangpura, Ahmedabad 380 009, India

J.D. Callen
Oak Ridge National Laboratory,
P.O. Box Y,
Oak Ridge, TN 37830, USA

Present address:
Nuclear Engineering Department,
University of Wisconsin,
Madison, WI 53706, USA

B. Coppi
Massachusetts Institute of Technology,
Room 26-201,
Cambridge, MA 02139, USA

G. Culler
Nuclear Engineering Department,
University of Wisconsin,
Madison, WI 53706, USA

S. Cuperman*
Department of Physics and Astronomy,
Tel Aviv University,
Ramat Aviv, Israel
J.M. Dawson
Department of Physics,
University of California, Los Angeles,
Los Angeles, CA 90024, USA

O. De Barbieri
CENG, Dph/DFC/SIG,
Cedex 85, 34041 Grenoble Cedex, France

D.F. Düchs
Max-Planck-Institut für Plasmaphysik,
8046 Garching, Federal Republic of Germany

A. El Nadi*
Electrical Engineering Department,
Faculty of Engineering,
Cairo University, Giza, Egypt

F. Engelmann*
Associatie Euratom-FOM,
FOM-Instituut voor Plasmafysica "Rijnhuizen",
Postbus 7, 3430 AA Nieuwegein, Netherlands

A.A. Galeev
Space Research Institute,
Academy of Sciences of the USSR,
117810 Moscow, USSR

G. Haerendel
Max-Planck-Institut für Extraterrestrische Physik,
8046 Garching, Federal Republic of Germany

M.H.A. Hassan*
(Resident Co-ordinator)
School of Mathematical Sciences,
University of Khartoum,
P.O. Box 321, Khartoum, Sudan

T.Hatori
Institute of Plasma Physics,
Nagoya University,
Nagoya 464, Japan

S. Jardin
Plasma Physics Laboratory,
Princeton University,
James Forrestal Campus,
P.O. Box 451, Princeton, NJ 08540, USA

T. Kamimura
Institute of Plasma Physics,
Nagoya University,
Nagoya, Japan

P.K. Kaw
Plasma Physics Laboratory,
Princeton University,
James Forrestal Campus,
P.O. Box 451, Princeton, NJ 08540, USA

C.F. Kennel
Department of Physics,
University of California, Los Angeles,
Los Angeles, CA 90024, USA
J. Killeen* National MFE Computer Center, Lawrence Livermore Laboratory, University of California, P.O. Box 5509, Livermore, CA 94550, USA

O.N. Krokhin P.N. Lebedev Physical Institute, Leninskie Prospect 53, 117924 Moscow, USSR

A.N. Lebedev P.N. Lebedev Physical Institute, Leninskie Propekt 53, 117924 Moscow, USSR

B. Lehnert* Royal Institute of Technology, S-100 44 Stockholm, Sweden

S. Mahajan Fusion Research Center, University of Texas at Austin, Austin, TX 78712, USA

J.H. Malmberg Department of Physics, B-019, University of California, San Diego, La Jolla, CA 92093, USA

A. Nishida Institute of Space and Aeronautical Science, University of Tokyo, Komaba, Meguro-ku, Tokyo, Japan

K. Nishihara Institute of Laser Engineering, Osaka University, Suita, Osaka, Japan

A.A. Offenberger* Department of Electrical Engineering, University of Alberta, Edmonton, T6G 2E1 Alberta, Canada

N. Rostoker Physics Department, University of California, Irvine, Irvine, CA 92717, USA

A.A. Rukhadze* P.N. Lebedev Physical Institute, Leninskie Prospect 53, 117924 Moscow, USSR

P.H. Sakanaka* Instituto de Fisica “Gleb Wataghin”, Universidad Estudal de Campinas, 13100 Campinas (São Paulo), Brazil

N. Sato Department of Electronic Engineering, Tohoku University, 980 Sendai, Japan
T. Sato
Institute of Geophysical Research, Faculty of Science, University of Tokyo, Hongo, Bunkyo-ku, Tokyo, Japan

A. Sen*
Physical Research Laboratory, Navrangpura, Ahmedabad 380 009, India

A. Skorupski*
Institute of Nuclear Research, Hoża 69, 00—681 Warsaw, Poland

R.N. Sudan*
Laboratory of Plasma Studies, Cornell University, Ithaca, N.Y. 24850, USA

V.S. Vlasenkov
IAEA, P.O. Box 100, A-1400 Vienna

R.B. White
Plasma Physics Laboratory, Princeton University, James Forrestal Campus, P.O. Box 451, Princeton, NJ 08540, USA

M.C. Wirth
Department of Physics, University of California, Davis, Davis, CA 95616, USA

L. Wood
Lawrence Livermore Laboratory, University of California, P.O. Box 808, Livermore, CA 94550, USA
# LIST OF PARTICIPANTS

<table>
<thead>
<tr>
<th>Name</th>
<th>Institution</th>
<th>Country</th>
</tr>
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<tbody>
<tr>
<td>I. Abonyi</td>
<td>Institute of Theoretical Physics, Roland Eötvös University, Puskin u. 5-7, H-1088 Budapest, Hungary</td>
<td>Hungary</td>
</tr>
<tr>
<td>P.S. Abrol</td>
<td>Physics Department, University of Rajasthan, Jaipur 302004, India</td>
<td>India</td>
</tr>
<tr>
<td>J.M. Akkermans</td>
<td>FOM-Institute for Plasma Physics &quot;Rijnhuizen&quot;, Postbus 7, 3430 AA Nieuwegein, Netherlands</td>
<td>Netherlands</td>
</tr>
<tr>
<td>F.O. Akuffo</td>
<td>Department of Mechanical Engineering, University of Science and Technology, Kumasi, Ghana</td>
<td>Ghana</td>
</tr>
<tr>
<td>K.H. Alfsen</td>
<td>Institute of Physics, University of Oslo, Blindern, Oslo 3, Norway</td>
<td>Norway</td>
</tr>
<tr>
<td>A.M. Al-Hassoun</td>
<td>Physics Department, College of Science, University of Baghdad, Baghdad, Iraq</td>
<td>Iraq</td>
</tr>
<tr>
<td>I. Azhar</td>
<td>Pakistan Institute of Nuclear Science and Technology, P.O. Nilore, Rawalpindi, Pakistan</td>
<td>Pakistan</td>
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<tr>
<td>T.J. Blenski</td>
<td>Computing Centre Cyfronet, Institute of Nuclear Research, Swierk, 05-400 Otwock, Poland</td>
<td>Poland</td>
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<tr>
<td>M. Cercek</td>
<td>Jožef Štefan Institute, P.O. Box 199, 61000 Ljubljana, Yugoslavia</td>
<td>Yugoslavia</td>
</tr>
<tr>
<td>R.K. Chajlani</td>
<td>School of Studies in Physics, Vikram University, Ujjain, M.P. 456 010, India</td>
<td>India</td>
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<tr>
<td>Y.H. Chen</td>
<td>Department of Physics, University of Malaya, Kuala Lumpur 23-11, Malaysia</td>
<td>Malaysia</td>
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<tr>
<td>A.-C. Chew</td>
<td>Department of Physics, University of Malaya, Kuala Lumpur 23-11, Malaysia</td>
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<tr>
<td>U. De Angelis</td>
<td>Istituto di Fisica Sperimentale, Università di Napoli, Via Tari 3, I-80100 Naples, Italy</td>
<td>Italy</td>
</tr>
<tr>
<td>C.J. Diaz</td>
<td>International Seminar in Physics, University of Uppsala, Box 530, S-751 21 Uppsala, Sweden</td>
<td>Colombia</td>
</tr>
<tr>
<td></td>
<td>Permanent address: Department of Physics, Universidad del Valle, Apartado Aéreo 2188, Cali, Colombia</td>
<td></td>
</tr>
<tr>
<td>A. Douiri</td>
<td>Département de physique, Faculté des sciences, Université Moham, Avenue Ibn Batota, Rabat, Morocco</td>
<td>Morocco</td>
</tr>
<tr>
<td>T. Duracz</td>
<td>Computing Centre Cyfronet, Institute of Nuclear Research, Swierk, 05–400 Otwock, Poland</td>
<td>Poland</td>
</tr>
<tr>
<td>I. Durrani</td>
<td>Nuclear Physics Division, Pakistan Institute of Nuclear Science and Technology, P.O. Nilore, Rawalpindi, Pakistan</td>
<td>Pakistan</td>
</tr>
<tr>
<td>M.K. El-Fayoumi</td>
<td>Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City, Cairo, Egypt</td>
<td>Egypt</td>
</tr>
<tr>
<td>S.M. El Khoga</td>
<td>Science Centre for Advancement of Postgraduate Studies, Horreya Avenue, Shatby, Alexandria, Egypt</td>
<td>Egypt</td>
</tr>
<tr>
<td>C. Ferro Fontan</td>
<td>Laboratorio de Física del Plasma, Ciudad Universitaria – Pab. 1, 1428 Buenos Aires, Argentina</td>
<td>Argentina</td>
</tr>
<tr>
<td>A. Forlani</td>
<td>Istituto di Fisica Sperimentale, Università di Napoli, Via Antonio Tari 3, I-80100 Naples, Italy</td>
<td>Italy</td>
</tr>
</tbody>
</table>
LIST OF PARTICIPANTS

A.A. Gabr  
Physics Department, Faculty of Science,  
Cairo University,  
Cairo, Egypt

R.O. Genga  
Department of Physics,  
Boston College,  
Chestnut Hill, MA 02167, USA

J. Goedert  
Departamento de Fisica,  
Universidade Federal da Paraiba,  
58000-João Pessoa, Brazil

S.K. Goel  
Laser Section,  
Bhabha Atomic Research Centre,  
Trombay, Bombay 400 085, India

P.P. Goldstein  
Nuclear Theory Department,  
Institute for Nuclear Research,  
Hoża 69, 00-681 Warsaw, Poland

S.K. Guharay  
Saha Institute of Nuclear Physics,  
92 Acharya Prafulla Chandra Road,  
Calcutta 700 009, India

S.C. Gupta  
Physics Department,  
Punjabi University,  
Patiala 147002, India

S. Gurung  
Physics Department,  
Tribhuvan University,  
Kirtipur Campus, Katmandu, Nepal

S.A.M. Hamadto  
Physics Department, Faculty of Science,  
University of Khartoum,  
Khartoum, Sudan

M. Hamdan  
Higher Petroleum Institute,  
P.O.Box 201, Tobruk, Libya

K. Hanatani  
Plasma Physics Laboratory,  
Kyoto University,  
Gokasho, Uji, Kyoto, Japan

J.E. Herrera  
Centro de Estudios Nucleares,  
Circuito Exterior, C.U.,  
Mexico 20, D.F., Mexico

Y.P. Ho  
Institute of Plasma Physics,  
Chinese Academy of Sciences,  
Hefei (Hofei), Anhui (Anhwei) Province,  
People's Republic of China

Libya/Egypt

People's Republic of China

Japan

Mexico

USA/Kenya
LIST OF PARTICIPANTS

H. Hojo
Faculty of Science,
Hiroshima University,
1-1-89 Higashisenda-machi,
Hiroshima, Japan

M. Imamuddin
Department of Physics,
Jahangirnagar University,
Savar, Dacca, Bangladesh

B. Keita
Département de physique,
Faculté des sciences,
Université de Dakar,
Dakar-Fann, Senegal

A.H. Khater
Physics Department,
Universitaire Instelling Antwerpen,
Universiteitsplein 1,
2610 Wilrijk, Antwerp, Belgium

Permanent address
Mathematics Department,
Faculty of Science,
Assiut University,
Assiut, Egypt

V.S. Krishan
Indian Institute of Astrophysics,
Bangalore – 560 034, India

A.M. Kurbatov
V.A. Steklov Mathematical Institute,
117333 Moscow, USSR

E. Leal
Departamento de Física,
Universidad Simón Bolivar,
Apartado 80659, Caracas – 108,
Venezuela

J. Liu
Plasma Laboratory,
Institute of Physics,
Chinese Academy of Sciences,
Beijing (Peking), People's Republic of China

M. Lontano
Laboratorio de Fisica del Plasma,
Associazione CNR-Euratom,
Via Bassini 15, I-20133 Milan, Italy

M. Maheswaran
Department of Mathematics,
University of Peradeniya,
Peradeniya, Sri Lanka
E. Maschke  
Association Euratom-CEA,  
Dph-PFC/STGI,  
BP N° 6, F-92260 Fontenay-aux-Roses, France  

G. Medrano  
Departamento de Física Aplicada (C–XII),  
Universidad Autónoma,  
Madrid (Cantoblanco), Spain  

S. Mercurio  
Associazione Euratom-CNEN,  
I-00044 Frascati (Rome), Italy  

D. Merlini  
Department of Electrical Engineering,  
Northeastern University,  
Boston, MA 02115, USA  

J.H. Misguich  
STGI/CEA,  
P.B. N° 6, F-92260 Fontenay-aux-Roses, France  

N.R. Mossaad  
Science Centre for Advancement of Post-Graduate Studies,  
Alexandria University,  
Horreya Avenue, Shatby, Alexandria, Egypt  

G. Murtaza  
Department of Physics,  
Quaid-i-Azam University,  
Islamabad, Pakistan  

N. Nagesha Rao  
Theoretical Physics Section,  
Physical Research Laboratory,  
Navrangpura, Ahmedabad 380 009, India  

L.C. Nascimento  
Instituto de Física,  
Universidade de São Paulo,  
C.P. 20516, CEP 05508, São Paulo, Brazil  

K. Niederl  
Institut für Theoretische Physik und Reaktorphysik,  
Technische Universität Graz,  
Petersgasse 16, A-8010 Graz, Austria  

G. Pastore  
Istituto di Fisica Sperimentale,  
Università di Napoli,  
Via Tari 3, I-80100 Naples, Italy
<table>
<thead>
<tr>
<th>Name</th>
<th>Institution</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>S. Poberaj</td>
<td>Jožef Štefan Institute, P.O. Box 199, 61001 Ljubljana, Yugoslavia</td>
<td>Yugoslavia</td>
</tr>
<tr>
<td>R. Pozzoli</td>
<td>Laboratorio de Fisica del Plasma, Associazione Euratom-CNR, Via Bassini 15, I-20133 Milan, Italy</td>
<td>Italy</td>
</tr>
<tr>
<td>W. Rozmus</td>
<td>Institute for Nuclear Research, Hoża 69, 00-681 Warsaw, Poland</td>
<td>Poland</td>
</tr>
<tr>
<td>H. Rucker</td>
<td>Institute of Space Research, Austrian Academy of Sciences, Balbärthgasse 1, A-8010 Graz, Austria</td>
<td>Austria</td>
</tr>
<tr>
<td>A. Rueda</td>
<td>Departamento de Física, Universidad de Los Andes, Apartado Aereo 4976, Bogotá 1, D.E., Colombia</td>
<td>Colombia</td>
</tr>
<tr>
<td>J.J. Santamarina</td>
<td>Istituto de Física, Universidad Austral de Chile, Casilla 567, Valdivia, Chile</td>
<td>Chile/Argentina</td>
</tr>
<tr>
<td>Y.C. Saxena</td>
<td>Physical Research Laboratory, Navrangpura, Ahmedabad – 380009, India</td>
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</tr>
<tr>
<td>A. Sengupta</td>
<td>Department of Mechanical Engineering, Indian Institute of Technology, Kanpur – 208016, U.P., India</td>
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</tr>
<tr>
<td>S.C. Sharma</td>
<td>Department of Physics, University of Cape Coast, P.O. Box 034, Cape Coast, Ghana</td>
<td>Ghana/India</td>
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<tr>
<td>M. Sharma Devi</td>
<td>Department of Physics, Gauhati University, Gauhati – 781014, Assam, India</td>
<td>India</td>
</tr>
<tr>
<td>C.M. Singh</td>
<td>Physical Research Laboratory, Navrangpura, Ahmedabad – 380009, India</td>
<td>India</td>
</tr>
<tr>
<td>A.J. Smith</td>
<td>Department of Physics, Njala University College, Private Mail Bag, Freetown, Sierra Leone</td>
<td>Sierra Leone</td>
</tr>
<tr>
<td>D.S. Spicer</td>
<td>Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA</td>
<td>USA</td>
</tr>
</tbody>
</table>
LIST OF PARTICIPANTS

K.M. Srivastava  
Department of Mathematics,  
University of Roorkee,  
Roorkee – 247672, U.P., India

M.P. Srivastava  
Department of Physics and Astrophysics,  
University of Delhi,  
Delhi – 110007, India

A.S. Sulastri  
Puslit Gama Batan (Gama Research Centre),  
P.O. Box 8, Yogyakarta, Indonesia

N.A. Tahir  
Department of Natural Philosophy,  
The University,  
Glasgow, G12 8QQ, Scotland,  
United Kingdom

K.L. Tan  
Physics Department,  
University of Singapore,  
Bukit Timah, Singapore 10

R. Tavzes  
Jožef Štefan Institute,  
P.O. Box 199, 61001 Ljubljana,  
Yugoslavia

D.P. Tewari  
Department of Physics,  
Indian Institute of Technology,  
New Delhi – 110029, India

C. Tezcan  
Physics Department,  
Middle East Technical University,  
Inönü Bulvari, Ankara, Turkey

V. Thevendran  
Department of Engineering Mathematics,  
Faculty of Engineering,  
University of Peradeniya,  
Peradeniya, Sri Lanka

S.-T. Tsai  
Plasma Laboratory,  
Institute of Physics,  
Chinese Academy of Sciences,  
Beijing (Peking), People's Republic of China

J.W. Vasquez  
Departamento Academico de Física,  
Universidad Nacional Mayor de San Marcos,  
Lima, Peru

F.G. Verheest  
Rijksuniversiteit Gent,  
Krijghsnaan 271-S9, B-9000 Gent, Belgium
LIST OF PARTICIPANTS

K.S. Viswanathan  
Department of Physics,  
Kerala University,  
Kariavattum Campus, Trivandrum-17, India  
India

I. Weiss  
Department of Physics and Astronomy,  
Tel-Aviv University,  
Ramat-Aviv, Tel-Aviv, Israel  
Israel

R.Z. Yahel  
Max-Planck-Institut für Physik und Astrophysik,  
Institut für Extraterrestrische Physik,  

Y. Yasaka  
Department of Electronics,  
Faculty of Engineering,  
Kyoto University,  
Kyoto 606, Japan  
Japan

L.F. Ziebell  
Instituto de Fisica,  
Universidade Federal do Rio Grande do Sul,  
Rua Luiz Englert s/no,  
9000 – Porto Alegre – RS, Brazil  
Brazil
FACTORS FOR CONVERTING SOME OF THE MORE COMMON UNITS TO INTERNATIONAL SYSTEM OF UNITS (SI) EQUIVALENTS

NOTES:
1. SI base units are the metre (m), kilogram (kg), second (s), ampere (A), kelvin (K), candela (cd) and mole (mol).
2. ▲ indicates SI derived units and those accepted for use with SI;
   △ indicates additional units accepted for use with SI for a limited time.
3. The correct abbreviation for the unit in column 1 is given in column 2.
4. -X- indicates conversion factors given exactly; other factors are given rounded, mostly to 4 significant figures.
5. = indicates a definition of an SI derived unit.

The following conversion table is provided for the convenience of readers and to encourage the use of SI units.

<table>
<thead>
<tr>
<th>Column 1 Multiply data given in:</th>
<th>Column 2</th>
<th>Column 3</th>
<th>Column 4 to obtain data in:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radiation units</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>▶ becquerel</td>
<td>1 Bq</td>
<td>(has dimensions of s⁻¹)</td>
<td></td>
</tr>
<tr>
<td>disintegrations per second (= dis/s)</td>
<td>1 s⁻¹</td>
<td>= 1.00 × 10⁶ Bq *</td>
<td></td>
</tr>
<tr>
<td>▶ curie</td>
<td>1 Ci</td>
<td>= 3.70 × 10⁶ Bq *</td>
<td></td>
</tr>
<tr>
<td>▶ roentgen</td>
<td>1 R</td>
<td>= 2.58 × 10⁴ C/kg *</td>
<td></td>
</tr>
<tr>
<td>▶ gray</td>
<td>1 Gy</td>
<td>= 1.00 × 10² J/kg *</td>
<td></td>
</tr>
<tr>
<td>▶ rad</td>
<td>1 rad</td>
<td>= 1.00 × 10⁻² Gy *</td>
<td></td>
</tr>
<tr>
<td>sievert (radiation protection only)</td>
<td>1 Sv</td>
<td>= 1.00 × 10⁻² J/kg *</td>
<td></td>
</tr>
<tr>
<td>rem (radiation protection only)</td>
<td>1 rem</td>
<td>= 1.00 × 10⁻² J/kg *</td>
<td></td>
</tr>
<tr>
<td>Mass</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>▶ unified atomic mass unit ((\frac{1}{12}) of the mass of (^{12})C)</td>
<td>1 u</td>
<td>= 1.660 57 × 10⁻²⁷ kg, approx.</td>
<td></td>
</tr>
<tr>
<td>▶ tonne (= metric ton)</td>
<td>1 t</td>
<td>= 1.00 × 10³ kg *</td>
<td></td>
</tr>
<tr>
<td>pound mass (avoirdupois)</td>
<td>1 lbm</td>
<td>= 4.536 × 10⁻¹ kg</td>
<td></td>
</tr>
<tr>
<td>ounce mass (avoirdupois)</td>
<td>1 ozm</td>
<td>= 2.835 × 10⁻¹ g</td>
<td></td>
</tr>
<tr>
<td>ton (long) (= 2240 lbm)</td>
<td>1 ton</td>
<td>= 1.016 × 10³ kg</td>
<td></td>
</tr>
<tr>
<td>ton (short) (= 2000 lbm)</td>
<td>1 short ton</td>
<td>= 9.072 × 10² kg</td>
<td></td>
</tr>
<tr>
<td>Length</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>statute mile</td>
<td>1 mile</td>
<td>= 1.609 × 10⁶ km</td>
<td></td>
</tr>
<tr>
<td>nautical mile (international)</td>
<td>1 n mile</td>
<td>= 1.852 × 10⁶ km *</td>
<td></td>
</tr>
<tr>
<td>yard</td>
<td>1 yd</td>
<td>= 9.144 × 10⁻¹ m *</td>
<td></td>
</tr>
<tr>
<td>foot</td>
<td>1 ft</td>
<td>= 3.048 × 10⁻¹ m *</td>
<td></td>
</tr>
<tr>
<td>inch</td>
<td>1 in</td>
<td>= 2.54 × 10⁻¹ mm *</td>
<td></td>
</tr>
<tr>
<td>mil (= 10⁻³ in)</td>
<td>1 mil</td>
<td>= 2.54 × 10⁻² mm *</td>
<td></td>
</tr>
<tr>
<td>Area</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>▶ hectare</td>
<td>1 ha</td>
<td>= 1.00 × 10⁶ m² *</td>
<td></td>
</tr>
<tr>
<td>▶ barn (effective cross-section, nuclear physics)</td>
<td>1 b</td>
<td>= 1.00 × 10⁻²⁸ m² *</td>
<td></td>
</tr>
<tr>
<td>square mile, (statute mile)²</td>
<td>1 mile²</td>
<td>= 2.590 × 10⁵ km²</td>
<td></td>
</tr>
<tr>
<td>acre</td>
<td>1 acre</td>
<td>= 4.047 × 10⁵ m²</td>
<td></td>
</tr>
<tr>
<td>square yard</td>
<td>1 yd²</td>
<td>= 8.361 × 10⁻¹ m²</td>
<td></td>
</tr>
<tr>
<td>square foot</td>
<td>1 ft²</td>
<td>= 9.290 × 10⁻² m²</td>
<td></td>
</tr>
<tr>
<td>square inch</td>
<td>1 in²</td>
<td>= 6.452 × 10⁻² mm²</td>
<td></td>
</tr>
<tr>
<td>Volume</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>▶ litre</td>
<td>1 l or 1 ltr</td>
<td>= 1.00 × 10⁻³ m³ *</td>
<td></td>
</tr>
<tr>
<td>cubic yard</td>
<td>1 yd³</td>
<td>= 7.646 × 10⁻³ m³</td>
<td></td>
</tr>
<tr>
<td>cubic foot</td>
<td>1 ft³</td>
<td>= 2.832 × 10⁻² m³</td>
<td></td>
</tr>
<tr>
<td>cubic inch</td>
<td>1 in³</td>
<td>= 1.639 × 10⁻⁶ mm³</td>
<td></td>
</tr>
<tr>
<td>gallon (imperial)</td>
<td>1 gal (UK)</td>
<td>= 4.546 × 10⁻³ m³</td>
<td></td>
</tr>
<tr>
<td>gallon (US liquid)</td>
<td>1 gal (US)</td>
<td>= 3.785 × 10⁻³ m³</td>
<td></td>
</tr>
<tr>
<td>Velocity, acceleration</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>foot per second (= fps)</td>
<td>1 ft/s</td>
<td>= 3.048 × 10⁻¹ m/s *</td>
<td></td>
</tr>
<tr>
<td>foot per minute</td>
<td>1 ft/min</td>
<td>= 5.08 × 10⁻³ m/s *</td>
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</tr>
<tr>
<td>mile per hour (= mph)</td>
<td>1 mile/h</td>
<td>= 1.609 × 10⁴ km/h</td>
<td></td>
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<tr>
<td>knot (international)</td>
<td>1 knot</td>
<td>= 1.852 × 10³ km/h *</td>
<td></td>
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<tr>
<td>free fall, standard, g</td>
<td>1 g</td>
<td>= 9.807 × 10⁵ m/s² *</td>
<td></td>
</tr>
<tr>
<td>foot per second squared</td>
<td>1 ft/s²</td>
<td>= 3.048 × 10⁻¹ m/s² *</td>
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</tbody>
</table>

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<tr>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
<th>Column 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, volumetric rate</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>pound mass per cubic inch</td>
<td>1 lbm/in³ = 2.768 × 10⁶ kg/m³</td>
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<tr>
<td>pound mass per cubic foot</td>
<td>1 lbm/ft³ = 1.062 × 10⁹ kg/m³</td>
<td></td>
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</tr>
<tr>
<td>cubic feet per second</td>
<td>1 ft³/s = 1.832 × 10⁻³ m³/s</td>
<td></td>
<td></td>
</tr>
<tr>
<td>cubic feet per minute</td>
<td>1 ft³/min = 4.719 × 10⁻⁴ m³/s</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Force</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>newton</td>
<td>1 N = 1.00 × 10⁶ m·kg/s²</td>
<td></td>
<td></td>
</tr>
<tr>
<td>dyne</td>
<td>1 dyn = 1.00 × 10⁻⁶ N</td>
<td></td>
<td></td>
</tr>
<tr>
<td>kilogram force (= kilopond (kpl))</td>
<td>1 kgf = 9.807 × 10⁶ N</td>
<td></td>
<td></td>
</tr>
<tr>
<td>poundal</td>
<td>1 pdl = 1.383 × 10⁻⁷ N</td>
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<td></td>
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<tr>
<td>pound force (avoirdupois)</td>
<td>1 lbf = 4.448 × 10⁶ N</td>
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<td></td>
</tr>
<tr>
<td>ounce force (avoirdupois)</td>
<td>1 ozf = 2.780 × 10⁵ N</td>
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<td></td>
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<tr>
<td>Pressure, stress</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>atmosphere, standard</td>
<td>1 atm = 1.01325 × 10⁵ Pa</td>
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<td></td>
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<tr>
<td>bar</td>
<td>1 bar = 1.00 × 10⁵ Pa</td>
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<td></td>
</tr>
<tr>
<td>centimetres of mercury (0°C)</td>
<td>1 cmHg = 1.333 × 10⁻⁵ Pa</td>
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<td></td>
</tr>
<tr>
<td>dyne per square centimetre</td>
<td>1 dyn/cm² = 1.00 × 10⁻⁵ Pa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>feet of water (4°C)</td>
<td>1 ftH₂O = 2.989 × 10⁻³ Pa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>inches of mercury (0°C)</td>
<td>1 inHg = 3.386 × 10⁻² Pa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>inches of water (4°C)</td>
<td>1 inH₂O = 2.491 × 10⁻² Pa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>kilogram force per square centimetre</td>
<td>1 kgf/cm² = 9.087 × 10⁴ Pa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>pound force per square foot</td>
<td>1 lbf/ft² = 4.788 × 10⁻² Pa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>pound force per square inch (= psi)</td>
<td>1 lbf/in² = 6.895 × 10³ Pa</td>
<td></td>
<td></td>
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<tr>
<td>torr (0°C) (≈ mmHg)</td>
<td>1 torr = 1.333 × 10⁻² Pa</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Energy, work, quantity of heat</td>
<td></td>
<td></td>
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<tr>
<td>joule (≡ W·s)</td>
<td>1 J = 1.00 × 10⁶ N·m</td>
<td></td>
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<tr>
<td>electronvolt</td>
<td>1 eV = 1.602 19 × 10⁻¹⁹ J, approx.</td>
<td></td>
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<tr>
<td>British thermal unit (International Table)</td>
<td>1 Btu = 1.055 × 10⁶ J</td>
<td></td>
<td></td>
</tr>
<tr>
<td>calorie (thermochemical)</td>
<td>1 cal = 4.184 × 10⁶ J</td>
<td></td>
<td></td>
</tr>
<tr>
<td>calorie (International Table)</td>
<td>1 cal_I = 4.187 × 10⁶ J</td>
<td></td>
<td></td>
</tr>
<tr>
<td>erg</td>
<td>1 erg = 1.00 × 10⁻⁷ J</td>
<td></td>
<td></td>
</tr>
<tr>
<td>foot-pound force</td>
<td>1 ft·lbf = 1.356 × 10⁶ J</td>
<td></td>
<td></td>
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<tr>
<td>kilowatt-hour</td>
<td>1 kW·h = 3.60 × 10⁶ J</td>
<td></td>
<td></td>
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<tr>
<td>kiloton explosive yield (PNE) (≡ 10¹² g-cal)</td>
<td>1 kt yield = 4.2 × 10¹² J</td>
<td></td>
<td></td>
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<tr>
<td>Power, radiant flux</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>watt</td>
<td>1 W = 1.00 × 10⁶ J/s</td>
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<tr>
<td>British thermal unit (International Table) per second</td>
<td>1 Btu/s = 1.055 × 10³ W</td>
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<td></td>
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<tr>
<td>calorie (International Table) per second</td>
<td>1 cal/s = 4.187 × 10⁶ W</td>
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<tr>
<td>foot-pound force/second</td>
<td>1 ft·lbf/s = 1.356 × 10⁶ W</td>
<td></td>
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<tr>
<td>horsepower (electric)</td>
<td>1 hp = 7.46 × 10³ W</td>
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<td></td>
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<tr>
<td>horsepower (metric) (= ps)</td>
<td>1 ps = 7.355 × 10³ W</td>
<td></td>
<td></td>
</tr>
<tr>
<td>horsepower (550 ft-lbf/s)</td>
<td>1 hp = 7.457 × 10³ W</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Temperature</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>temperature in degrees Celsius, t</td>
<td>t = T - T₀</td>
<td></td>
<td></td>
</tr>
<tr>
<td>where T is the thermodynamic temperature in kelvin and T₀ is defined as 273.15 K</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>degree Fahrenheit</td>
<td>t₁₀₀ = 32</td>
<td></td>
<td></td>
</tr>
<tr>
<td>degree Rankine</td>
<td>T₁₀₀ =</td>
<td></td>
<td></td>
</tr>
<tr>
<td>degrees of temperature difference</td>
<td>ΔT = ΔT₁₀₀</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 Btu/in²·s·°F (International Table Btu)</td>
<td>5.192 × 10² W·m⁻¹·K⁻¹</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 Btu/ft²·s·°F (International Table Btu)</td>
<td>6.231 × 10³ W·m⁻¹·K⁻¹</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 cal/ft²·s·°C</td>
<td>4.187 × 10⁵ W·m⁻¹·K⁻¹</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a atm abs, ata: atmospheres absolute; b lbf/in² (g) (≡ psig): gauge pressure; atm (g), atu: atmospheres gauge lbf/in² abs (≡ psla): absolute pressure.

c The abbreviation for temperature difference, deg (≡ degK = degC), is no longer acceptable as an SI unit.
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